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An Extended Alpha Power Family of Distributions: Its Applications to the Scientific Data

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Abstract: In this study, a Novel Extended Alpha Power (NEAP) family of distributions is introduced to improve efficiency of the existing class of lifetime distributions. A sub-mode of NEAP is further studied using Weibull distribution as an input model. This new version of distribution is referred as novel extended alpha power Weibull (NEAPW) distribution. The new distribution is suitable for modeling both monotone and non-monotone type data. Various statistical characteristics of the suggested model are estimation of parameters, the order statistics, mean residual, quantile function, and moments are obtained. A simulation study of the novel distribution is also conducted. The usefulness and effectiveness of the novel model is established by investigating two real data sets from the field of basic sciences.

1. Introduction

In the statistical distribution theory, adding further parameter(s) to the present family of distribution is a common practice. Usually adding further parameter(s) brings enhanced flexibility to probability function. In the recent years, the development of new distributions and new family of distributions has become popular. This is because, the classical distribution is insufficient for modeling the real data set. In this connection, Azzalini [1] presented skewed normal distribution. Mudholkar and Srivastava [2] presented a new technique to incorporate a new parameter to the baseline distributions. Cordeiro et al. [3] developed a new method known as exponentiated generalized class of distribution. Alzaatreh et al. [4] generated a family of

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distributions. A new technique was suggested by Marshal and Olkin [5]. Rehman et al. [6] introduced generalize transmuted family of distribution and they gave the as K-transmuted family. Mahadavi and Kundu [7] produced a new technique for developing distribution(s) and the called it as alpha power transformation (APT) technique. Khan and King worked on transmuted modified Weibull distribution, Alizadeh et al. [9] produced generalized transmuted family of distribution, Afify et al. [10] developed the Kumaraswamy transmuted G family of distributions and Bourguignon et al. [11] produced general result for transmuted family of distribution. Oluyede and Yang [12] have studied Beta generated family of distribution. Besides this Rashid and Jan [13] have introduced a new family of distribution by compounding Lindly distribution with power series distribution.

Corderio and Castro [14] proposed Kumaraswamy type-1 class of distributions, and is given by

$$G(x)=1-\left(1-\left(F(x)\right)^{\alpha}\right)^{\beta} \qquad \alpha,\beta>0 \tag{1}$$

A new family of distribution called Kumaraswamy type 2 was proposed by Tahir and Nadarajah [15]. The cumulative distribution function (CDF) of the proposed family of distribution is defined by

$$F(x)=1-\left(1-\left(1-G(x)\right)^{\beta}\right)^{\lambda} \qquad \beta,\lambda>0 \tag{2}$$

A new generated family of distribution presented by Zografos and Balakrishnan [16] is written as

$$G(x)=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_{0}^{-\ln\left(1-G(x)\right)} t^{\alpha-1}e^{-\beta t}\,dt \tag{3}$$

The alpha power transformation technique [7] is defined as follows:

The CDF of the APT is given as

$$G(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1} & \alpha>0,\alpha\neq 1\\ F(x) & \alpha=1\end{cases} \tag{4}$$

In this study, an effort will be made to develop new class of distribution called a novel extended alpha power (NEAP) family of distributions. The particular case of this class is studied by inputting the Weibull distribution referred as Novel Extended Alpha Power Weibull (NEAPW) distribution. The new proposal would help in modeling the phenomenon taking both monotone and non-monotone hazard rate forms.

The Proposed Family

The CDF of the new family is defined by the following expression

$$G(x)=\begin{cases}\dfrac{(F(x)+1)^{\alpha}-1}{2^{\alpha}-1} & \alpha>0,\alpha\neq 1,x>0\\ F(x) & \alpha=1\end{cases} \tag{5}$$

The probability density function (PDF) corresponding to (5) is given by

$$f(x)=\begin{cases}\dfrac{\alpha}{2^{\alpha}-1}\left(f(x)\left(F(x)+1\right)^{\alpha-1}\right) & if\ \ \alpha,x>0,\alpha\neq 1\\ f(x) & if\ \alpha=1\end{cases} \tag{6}$$

The corresponding survival function (SF), hazard rate function (HRF) and reversed hazard rate function (RHRF) of the NEAP family of distributions are obtain as

$$s(x)=\begin{cases}\dfrac{2^{\alpha}-\left(F(x)+1\right)^{\alpha}}{2^{\alpha}-1} & if\ \alpha,x>0,\alpha\neq 1\\ s(x) & if\ \ \alpha=1\end{cases} \tag{7}$$

$$h(x)=\begin{cases}\dfrac{\alpha f(x)\left(F(x)+1\right)^{\alpha-1}}{2^{\alpha}-\left(F(x)+1\right)^{\alpha}} & if\ \ \alpha,x>0,\ \alpha\neq 1\\ h(x) & if\ \alpha=1\end{cases} \tag{8}$$

$$r(x)=\begin{cases}\dfrac{\alpha f(x)\left(F(x)+1\right)^{\alpha-1}}{\left(F(x)+1\right)^{\alpha}-1} & if\ \alpha,x>0,\alpha\neq 1\\ r(x) & if\ \ \alpha=1\end{cases} \tag{9}$$

The NEAP family is appealing, flexible and effective method for inducing an extra parameter(s) to generalize the baseline distribution. The proposed distribution claims to have superior flexibility. It efficiently models lifetime data sets having monotonically increasing and decreasing failure rates.

Weibull distribution

The PDF and CDF of the Weibull distribution respectively is given as

$$f(x)=\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}} \qquad \lambda,\beta,x>0 \tag{10}$$

$$F(x)=1-e^{-\lambda x^{\beta}} \qquad x>0 \tag{11}$$

The Proposed Model: Novel Extended Alpha Power Weibull (NEAPW) distribution

Let $X\sim NEAPW(\alpha,\beta,\lambda)$. Then its PDF and CDF can be written as

$$f(x)=\begin{cases}\dfrac{\alpha\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}(2-e^{-\lambda x^{\beta}})^{\alpha-1}}{2^{\alpha}-1} & x,\alpha,\beta,\ \lambda>0,\ \ \alpha\neq 1\\ f(x)_{W} & \alpha=1\end{cases} \tag{12}$$

$$F(x)=\begin{cases}\dfrac{(2-e^{-\lambda x^{\beta}})^{\alpha}-1}{2^{\alpha}-1} & x,\alpha,\beta,\ \lambda>0 \quad \alpha\neq 1\\ F(x)_W & \alpha=1\end{cases} \tag{13}$$

Figure (1) gives the shape forms of the PDF of NEAPW distribution for different values of parameters. We can see from this figure that NEAPW distribution has both increasing and decreasing pattern.

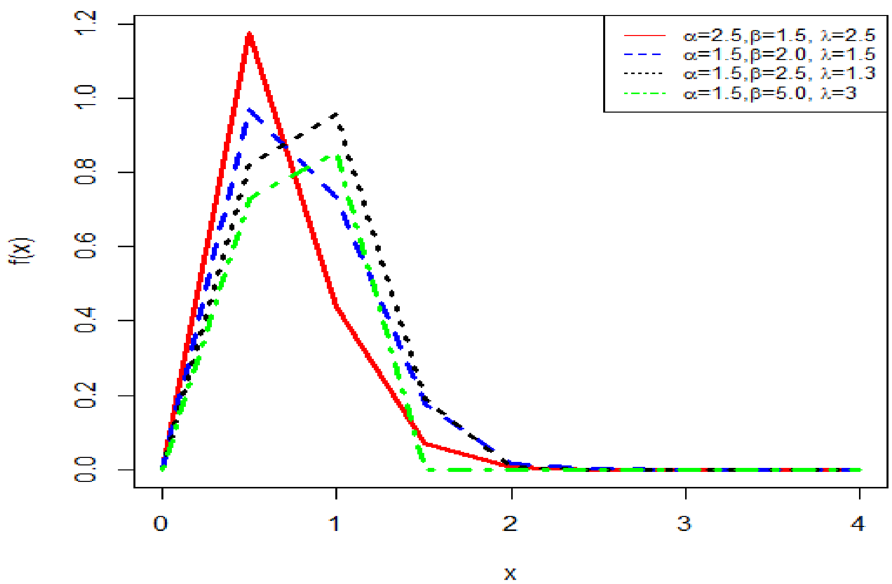

Fig 1: graph of the PDF of NEAPW

The associated SF, HF and RHRF, respectively, are given below

$$S(x)=\begin{cases}\dfrac{2^{\alpha}-(2-e^{-\lambda x^{\beta}})^{\alpha}}{2^{\alpha}-1} & x,\alpha,\beta,\lambda>0,\alpha\neq 1\\ S(x)_W & \alpha=1\end{cases} \tag{14}$$

$$h(x)=\begin{cases}\dfrac{\alpha\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}(2-e^{-\lambda x^{\beta}})^{\alpha-1}}{2^{\alpha}-(2-e^{-\lambda x^{\beta}})^{\alpha}} & x,\alpha,\beta,\lambda>0,\alpha\neq 1\\ h(x)_W & \alpha=1\end{cases} \tag{15}$$

$$r(x)=\begin{cases}\dfrac{\alpha\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}(2-e^{-\lambda x^{\beta}})^{\alpha-1}}{(2-e^{-\lambda x^{\beta}})^{\alpha}-1} & x,\alpha,\beta,\lambda>0,\alpha\neq 1\\ r(x)_W & \alpha=1\end{cases} \tag{16}$$

Figure 2 displays the SF of the NEAPW distribution for various values of the parameters.

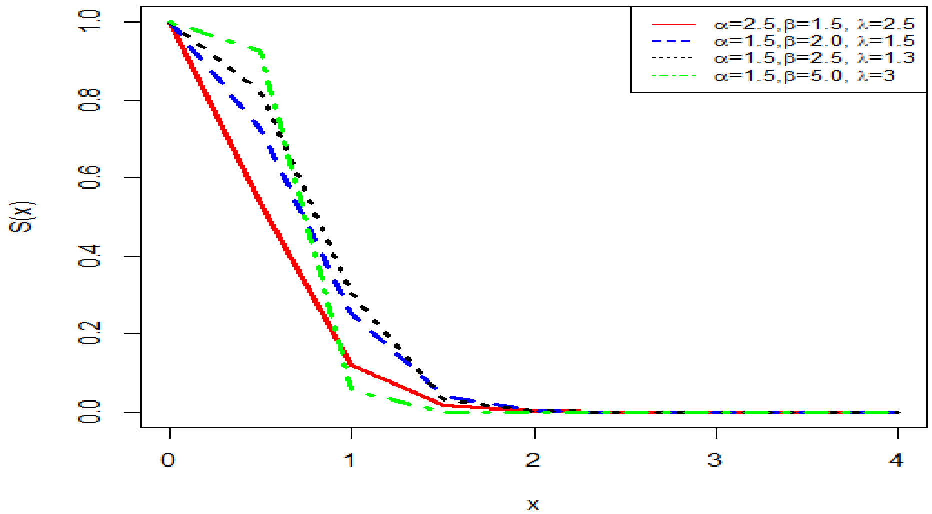

Fig 2: graph of the survival function

Special Cases of the NEAPW Distribution

We present various sub-models of the NEAPW in Table 1.

Table 1: Special models of the NEAPW distribution

α	β	λ	Reduced Models
_	_	1	NEAPW with one parameter
1	_	_	Weibull distribution
1	_	1	Standard Weibull distribution
_	2	_	NEAP Rayleigh distribution
_	1	_	NEAP exponential distribution
1	1	_	exponential distribution

Quantile function

The quantile function is very useful in conducting simulation study. It is used to measure the central tendency and range of values that a random variable may assume. Let a random variable X follow the NEAPW distribution, then the quantile function of the NEAPW distribution can be obtained by

$$F(x)=q$$

Using equation (13), we get the quantile function of the NEAPW distribution

$$-e^{-\lambda x^{\beta}}=\left(q\left(2^{\alpha}-1\right)+1\right)^{\frac{1}{\alpha}}-2$$

$$\lambda x^{\beta}=\log\left(\left(q\left(2^{\alpha}-1\right)+1\right)^{\frac{1}{\alpha}}-2\right)$$

$$x=\left(\frac{1}{\lambda}\log\left(\left(q\left(2^{\alpha}-1\right)+1\right)^{\frac{1}{\alpha}}-2\right)\right)^{\frac{1}{\beta}} \tag{17}$$

First Quartile

When $q=\frac{1}{4}$, then it is called as the first quartile, which is defined as follow

Second Quartile

When $q=\frac{1}{2}$, then it is called as second quartile. It is also called as median. We can define the median as

$$Q_{(x)}\left(\frac{1}{2}=0.5\right)=\left(\frac{1}{\lambda}\left(\log\left(\frac{2^{\alpha}+1}{2}\right)^{\frac{1}{\alpha}}-2\right)\right)^{\frac{1}{\beta}} \tag{19}$$

Third Quartile

The quantile of order $\frac{3}{4}$ is called the third quartile, which can be written as bellow

$$Q_{(x)}\left(\frac{3}{4}=0.75\right)=\left(\frac{1}{\lambda}\left(\log\left(\frac{3.2^{\alpha}+3}{4}\right)^{\frac{1}{\alpha}}-2\right)\right)^{\frac{1}{\beta}} \tag{20}$$

It is interesting to note that the interval from Q_1 to Q_3 is known as the interquartile range (IQR). The expression for interquartile range is given as

$$IQR=Q_3-Q_1$$

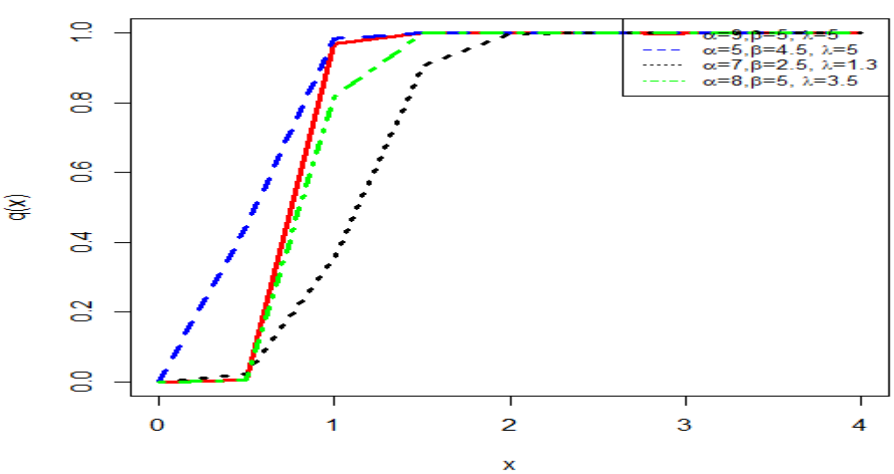

Fig 3: Graph of the Quantile function

The Moment Generating Function

Moments generating function of the NEAPP distribution is defined by

$$M_x\left(t\right)=E\left(e^{tx}\right)=\int_0^\infty e^{tx} f(x)dx \tag{21}$$

Inserting equation (12) into equation (21), we have

$$M_x(t)=\int_0^\infty e^{tx}\,\frac{\beta\lambda x^{\beta-1}e^{-\lambda x^\beta}\left(2-e^{-\lambda x^\beta}\right)}{\left(2^\alpha-1\right)}dx \tag{22}$$

We know that

$$e^{tx}=\sum_{r=0}^\infty \frac{t^r}{r!}x^r$$

Equation (22) become as

$$M_x(t)=\int_0^\infty \sum_{r=0}^\infty \frac{t^r x^r}{r!}\left(\frac{\alpha\beta\lambda x^{\beta-1}e^{-\lambda x^\beta}\left(2-e^{-\lambda x^\beta}\right)^{\alpha-1}dx}{\left(2^\alpha-1\right)}\right) \tag{23}$$

Substituting $y=x^\beta$ in equation(23) to have

$$M_x(t)=\int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!}\left(\frac{\left(y\right)^{\frac{r}{\beta}}\alpha\lambda e^{-\lambda y}\left(2-e^{-\lambda y}\right)^{\alpha-1}dy}{(2^\alpha-1)}\right) \tag{24}$$

After simplifying equation (24), we get

$$M_x(t)=\frac{1}{\left(2^\alpha-1\right)}\int_1^0 \sum_{r=0}^\infty \frac{t^r}{r!}\left(-\frac{1}{\lambda}\log z\right)^m \alpha\left(2-z\right)^{\alpha-1}\left(-dz\right)$$

$$M_x(t)=\frac{\alpha}{\left(2^\alpha-1\right)}\left(\frac{-1}{\lambda}\right)^m \int_0^1 \sum_{r=0}^\infty \frac{t^r}{r!}\left(\log z\right)^m\left(2-z\right)^{\alpha-1}dz \tag{25}$$

$$M_x(t)=\frac{\alpha}{\left(2^\alpha-1\right)}\left(\frac{-1}{\lambda}\right)^m \sum_{r=0}^\infty \sum_{k=0}^{\alpha-1} \frac{t^r}{r!}\binom{\alpha-1}{k}\left(2\right)^{\alpha-1-k}\left(-1\right)^k \int_0^1 \left(\log z\right)^m z^k dz \tag{26}$$

Solving the integral part of the equation (26)

$$\int_0^1 \left(\log z\right)^m z^k dz=\int_{-\infty}^0 u^m e^{uk}e^u du$$

$$\int_0^1 \left(\log z\right)^m z^k dz=-\int_0^\infty u^m e^{u(k+1)}du$$

$$\int_0^\infty u^m e^{u(k+1)} du = \frac{(k+1)^{-m-1} \overline{\left|(m+1),-(k+1)u\right.}}{(-1)^m}$$

The final expression for MGF is

$$M_X(t) = \frac{\alpha}{(2^\alpha - 1)} \left(\frac{-1}{\lambda}\right)^m \sum_{r=0}^{\infty} \sum_{k=0}^{\alpha-1} \frac{t^r}{r!} \binom{\alpha-1}{k} (2)^{\alpha-1-k} (-1)^{k+1} \left(\frac{(k+1)^{-m-1} \overline{\left|(m+1),-(k+1)u\right.}}{(-1)^m} \right) \tag{27}$$

Bonferroni curve

Bonferroni curve is frequently used in different fields, including economics, to study income and poverty and in the life time analysis to study survivorship and reliability. Paranaiba et al. [17] defined it as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx \tag{28}$$

The integral part of equation (28) can be expressed as

$$\int_0^q x f(x) dx = \mu - T(q)$$

Where $\mu = E(x)$ and $q = F^{-1}(p)$ is the quantile function (QF).

We know that

$$T(q) = \int_q^\infty x f(x) dx$$

Using equation (12)

$$T(q) = \int_q^\infty x f(x) dx = \int_q^\infty x \left(\frac{\alpha\beta\lambda x^\beta e^{-\lambda x^\beta} \left(2 - e^{-\lambda x^\beta}\right)^{\alpha-1}}{(2^\alpha - 1)} \right) dx \tag{29}$$

By applying transformation $y = e^{-\lambda x^\beta}$, equation (29) reduces to

$$T(q) = \frac{\alpha}{(2^\alpha - 1)} \int_\infty^{e^{-\lambda q^\beta}} \left(\frac{-1}{\lambda} \log y \right)^{\frac{-1}{\beta}} (2 - y)^{\alpha-1} dy \tag{30}$$

$$T(q) = \frac{\alpha}{(2^\alpha - 1)} \sum_{i=1}^{\alpha-1} \binom{\alpha-1}{i} (2)^{\alpha-1-i} \int_\infty^{e^{-\lambda q^\beta}} \left(\frac{-1}{\lambda} \log y \right)^{\frac{-1}{\beta}} (-y)^{-i} dy \tag{31}$$

Using computer software, we can solve the above equation.

Lorenz Curve

Like Bonferroni curve, Lorenz curve is also used to study income and poverty, reliability an demographic phenomenon, insurance and medicine. It is defined as

$$L(p)=\frac{1}{\mu}\int_{0}^{q}xf(x)dx$$

$$L(p)=\frac{1}{\mu}\left(\mu-T(q)\right)$$

$$L(p)=1-\frac{T(q)}{\mu} \tag{32}$$

Inserting equation (31) in the equation (32), we can get the Lorenz curve

Parameter Estimation

The parameters of the probability distribution function are not known and it is needed to estimate these parameters on the bases of information obtained from a sample. Here, we consider the Maximum likelihood Estimation (MLE) method to estimate the parameters. Let $x_1, x_2, x_3, \ldots x_n$ be n observed values from NEAPW distribution, then the likelihood function is defined as

$$L=\prod_{i=1}^{n}f(x)$$

$$L=\prod_{i=1}^{n}\left(\frac{\alpha\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}}{2^{\alpha}-1}\right)\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}$$

The expression for log likelihood function is given as

$$\log L=\sum_{i=0}^{n}\left(\log\alpha+\log\beta+\log\lambda+(\beta-1)\log x-\lambda x^{\beta}-\log\left(2^{\alpha}-1\right)+(\alpha-1)\log\left(2-e^{-\lambda x^{\beta}}\right)\right) \tag{33}$$

Differentiating equation (33) with respect to α, β and λ, we obtain

$$\frac{d\log L}{d\alpha}=\sum_{i=0}^{n}\left(\frac{1}{\alpha}-\frac{2^{\alpha}\log(2)}{\left(2^{\alpha}-1\right)}+\log\left(2-e^{-\lambda x^{\beta}}\right)\right)$$

$$\frac{d\log L}{d\alpha}=\frac{n}{\alpha}-\frac{n2^{\alpha}\log(2)}{\left(2^{\alpha}-1\right)}+\sum_{i=1}^{n}\log\left(2-e^{-\lambda x^{\beta}}\right) \tag{34}$$

$$\frac{d\log L}{d\beta}=\sum_{i=0}^{n}\left(\frac{1}{\beta}+\log x-\lambda x^{\beta}\log x+\frac{(\alpha-1)}{\left(2-e^{-\lambda x^{\beta}}\right)}\left(-e^{-\lambda x^{\beta}}\right)\left(-\lambda x^{\beta}\log x\right)\right)$$

$$\frac{d\log L}{d\beta}=\frac{n}{\beta}+\sum_{i=0}^{n}\log x-\lambda\sum_{i=1}^{n}x^{\beta}\log x+\lambda(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\lambda x^{\beta}}\lambda x^{\beta}\log x}{\left(2-e^{-\lambda x^{\beta}}\right)} \tag{35}$$

$$\frac{d\log L}{d\lambda}=\sum_{i=0}^{n}\left(\frac{1}{\lambda}-x^{\beta}+\frac{(\alpha-1)}{\left(2-e^{-\lambda x^{\beta}}\right)}\left(-e^{-\lambda x^{\beta}}\left(-x^{\beta}\right)\right)\right)$$

$$\frac{d\log L}{d\lambda}=\frac{n}{\beta}+\sum_{i=1}^{n}x^{\beta}+(\alpha-1)\sum_{i=1}^{n}\frac{x^{\beta}e^{-\lambda x^{\beta}}}{\left(2-e^{-\lambda x^{\beta}}\right)} \tag{36}$$

Setting the above equations (34, 35,36) to zero and resolving them mathematically produce MLEs of the parameters α, β and λ. But these equations are not in closed form, therefore a appropriate mathematical algorithm must be used to achive the MLEs of the parameters

Asymptotic Confidence Bound

To find the asymptotic confidence bound for the population parameters of the NEAPP, we have to find the higher order partial derivative of the above equations.

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{bmatrix}\rightarrow N\left[\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix},\begin{pmatrix}v_{11} & v_{12} & v_{13}\\ v_{21} & v_{22} & v_{23}\\ v_{31} & v_{32} & v_{33}\end{pmatrix}\right]$$

v_{ij} is the variance- covariance matrix

$$V=\begin{pmatrix}v_{11} & v_{12} & v_{13}\\ v_{21} & v_{22} & v_{23}\\ v_{31} & v_{32} & v_{33}\end{pmatrix}=\begin{pmatrix}I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}^{-1}$$

All the second ordered derivatives are exist, which can obtained by the following relation

$$I_{11}=E\left(\frac{\partial^{2}\log L}{\partial\alpha^{2}}\right)\qquad I_{12}=I_{21}=E\left(\frac{\partial\log L}{\partial\alpha\partial\beta}\right)\qquad I_{22}=E\left(\frac{\partial^{2}\log L}{\partial\beta^{2}}\right)$$

$$I_{13}=I_{31}=E\left(\frac{d\log L}{\partial\alpha\partial\lambda}\right)\qquad I_{33}=E\left(\frac{\partial\log L}{\partial\lambda^{2}}\right)\qquad I_{23}=I_{32}=E\left(\frac{\partial logL}{\partial\beta\partial\lambda}\right)$$

through variance-covariance matrix, we can find the 100(1-α) for the parameters α , β , λ which is given as

$$\hat{\alpha}\pm Z_{\frac{\alpha}{2}}S.E\left(\hat{\alpha}\right)\qquad \hat{\lambda}\pm Z_{\alpha/2}S.E\left(\hat{\lambda}\right)\qquad \hat{\beta}\pm Z_{\alpha/2}S.E\left(\hat{\beta}\right)$$

Order statistics

In statistical applications, the extreme value of a distribution plays an important role. The pdf $f_{i:n}(x)$ of the i^{th} order statistics of random sample $x_1,x_2,x_3,\ldots,x_n$ from NEAPW distribution is given by the expression

$$f_{i:n}(x)=\frac{n!}{(i-1)(n-i)}(F(x))^{i-1}f(x)\left(1-F(x)\right)^{n-i} \tag{37}$$

Using the PDF and CDF of the NEAPW distribution

$$f_{i;n}\left(x\right)=\frac{n!}{(i-1)!(n-i)!}\left(\frac{\left(2-e^{-\lambda x^{\beta}}\right)-1}{\left(2^{\alpha}-1\right)}\right)^{i-1}\left(\frac{\alpha\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}}{\left(2^{\alpha}-1\right)}\right)\left(1-\frac{\left(2-e^{-\lambda x^{\beta}}\right)-1}{\left(2^{\alpha}-1\right)}\right)^{n-i} \tag{38}$$

After simplification the equation (38), we have the final result as

$$f_{i;n}\left(x\right)=\frac{\alpha\beta\lambda n!}{\left(2^{\alpha}-1\right)^{n}(i-1)!(n-i)!}\left(\left(2-e^{-\lambda x^{\beta}}\right)-1\right)^{i-1}\left(x^{\beta-1}e^{-\lambda x^{\beta}}\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}\right)\left(2^{\alpha}-\left(2-e^{-\lambda x^{\beta}}\right)\right)^{n-i} \tag{39}$$

The first order statistic is given as

$$f_{1}\left(x\right)=\frac{\alpha\beta\lambda n!}{\left(2^{\alpha}-1\right)^{n}(n-1)!}\left(x^{\beta-1}e^{-\lambda x^{\beta}}\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}\right)\left(2^{\alpha}-\left(2-e^{-\lambda x^{\beta}}\right)\right)^{n-1} \tag{40}$$

The nth order statistic is given by

$$f_{n}\left(x\right)=\frac{\alpha\beta\lambda n!}{\left(2^{\alpha}-1\right)^{n}(n-1)!}\left(\left(2-e^{-\lambda x^{\beta}}\right)-1\right)^{n-1}\left(x^{\beta-1}e^{-\lambda x^{\beta}}\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}\right) \tag{41}$$

The r^{th} moments

Let x be a random variable follow NEAPW distribution, then the r^{th} moments say $\mu_r^{/}$ is defined as

$$\mu^{/}{}_{r}=E\left(x^{r}\right)=\int_{0}^{\infty}x^{r}f(x)dx \tag{41}$$

Substituting the PDF of NEAPW distribution in equation (41), we have

$$\mu^{/}{}_{r}=\int_{0}^{\infty}x^{r}\left(\frac{\alpha\beta\lambda x^{\beta-1}e^{-\lambda x^{\beta}}}{2^{\alpha}-1}\right)\left(2-e^{-\lambda x^{\beta}}\right)^{\alpha-1}dx \tag{42}$$

Let $y=x^{\beta}$, then using the transformation, equation (42) reduced to

$$\mu^{/}{}_{r}=\frac{\alpha}{\left(2^{\alpha}-1\right)}\int_{0}^{\infty}x^{\frac{r}{\beta}}e^{-\lambda y}\left(2-e^{-\lambda y}\right)^{\alpha-1}dy \tag{43}$$

$$\mu^{/}{}_{r}=\frac{\alpha}{\left(2^{\alpha}-1\right)}\left(\frac{-1}{\lambda}\right)^{\frac{r}{\beta}}\int_{0}^{1}(\log z)^{\frac{r}{\beta}}\left(2-z\right)^{\alpha-1}dz \tag{44}$$

$$\mu_{r}^{/}=\frac{\alpha}{\left(2^{\alpha}-1\right)}\left(\frac{-1}{\lambda}\right)^{m}\sum_{k=0}^{\alpha-1}\binom{\alpha-1}{k}(2)^{\alpha-1-k}(-1)^{k}\int_{-\infty}^{0}\left(\log z\right)^{m}z^{k}dz \tag{45}$$

Solving the integral part of the equation (45), we obtain the result as

$$-\int_0^\infty u^m e^{uk} e^u du = -\int_0^\infty u^m e^{u(k+1)} du = \frac{(k+1)^{-m-1}\overline{\left| m+1,-(k+1)u \right.}}{(-1)^m}$$

Where

$-k-1\neq 0,\qquad -m-1\neq 0,\qquad -m-2\neq 0$.

After solving the integral part, we have

$$\mu_r^{/} = \frac{\alpha}{(2^\alpha-1)}\left(\frac{-1}{\lambda}\right)^m \sum_{k=0}^{\alpha-1}\binom{\alpha-1}{k}(2)^{\alpha-1-k}(-1)^{k+1}\left(\frac{(k+1)^{-m-1}\overline{\left| m+1,-(k+1)u \right.}}{(-1)^m}\right)$$

Replace $m=\frac{r}{\beta}$, we have the final result for r^{th} moments

$$\mu_r^{/} = \frac{\alpha}{(2^\alpha-1)}\left(\frac{-1}{\lambda}\right)^{\frac{r}{\beta}} \sum_{k=0}^{\alpha-1}\binom{\alpha-1}{k}(2)^{\alpha-1-k}(-1)^{k+1}\left(\frac{(k+1)^{-\frac{r}{\beta}-1}\overline{\left| \frac{r}{\beta}+1,-(k+1)u \right.}}{(-1)^{\frac{r}{\beta}}}\right) \tag{46}$$

Simulation study

The simulation study has been carried out for obtaining average MLE and mean square error (MSE). The simulated data were generated 1000 times for different sample size. The random number were generated by the following expression

$$X = \left[\frac{\log((u(2^\alpha-1)+1)^{\frac{1}{\alpha}}-2)}{\lambda}\right]^{\frac{1}{\beta}}$$

Where the bias and MSE were calculated by the expressions below

$$Bias = \frac{1}{w}\sum_{i=}^{w}(\hat{\beta}_i-\beta) \quad \text{and} \quad MSE = \frac{1}{w}\sum_{i=0}^{w}(\hat{\beta}_i-\beta)^2 .$$

From the results presented in table 2, we observe that the MSE and bias decrease as we increase the sample size. This clearly shows the consistency of MLE's.

Table 2: Bias and MSE for the parameters α, β, λ

α	B	λ	N	MSE(α)	MSE(β)	MSE(λ)	Bias(α)	Bias(β)	Bias(λ)
2	2	2	10	7.8790	1.3168	2.0871	1.2780	0.6434	0.1125
			50	5.3111	0.2199	0.3447	0.5855	0.1728	0.1044
			130	3.1349	0.1101	0.2076	0.3924	0.0938	0.0965
			240	1.6096	0.0589	0.1103	0.0907	0.0354	0.0258
1.3	2.0	1.7	10	6.1209	1.1290	1.5102	1.1633	0.5717	0.1102
			50	3.5228	0.0775	0.2147	0.5796	0.1102	0.0913
			130	2.9552	0.0875	0.1781	0.3032	0.0619	0.0612

			240	1.0601	0.0380	0.0759	0.1043	0.0290	0.0209
1.3	1.5	1.7	10	5.9292	0.6375	1.4136	1.1623	0.4299	0.1088
			50	3.9495	0.0891	0.2583	0.4976	0.0985	0.0610
			130	2.5735	0.0459	0.1606	0.2158	0.0388	0.0416
			240	1.3363	0.0255	0.0900	0.1247	0.0239	0.0259
7	3.0	5	10	19.2454	2.9694	11.5522	0.4027	0.8435	1.5416
			50	11.5156	0.6566	0.6982	0.4423	0.1994	0.2621
			130	4.8813	0.3715	0.2262	0.1899	0.1717	0.0358
			240	1.6850	0.0895	0.0960	0.0818	0.0367	0.0343

Applications

We consider two real data sets to illustrate the usefulness of the NEAPW distribution. The two data sets correspond to "losses due to wind catastrophes recorded in 1977 [18]" and "breaking stress of corban fibers from two different materials [19]". These data sets have been fitted by NEAPW distribution and some other extensions of the Weibull distribution. Some goodness of fit like, LogL, AIC, BIC, HQIC, CAIC and p- value have been produced. The model with minimum goodness of fit is said to provide better fit to the data.

Different goodness of fits are defined by

$$AIC = -2l(\hat{\vartheta}) + \Psi$$

$$BIC = -2l(\hat{\vartheta}) + \Psi \log(n)$$

$$HQIC = -2l(\hat{\vartheta}) + 2\Psi \log(\log(n))$$

$$CAIC = -2l(\hat{\vartheta}) + \frac{2\Psi n}{n - \Psi - 1}$$

Where $l(\hat{\vartheta})$ gives the log-likelihood function, Ψ, ϑ represents the number of parameters, and n shows the sample size.

Data set 1: The first data set correspond to losses due to wind catastrophes. The analysis is based on this data set.

Table 3: MLEs and p-values for data 1

Distributions	MLEs $\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-logL	P value
NEAPW	**18.301**	**0.4444**	**1.2252**	**119.8839**	**0.2520**
Weibull	1.0020	0.1120	-	124.0191	0.0838
APW	0.2107	1.1300	0.0657	123.1877	0.0236
NAPTW	16.7974	0.5925	0.6263	122.9424	0.0777

Table 4: Measure of AIC, BIC, HQIC, and CAIC for data 1

Distribution	AIC	CAIC	BIC	HQIC
NEAPW	**245.7679**	**246.4536**	**250.7586**	**247.5585**
Weibull	252.0382	252.3715	255.3653	353.2320
APW	252.3755	253.0612	257.3661	254.1661
NAPTW	351.8848	252.5705	256.8755	253.6754

Table 3 gives MLE's and p-values of various distributions for data set1. In table 4, we present the AIC, CAIC, BIC, and HQIC values for the data set 1. From table 4 it is clear that the NEAPW distribution has minimum values for all model selection criteria than the other distributions. Hence NEAPW distribution provides better fit.

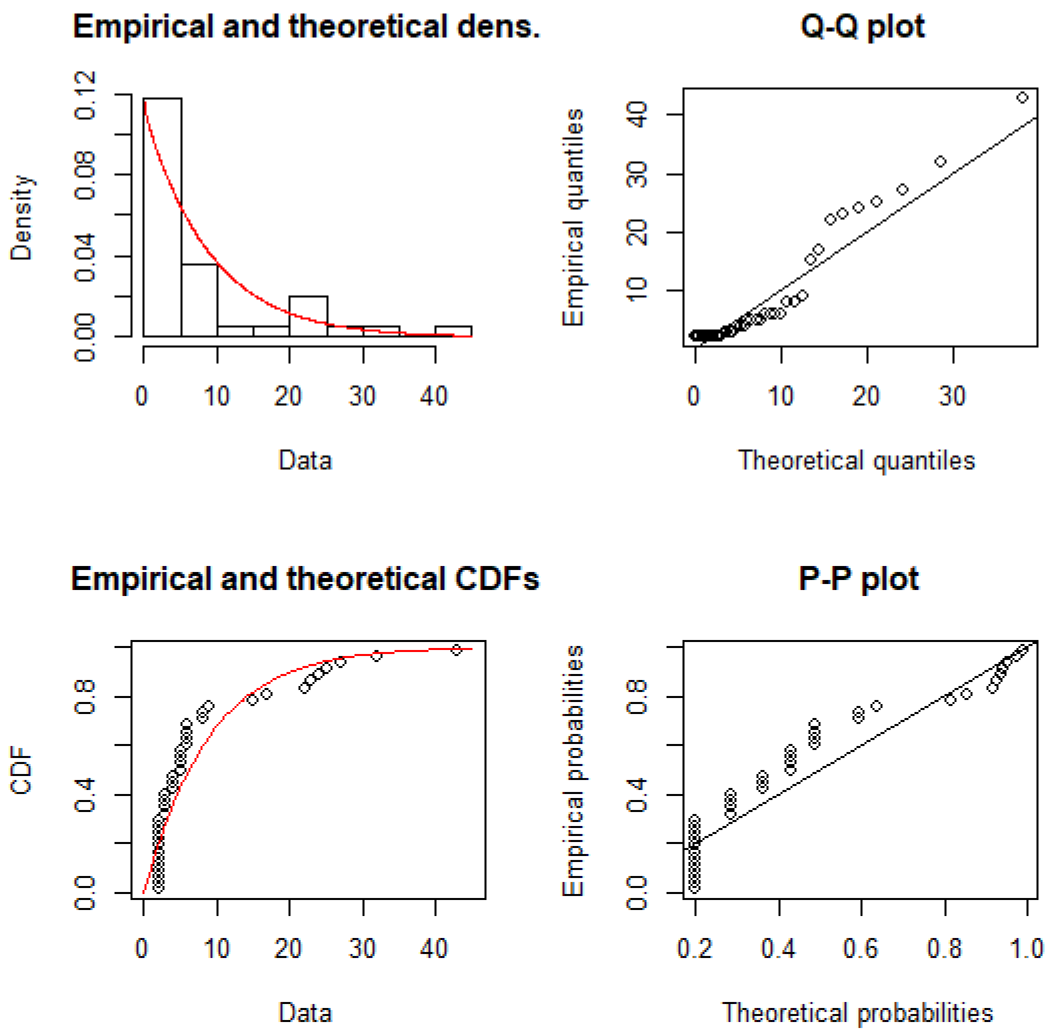

Fig 4: theoretical and empirical PDF and CDF of the NEAPW for data set 1

Figure 4 illustrates the theoretical and empirical PDF, CDF, Q-Q and P-P plot of the NEAPW distribution for the data set 1. The graphs clearly show better fit of the data set 1.

Data set 2

The following analysis are based on data set 2.

Table 5: MLEs and p- values for data 2

Distribution	MLEs			Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-logL	p-value
NEAPW	25.5530	0.6032	1.5700	128.6708	0.0011
Weibull	0.9755	0.2722	_	157.8819	0.0075
APW	16.4609	0.7850	0.7131	147.4325	0.0005
NAPTW	0.3757	1.1851	0.0874	148.6812	0.0025
IW	0.4840	1.6000	_		0.0016

Table 6: Measure of AIC, BIC, HQIC, and CAIC for data 2

Distribution	AIC	CAIC	BIC	HQIC
NEAPW	**263.3411**	**263.7220**	**269.9551**	**265.9583**
Weibull	319.7639	319.9514	324.1733	321.5087
APW	300.8651	301.2460	307.4791	303.4823
NAPTW	303.3623	303.7433	309.9764	305.9795
IW	265.1506	265.3381	269.5599	266.8954

Table 5 gives MLE's and p-values of various distributions for data set 2. In table 6, we present the AIC, CAIC, BIC, and HQIC values for the data set 2. From table 6 it is clear that the NEAPW distribution has minimum values as compared to other distributions. Hence NEAPW distribution provides better fit.

Figure 5 and 6, present the TTT plots for the data set 1and data set 2.

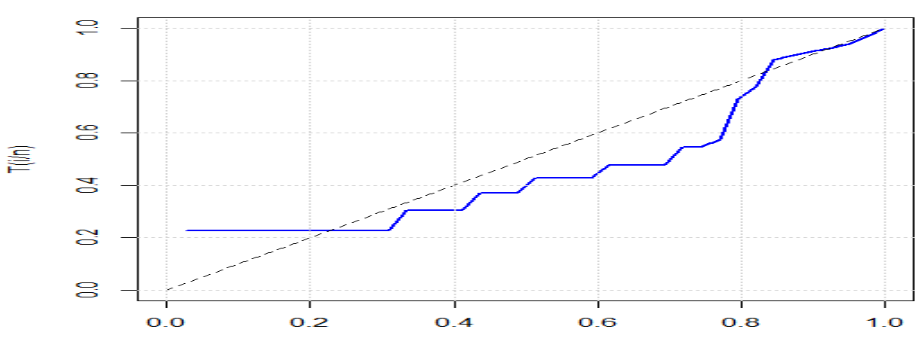

Fig 5: TTT plot for the data 1

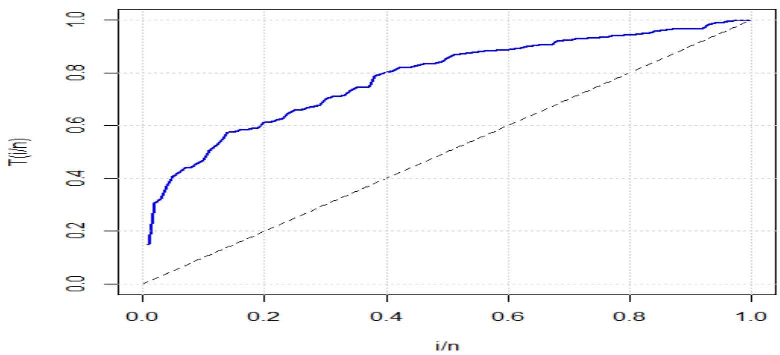

Fig 6: TTT plot for the data 2

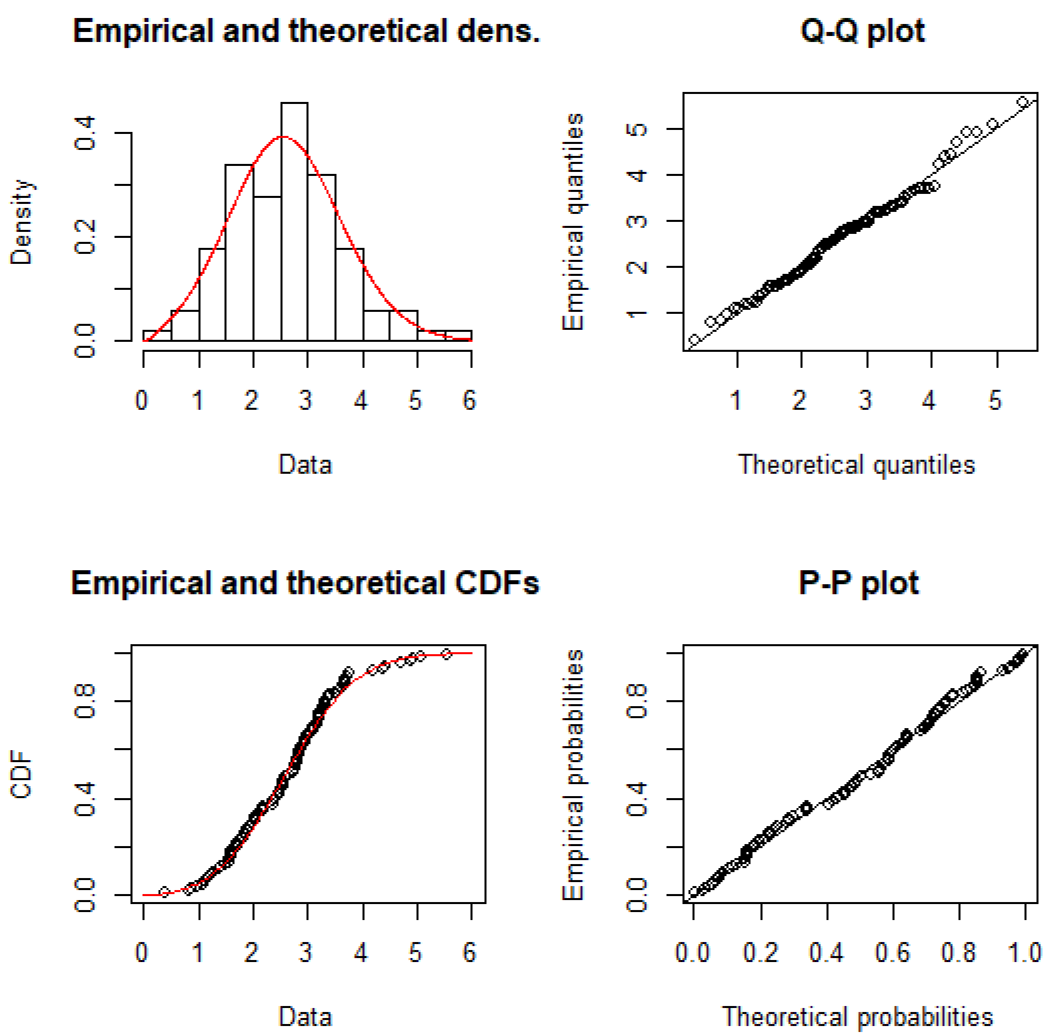

Fig 7: theoretical and empirical PDF and CDF of the **NEAPW** for data set 2

Figure 7 illustrates the theoretical and empirical PDF, CDF, Q-Q and P-P plot for data set 2 of the NEAPW distribution. The graphs clearly show better fit of the data set 2.

Conclusion

In this study, we developed a new model known as Novel Extended Alpha Power Weibull (NEAPW) distribution and examined its properties. These include probability density function, cumulative distribution function, survival, hazard and reversed hazard functions. The quantile function and r^{th} moments were also derived. Simulation studies were carried out to examine consistency of the MLE. It was found that with the increasing of sample size the MSE and Bias decreased. The new model was applied to two real data sets. Based on the findings we concluded that the new constructed model gave better result as compared to other forms of Pareto distribution.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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