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Best Proximity Point Results for Multivalued Non-Self Mappings in O-Complete Metric Space

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Abstract. The main objective of this paper is to find the sufficient conditions for the existence of best proximity points for multivalued non-self mapping in the setting of O-complete metric space. We prove the existence of best proximity point by introducing the new concept called proximal relation in O-sets along with various contraction conditions on multivalued non-self mappings. In addition, we provide an example to support our main result.

1. Introduction

Let A be non empty subset of a metric space (X, D) and let $\Gamma : A \to X$ be a mapping. We say that Γ has a fixed point in A if the fixed point equation $\Gamma a = a$ has at least one solution. That is, $a \in A$ is a fixed point of Γ if $D(a, \Gamma a) = 0$. Now, consider the case where the equation $\Gamma a = a$ does not possess a solution. Then we have $D(a, \Gamma a) > 0$ for all $a \in A$. In this case, our aim is to find an element $a \in A$ such that $D(a, \Gamma a)$ is minimum. The concept of best approximation theory and the theorems regarding the best proximity point are studied in this case. The best approximation theorem due to Ky Fan [13] states that

Theorem 1.1. *([13]) Let A be a nonempty compact convex subset of a normed linear space X and $\Gamma : A \to X$ be a continuous function. Then there exists $a \in A$ such that $\|a - \Gamma a\| = D(\Gamma a, A) := \inf\{\|\Gamma a - m\| : m \in A\}$.*

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The element $a \in A$ mentioned in Theorem 1.1 is called a best approximant of Γ in A. Note that if $a \in A$ is a best approximant, then $\|a - \Gamma a\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{a \in A} \|a - \Gamma a\|$ has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets A, B of a metric space (X, D) and a mapping $\Gamma : A \to B$. The natural question is whether one can find an element $a_0 \in A$ such that $D(a_0, \Gamma a_0) = \min\{D(a, \Gamma a) : a \in A\}$. Since $D(a, \Gamma a) \geq D(A, B) = \inf\{D(a, b) : a \in A, b \in B\}$, the optimal solution to the problem of minimizing the real valued function $a \to D(a, \Gamma a)$ over the domain A of the mapping Γ will be the one for which the value $D(A, B)$ is attained. A point $a_0 \in A$ is called a best proximity point of Γ if $D(a_0, \Gamma a_0) = D(A, B)$. Note that if $D(A, B) = 0$, then the best proximity point is nothing but a fixed point of Γ.

In the literature, a variety of generalized results can be found that establish the sufficient conditions for the existence of a best proximity point. Among these results, several notable contributions stand out.

One such contribution is the best proximity point theorem for contractive mappings by Sadiq Basha [8]. This result was inspired by the work of Anthony Eldred et al. [14], who focused on relatively non-expansive mappings. Sankar Raj and Veeramani [22] also provided an alternative treatment in this regard. Additionally, Sadiq Basha [10] obtained a best proximity point theorem for contractions, while V. Sankar Raj [23] proved best proximity point theorems for contractive non-self-mappings. The study was further extended by Abkar and Gabeleh [1,26].

Eldred and Veeramani [15] explored best proximity point theorems for various variants of contractions, and Haddadi and Moshtaghioun [17] also made significant contributions in this regard. Another important concept introduced was the P-property, which allowed for the investigation of the existence of best proximity points for weakly contractive mappings [1,23,26].

The existence and convergence of best proximity points have attracted the attention of many authors, as evidenced by numerous references (see ref. [5,6,9,11,12,18,19,21,23–25]). Furthermore, the existence of best proximity points has been studied within the framework of partially ordered metric spaces, as indicated in references [3,7,20].

Let X be a non-empty set such that $(X, D, \perp)$ is a O-metric space and let $A, B \subseteq X$. The following notions are used subsequently:

(1) $\mathcal{CB}(X)$: Set of all non-empty closed and bounded subsets of X.
(2) $\mathcal{K}(X)$: Set of all non-empty compact subsets of X.
(3) $\mathcal{B}$: Set of all non-empty subsets of B.
(4) $D(A, B) = \inf\{D(a, b) : a \in A, b \in B\}$.
(5) $\delta(A, B) = \sup\{D(a, b) : a \in A, b \in B\}$.
(6) $D(a, A) = \inf\{D(a, b) : b \in A\}$.
(7) $A_0 = \{a \in A : D(a, b) = D(A, B)\}$ for some $b \in B$.
(8) $B_0 = \{b \in B : D(a, b) = D(A, B)\}$ for some $a \in A$.

Suppose that $\Gamma : A \to 2^B$ is a multivalued non-self mapping. Analogously to the case of a single valued map, one can investigate the conditions to find an element $x_0 \in A$ such that $D(x_0, \Gamma x_0) = D(A, B)$. Such an element is called best proximity point for Γ.

For the existence of best proximity point theorems for multivalued non-self mappings, one can refer to the recent works [4, 27].

In the following theorem, R. K. Sharma and Sumit Chandok [28] have provided the sufficient conditions for the existence of fixed point for a multi-valued $\mathcal{F}$-contraction mapping in the setting of O-metric space.

Theorem 1.2. *[28] Let $(X, D, \perp)$ be an O-complete metric space and $\Gamma : X \to \mathcal{K}(X)$ be a multi-valued mapping on X. Assume that the following conditions are satisfied:*

(i) $\exists a_0 \in X$ *such that* $\{a_0\} \perp_1 \Gamma a_0$ *or* $\Gamma a_0 \perp_1 \{a_0\}$,

(ii) $\forall a, b \in X$, $a \perp b$ *implies* $\Gamma a \perp_1 \Gamma b$,

(iii) *If* $\{a_n\}$ *is an O- sequence in X such that* $a_n \to a^* \in X$, *then* $a_n \perp a^*$ *or* $a^* \perp a_n$ $\forall\, n \in \mathbb{N}$,

(iv) *If* $F \in \mathcal{F}$, $\exists\, \tau > 0$ *such that* $a, b \in X$ *with* $a \perp b$ *satisfying the following:*

$$H(\Gamma a, \Gamma b) > 0,\ \tau + F(H(\Gamma a, \Gamma b)) \leq F(D(a, b)).$$

Then, Γ *has atleast a fixed point.*

Research on the concept of orthogonal spaces (O-sets) is worth analyzing, as these spaces represent a more general framework that cannot be compared to partially ordered spaces (see [23]).

In the existing literature, there are best proximity point results for multivalued non-self mappings in various spaces such as metric spaces, b-metric spaces, partially ordered sets, CAT(0) spaces, etc. However, there are no existence results for such mappings in O-sets.

In this article, we attempt to extend the above theorem (Theorem 1.2) by considering a non-self multivalued map Γ.

Motivated by the above mentioned result, in this paper, we shall extend the result from fixed point to best proximity point for a multivalued non self mapping by defining a new concept of proximal relation in the Orthogonal set .

2. Preliminaries

Here we provide some definitions, notations, and concepts needed in the sequel.

Definition 2.1. *[28] Let* $F : (0, +\infty) \to \mathbb{R}$ *be a mapping satisfying the following:*
(F1) For all $a, b > 0$, $a > b \implies F(a) > F(b)$.
(F2) For every sequence $\{a_n\}$ *in* $\mathbb{R}^+$, $\lim_{n\to+\infty} a_n = 0$ *if and only if* $\lim_{n\to+\infty} F(a_n) = -\infty$.
(F3) $\exists k \in (0, 1)$ *such that* $\lim_{a\to 0^+} a^k F(a) = 0$.
If $\lim_{t\to 0^+} F(t) = -\infty$, *then by using (F1), we have* $F(t_n) \to -\infty \implies t_n \to 0$.

Let $\mathcal{F}$ *denote the family of functions* $F : (0, +\infty) \to \mathbb{R}$ *satisfying (F1) and (F3).*

Definition 2.2. *[28] Let (X, D) be a metric space. The Hausdorff-Pompeiu metric H, induced by metric D on X can be defined as: $\forall A, B \in \mathcal{CB}(X)$*

$$H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R},$$
$$H(A, B) = max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}$$

where, $D(a, A) = \inf\{D(a, b) : b \in A\}$.

Definition 2.3. *[16] Let $X \neq \phi$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition:*

$$\exists a_0 : (\forall b, b \perp a_0) \text{ or } (\forall b, a_0 \perp b)$$

then it is called an orthogonal set (briefly O-set) denoted by $(X, \perp)$.

Example 2.1. *Let $X = V$ be an inner product space. For a and $b \in V$, define*

$$\langle a, b\rangle = \sum_{j=1}^{\infty} a(j)b(j).$$

Define $\perp$ as $a \perp b$ if $\langle a, b\rangle = 0$. Then, for $a = (0, 0)$, it is an O-set.

Example 2.2. *[16] In graph theory, a wheel graph W_n is a graph with n vertices for each $n \geq 5$, formed by connecting a single vertex to all vertices of $(n-1)$-cycle. Let X be the set of all vertices of W_n for each $n \geq 4$. Define $a \perp b$ if there is a edge connecting from a to b. Then $(X, \perp)$ is an O-set.*

Definition 2.4. *[16] Let $(X, \perp)$ be O-set. A sequence $\{a_n\}$ is called an orthogonal sequence (O-sequence) if $a_n \perp a_{n+1}$ or $a_{n+1} \perp a_n, \forall n \in \mathbb{N}$.*

Example 2.3. *Let $M = \mathbb{R}$, define $a \perp b$ by $ab \leq a$ or b. Take $a_n = 1/n$, then a_n is an O- sequence, since $\forall n$, $a_n \perp a_{n+1}$.*

Definition 2.5. *[16] Let $(X, \perp, D)$ be an orthogonal metric space. Then X is said to be orthogonally complete (or O-complete) if every Cauchy O-sequence is convergent.*

Example 2.4. *[16] Let $X = [0, 1)$ and suppose that*

$$a \perp b \text{ if and only if } (a \leq b \leq \tfrac{1}{2} \text{ or } a = 0).$$

Then $(X, \perp)$ is an O-set. Clearly, X with the Euclidean metric is not complete metric space, but it is O-complete metric space. In fact, if $\{a_n\}$ is an arbitrary Cauchy O-sequence in X, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ for which $a_{n_k} = 0, \forall k \geq 1$ or there exists a monotone subsequence $\{a_{n_k}\}$ of $\{a_n\}$ for which $a_{n_k} \leq \frac{1}{2}, \forall k \geq 1$. It follows that $\{a_{n_k}\}$ converges to a point $a \in [0, 1] \subseteq X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{a_n\}$ is convergent.

Remark 2.1. *[16] Every complete metric space is O-complete metric space, but the converse is not true.*

Lemma 2.1. *[23] If A is an O-closed set of an O-complete metric space, then A is an O-complete metric space.*

Definition 2.6. *[23] Let $(X, \perp)$ be an O-set. Let M be any subset of X. Then M is an orthogonally closed set (or O-closed set) if any O-sequence $a_n \to a$, then $a \in M$.*

Example 2.5. *[23] Let $X = [0, \infty)$. The space $(X, \leq)$ is an O-set. Consider $M = [0, 1]$, which is O-closed.*

Remark 2.2. *[23] Every closed set is an O-closed set, but the converse is not true.*

Definition 2.7. *[20] Let A and B be any two non empty subsets of the metric space (X, D). Then a point $p \in A$ is said to be best proximity point of a single valued mapping $\Gamma : A \to B$ if $D(p, \Gamma p) = D(A, B)$.*

Example 2.6. *Let $X = \mathbb{R}$ with $D(a, b) = |a - b|$. Define $\Gamma : [0, 1] \to [4, 6]$ with $\Gamma(a) = 5 - a$. Here, $D(A, B) = D([0, 1], [4, 6]) = 3$. Then, for $a_0 = 1$, $\Gamma a_0 = 4$, which gives $D(a_0, \Gamma a_0) = 3 = D(A, B)$. Therefore, $a_0 = 1$ is a best proximity point for the given mapping.*

Definition 2.8. *[20] Let A and B be any two non empty subsets of the metric space (X, D). Then a point $p \in A$ is said to be best proximity point of a multivalued mapping $\Gamma : A \to \mathcal{B}$ if $D(p, \Gamma p) = D(A, B)$.*

Definition 2.9. *[20] Let (A, B) be a pair of non empty subsets of a metric space (X, D) with $A_0 \neq \phi$. Then (A, B) is said to have P-property if and only if*

$$D(a_1, b_1) = D(A, B) \text{ and } D(a_2, b_2) = D(A, B) \Longrightarrow D(a_1, a_2) = D(b_1, b_2)$$

where $a_1, a_2 \in A_0$ and $b_1, b_2 \in B_0$.

3. Main Results

First, let us define a new concept called *proximalrelation* between two non empty subsets of the O-set X. Then we prove the existence of best proximity point for contractive type multivalued non-self mapping.

Definition 3.1. *Let A and B be two non empty subsets of an orthogonal space $(X, D, \perp)$ such that $A_0 \neq \phi$. Let B_1 and B_2 be two non empty subsets of B_0. The proximal relation between B_1 and B_2 is defined as: $B_1 \perp_1 B_2$ if for every $b_1 \in B_1$ with $D(a_1, b_1) = D(A, B)$, there exists $b_2 \in B_2$ with $D(a_2, b_2) = D(A, B)$ such that $a_1 \perp a_2$ and $b_1 \perp b_2$.*

One can observe that when the orthogonal set becomes a partially ordered set, Definition 3.1 corresponds to Definition 1.9 in [20].

Now, we state and prove our main result.

Theorem 3.1. *Let $(X, D, \perp)$ be an O-complete metric space and let A and B be non-empty O-closed subsets of (X, D) such that $A_0 \neq \phi$ and (A, B) satisfies P- property. Let $\Gamma : A \to \mathcal{K}(B)$ be a multivalued mapping which satisfies the following conditions:*

(i) *$\exists a_0, a_1 \in A_0$ and $b_0 \in \Gamma a_0$ such that $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ with $a_0 \perp a_1$,*
(ii) *$\Gamma a_0 \subseteq B_0$, $\forall\, a_0 \in A_0$,*
(iii) *$\forall a, b \in A_0$, $\Gamma a \perp_1 \Gamma b$ whenever $a \perp b$,*
(iv) *If $F \in \mathcal{F}$, $\exists\, \tau > 0$ such that $\tau + F(\delta(\Gamma a, \Gamma b)) \leq F(D(a, b))$, for all $a, b \in A$ with $a \perp b$,*
(v) *If $\{x_n\}$ is an O- sequence in X such that $x_n \to x^*$, then $x_n \perp x^*$ or $x^* \perp x_n$, $\forall\, n \in \mathbb{N}$.*

Then, $\exists\, a^* \in A$ such that $D(a^*, \Gamma a^*) = D(A, B)$.

Proof. By (i), $\exists\ a_0, a_1\ in\ A_0$ and $b_0 \in \Gamma a_0$ such that $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ with $a_0 \perp a_1$. Adding condition (iii), we get $\Gamma a_0 \perp_1 \Gamma a_1$. That is, $\exists\ b_1 \in \Gamma a_1$ such that

$$D(a_2, b_1) = D(a_2, \Gamma a_1) = D(A, B) \text{ with } a_1 \perp a_2 \text{ and } b_0 \perp b_1.$$

In general, for each $n \in \mathbb{N}$, there exists $a_{n+1} \in A_0$ and $b_n \in \Gamma a_n$ such that $D(a_{n+1}, b_n) = D(A, B)$. Hence, we obtain

$$D(a_{n+1}, b_n) = D(a_{n+1}, \Gamma a_n) = D(A, B) \text{ for all } n \in \mathbb{N} \tag{3.1}$$

$$\text{with } \ a_n \perp a_{n+1} \ \text{ and } \ b_{n-1} \perp b_n.$$

If there exists n_0 such that $a_{n_0} = a_{n_0+1}$, then $D(a_{n_0+1}, \Gamma a_{n_0}) = D(a_{n_0}, \Gamma a_{n_0}) = D(A, B)$. This means that a_{n_0} is a best proximity point of Γ.

Thus, we can suppose that $a_n \neq a_{n+1}$, for all $n \in \mathbb{N}$. Since $D(a_{n+1}, b_n) = D(A, B)$ and $D(a_n, b_{n-1}) = D(A, B)$ and (A, B) has the P-property, we obtain

$$\begin{aligned} D(a_{n+1}, a_n) &= D(b_n, b_{n-1}), \qquad \forall n \in \mathbb{N} \cup \{0\} \\ &\leq \delta(\Gamma a_n, \Gamma a_{n-1}), \quad \text{where, } b_n \in \Gamma a_n \text{ and } b_{n-1} \in \Gamma a_{n-1}. \end{aligned} \tag{3.2}$$

Consider $F \in \mathcal{F}$. By $(F1)$ and (iv), we get

$$\begin{aligned} F(D(a_{n+1}, a_n)) &\leq F(\delta(\Gamma a_n, \Gamma a_{n-1})) \\ &\leq F(D(a_n, a_{n-1})) - \tau \\ &< F(D(a_n, a_{n-1})) \end{aligned} \tag{3.3}$$

Hence from the strictly increasing property of F, we get $D(a_{n+1}, a_n) < D(a_n, a_{n-1})$. Therefore, the sequence $\{D(a_{n+1}, a_n)\}$ is strictly decreasing sequence. Suppose that $t_n = D(a_{n+1}, a_n) \to t$, for some $t \geq 0$. Now, we have to prove that $t = 0$.
From (3.3), we get

$$\tau + F(t_n) \leq F(t_{n-1}). \tag{3.4}$$

Taking $n \to +\infty$ in (3.4), we get $\tau + F(t + 0) \leq F(t + 0)$, which is contradiction, and hence $t_n = D(a_{n+1}, a_n) \to 0$.

By $(F3)$ exists $k \in (0, 1)$ such that

$$\lim_{n \to +\infty} t_n^k F(t_n) = 0. \tag{3.5}$$

By (3.3), we get

$$F(t_n) \leq F(t_{n-1}) - \tau \leq F(t_{n-2}) - 2\tau \cdots \leq F(t_0) - n\tau. \tag{3.6}$$

From (3.6), the following holds for all $n \in \mathbb{N}$:

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k n\tau \leq 0. \tag{3.7}$$

Letting $n \to \infty$ in (3.7), we get $\lim_{n \to +\infty} n t_n^k = 0$. Hence there exists $n_1 \in \mathbb{N}$ such that $n t_n^k \leq 1, \forall n \geq n_1$. So, we have for all $n \geq n_1$:

$$t_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.8}$$

Now, for proving $\{a_n\}$ is a Cauchy O-sequence, let $m \geq n \geq n_1$, using (3.8) and triangle inequality, we have

$$D(a_n, a_m) \leq D(a_n, a_{n+1}) + D(a_{n+1}, a_{n+2}) + \ldots + D(a_{m-1}, a_m) \quad (3.9)$$

$$\leq t_n + t_{n+1} + \ldots + t_{m-1} \leq \sum_{i=1}^{m-1} t_i \leq \sum_{i=1}^{\infty} t_i \leq \sum_{i=1}^{\infty} (\frac{1}{i})^{\frac{1}{k}}. \quad (3.10)$$

Since the series $\sum_{i=1}^{\infty} (\frac{1}{i})^{\frac{1}{k}}$ is convergent, we get $D(a_n, a_m) \to 0$ as $n \to \infty$.

Here, $\{a_n\}$ is an O-sequence by construction. Thus, $\{a_n\}$ is a Cauchy O-sequence in A and hence converges to some element $a^* \in A$. Since $D(a_{n+1}, a_n) = D(b_n, b_{n-1})$, the sequence $\{b_n\}$ is also Cauchy O-sequence in B and converges to b^* in B. By the relation $D(a_{n+1}, b_n) = D(A, B)$ for all n, we conclude that $D(a^*, b^*) = D(A, B)$. We now claim that $b^* \in Ta^*$. Using (v), we get

$$b_n \perp b^* (\text{or})\ b^* \perp b_n, \forall n \in \mathbb{N}. \quad (3.11)$$

Suppose that $b^* \notin \Gamma a^*$. Then by (3.11) and (iv), we obtain

$$F(D(b_n, \Gamma a^*)) \leq F(\delta(\Gamma a_n, \Gamma a^*)) \quad (3.12)$$

$$\leq F(D(a_n, a^*)) - \tau \quad (3.13)$$

$$\leq F(D(a_n, a^*)). \quad (3.14)$$

Now using strictly increasing property of F and $\tau > 0$, we get $D(b_n, \Gamma a^*) < D(a_n, a^*)$.
Taking $n \to \infty$, we get $D(b^*, \Gamma a^*) \leq 0$. Since $D(b^*, \Gamma a^*) = 0$, we get $b^* \in \overline{\Gamma a^*} = \Gamma a^*$. Hence, we get $D(a^*, b^*) = D(a^*, \Gamma a^*) = D(A, B)$. That is a^* is the required best proximity point of the mapping Γ. □

Example 3.1. *Let* $X = \mathbb{R}^2$ *and for* $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^2$, *define* $u \perp v \iff u_i v_i \leq {u_i}^2$ *or* ${v_i}^2, \forall i \in \{1, 2\}$. *Then,* $(\mathbb{R}^2, \perp)$ *is an O-set. Moreover,* $(\mathbb{R}^2, \perp, D_1)$ *is an O-complete metric space with metric* D_1 *defined as* $D_1((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|$. *Let* $A = \{(-6, 0), (0, -6), (0, 5)\}$ *and* $B = \{(-1, 0), (0, -1), (0, 0), (-1, 1), (1, 1)\}$ *be an O-closed subset of X. Then,* $D(A, B) = 5, A_0 = A$ *and* $B_0 = B$. *Let* $\Gamma : A \to \mathcal{K}(B)$ *be defined as:*

$$\Gamma(u) = \begin{cases} \{(0, -1), (0, 0)\} & if\ u = (-6, 0) \\ \{(-1, 1), (0, 0), (-1, 0)\} & if\ u = (0, -6) \\ \{(1, 1), (-1, 1)\} & if\ u = (0, 5). \end{cases}$$

Since there exists $(-6, 0), (0, 5)$ *in* A_0 *and* $(0, 0) \in \Gamma(-6, 0)$ *such that :*

$$D((0, 5), (0, 0)) = D(A, B) = 5\ and\ (-6, 0) \perp (0, 5).$$

This satisfies condition (i). *Since* $B_0 = B$, $\Gamma u \subseteq B_0$, $\forall u \in A_0$, *the condition* (ii) *is satisfied. It is easy to claim the condition* (iii). *Now, for the condition* (iv), *choose* $F(t) = \ln t,\ t > 0$, *and for condition* (v), *choose the O-sequence as constant O-sequence for each of* $u \in A$. *Thus, it satisfies all the hypotheses of Theorem 3.1. Here,* $(0, 5)$ *is the required best proximity point of the mapping* Γ.

Corollary 3.1. *Let $(X, D, \perp)$ be an O-complete metric space and let A and B be non-empty O-closed subsets of (X, D) such that $A_0 \neq \phi$ and (A, B) satisfies P- property. Let $\Gamma : A \to \mathcal{K}(B)$ be a multivalued mapping which satisfies the following conditions:*

(i) $\exists\ a_0, a_1 \in A_0$ *and* $b_0 \in \Gamma a_0$ *such that* $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ *with* $a_0 \perp a_1$,

(ii) $\Gamma a_0 \subseteq B_0, \forall a_0 \in A_0$,

(iii) $\forall a, b \in A_0, \Gamma a \perp_1 \Gamma b$ *whenever* $a \perp b$,

(iv) *If* $F \in \mathcal{F}, \exists\ \tau_i > 0, i = 1, 2, 3$ *such that for all* $a, b \in A$ *with* $a \perp b$, *either of the following condition hold:*

$$\tau_1 + \delta(\Gamma a, \Gamma b) \leq D(a, b);$$
$$\tau_2 - \frac{1}{\delta(\Gamma a, \Gamma b)} \leq -\frac{1}{D(a, b)};$$
$$\tau_3 + \frac{1}{1 - \exp(\delta(\Gamma a, \Gamma b))} \leq \frac{1}{1 - \exp(D(a, b))}.$$

(v) *If* $\{a_n\}$ *is an O- sequence in X such that* $a_n \to a^*$, *then* $a_n \perp a^*$ *or* $a^* \perp a_n$, $\forall\ n \in \mathbb{N}$.

Then, $\exists a^* \in A$ *such that* $D(a^*, \Gamma a^*) = D(A, B)$.

Proof. Choose each functions as $F_1(r) = r, F_2(r) = (-\frac{1}{r})$ and $F_3(r) = (\frac{1}{1 - \exp r})$, where $r = D(a, b) > 0$ is strictly increasing on $(0, +\infty)$. The proof follows from Theorem 3.1. □

Further, Theorem 3.1 can be restricted to Γ as a single valued mapping by considering Γa as a singleton set for all $a \in A$.

Corollary 3.2. *Let $(X, D, \perp)$ be an O-complete metric space and let A and B be non-empty O-closed subsets of (X, D) such that $A_0 \neq \phi$ and (A, B) satisfies P- property. Let $\Gamma : A \to B$ be a multivalued mapping which satisfies the following conditions:*

(i) $\exists a_0, a_1 \in A_0$ *and* $b_0 \in B_0$ *such that* $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ *with* $a_0 \perp a_1$,

(ii) $\Gamma a_0 \in B_0, \forall\ a_0 \in A_0$,

(iii) $\forall a, b \in A_0, \Gamma a \perp \Gamma b$ *whenever* $a \perp b$,

(iv) *If* $F \in \mathcal{F}, \exists\ \tau > 0$ *such that for* $a, b \in A$ *with* $a \perp b$,

$$\tau + F(D(\Gamma a, \Gamma b)) \leq F(D(a, b))\ ,$$

(v) *If* $\{a_n\}$ *is an O- sequence in X such that* $a_n \to a^*$, *then* $a_n \perp a^*$ *or* $a^* \perp a_n$, $\forall\ n \in \mathbb{N}$.

Then, $\exists\ a^* \in A$ *such that* $D(a^*, \Gamma a^*) = D(A, B)$.

Theorem 3.2. *Let $(X, D, \perp)$ be an O-complete metric space and let A and B be non-empty O-closed subsets of (X, D) such that $A_0 \neq \phi$ and (A, B) satisfies P- property. Let $\Gamma : A \to \mathcal{K}(B)$ be a multivalued mapping which satisfies the following conditions:*

(i) $\exists a_0, a_1 \in A_0$ *and* $b_0 \in \Gamma a_0$ *such that* $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ *with* $a_0 \perp a_1$,

(ii) $\Gamma a_0 \subseteq B_0, \forall\ a_0 \in A_0$,

(iii) $\forall a, b \in A_0, \Gamma a \perp_1 \Gamma b$ *whenever* $a \perp b$,

(iv) $\exists \alpha > 0$ *such that* $\forall B_1, B_2 \in B_0$,

$$D(b_1, b_2) < \alpha H(B_1, B_2), \text{ where } b_1 \in B_1 \text{ and } b_2 \in B_2,$$

(v) *If* $F \in \mathcal{F}$ *with* $F(\alpha x) = \alpha F(x), \forall x \in X$, $\exists\ \tau > 0$ *such that for* $a, b \in A$ *with* $a \perp b$,

$$\tau + F(H(\Gamma a, \Gamma b)) \leq \tfrac{1}{\alpha} F(D(a, b)),$$

(vi) *If* $\{a_n\}$ *is an O- sequence in* X *such that* $a_n \longrightarrow a^*$, *then* $a_n \perp a^*$ *or* $a^* \perp a_n$, $\forall n \in \mathbb{N}$.

Then, $\exists\ a^* \in A$ *such that* $D(a^*, \Gamma a^*) = D(A, B)$.

Proof. Follow the proof of Theorem 3.1 till (3.2). Now by condition (iv), (3.2) becomes,

$$\begin{aligned} D(a_{n+1}, a_n) &= D(b_n, b_{n-1}) \\ &< \alpha H(\Gamma a_n, \Gamma a_{n-1}). \end{aligned}$$

Now using strictly increasing property of F and (v), we get

$$\begin{aligned} F(D(a_{n+1}, a_n)) &< F(\alpha\, H(\Gamma a_n, \Gamma a_{n-1})) \\ &< \alpha F(H(\Gamma a_n, \Gamma a_{n-1})) \\ &\leq \frac{\alpha}{\alpha} F(D(a_n, a_{n-1})) - \tau\alpha \\ &< F(D(a_n, a_{n-1})). \end{aligned}$$

Hence, we get $D(a_{n+1}, a_n) < D(a_n, a_{n-1})$. Now, by proceeding the same as the proof of Theorem 3.1, we obtain the result. □

Theorem 3.3. *Let* $(X, D, \perp)$ *be an O-complete metric space and let* A *and* B *be non-empty O-closed subsets of* (X, D) *such that* $A_0 \neq \phi$ *and* (A, B) *satisfies P- property. Let* $\Gamma : A \to \mathcal{K}(B)$ *be a multivalued mapping which satisfies the following conditions:*

(i) $\exists a_0, a_1 \in A_0$ *and* $b_0 \in \Gamma a_0$ *such that* $D(a_1, b_0) = D(a_1, \Gamma a_0) = D(A, B)$ *with* $a_0 \perp a_1$,

(ii) $\Gamma a_0 \subseteq B_0$, $\forall\ a_0 \in A_0$,

(iii) $\forall a, b \in A_0$, $\Gamma a \perp_1 \Gamma b$ *whenever* $a \perp b$,

(iv) $\exists \alpha > 0$ *such that* $\forall B_1, B_2 \in B_0$,

$$D(b_1, b_2) \ < \ \alpha H(B_1, B_2), \text{ where } b_1 \in\ B_1 \text{ and } b_2 \in\ B_2,$$

(v) $\exists \lambda \in (0, 1)$ *such that for* $a, b \in A$ *with* $a \perp b$,

$$H(\Gamma a, \Gamma b)) \leq \frac{\lambda}{\alpha}\, D(a, b),$$

(vi) *If* $\{a_n\}$ *is an O- sequence in* X *such that* $a_n \longrightarrow a^*$, *then* $a_n \perp a^*$ *or* $a^* \perp a_n$, $\forall\ n \in \mathbb{N}$.

Then, $\exists\ a^* \in A$ *such that* $D(a^*, \Gamma a^*) = D(A, B)$.

Proof. Follow the proof of Theorem 3.1 till (3.2). Now by conditions (iv) and (v), (3.2) becomes,

$$\begin{aligned} D(a_{n+1}, a_n) &= D(b_n, b_{n-1}) \\ &< \alpha H(\Gamma a_n, \Gamma a_{n-1}) \\ &\leq\ \alpha \frac{\lambda}{\alpha} D(a_n, a_{n-1}) \\ &= \lambda D(a_n, a_{n-1}). \end{aligned}$$

Since $D(a_{n+1}, a_n) < \lambda D(a_n, a_{n-1})$. In general , for each $n \in \mathbb{N}$, $D(a_{n+1}, a_n) < \lambda^n D(a_0, a_1)$. If $m, n \in \mathbb{N}$ and $n \geq m$, then

$$\begin{aligned} D(a_m, a_n) \leq & \ D(a_m, a_{m+1}) + D(a_{m+1}, a_{m+2}) + \ldots + D(a_{n-1}, a_n) \\ \leq & \ (\lambda^m D(a_0, a_1) + \ldots + \lambda^{n-1} D(a_0, a_1)) \\ \leq & \ \frac{\lambda^m}{1-\lambda} D(a_0, a_1). \end{aligned}$$

Since $\lambda \in (0, 1)$, we get $\{a_n\}$ is a Cauchy O- sequence in A. Now, by proceeding the same as the proof of Theorem 3.1, we get the result. □

4. Conclusions

The fixed point and best proximity point results ensure the existence of solutions to many problems in non-linear analysis. In our paper, we have given the existence of the best proximity point for multivalued non-self mapping in O-complete metric space. Also, we have given example to support our result.

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