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Global Strong Solutions for a One-Dimensional Bilayer Shallow Water Model With a Rigid-Lid Assumption

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Abstract. In this paper, we examine a one-dimensional viscous bilayer shallow water model under the rigid-lid assumption. Each layer is described by the one-dimensional shallow water equations. The work presented in [*Discrete and Continuous Dynamical Systems Series* B18(1), (2011), 361-383] established the stability of a similar model in the two-dimensional case. The primary focus of this study is to demonstrate the existence of global strong solutions for the proposed model within a periodic domain.

1. Introduction

Many flow phenomena are often modeled using the Navier-Stokes equations or their derivatives, such as the shallow water equations. However, in numerous situations, a single-layer model fails to adequately capture the dynamics of the flow. For instance, in cases like the Strait of Gibraltar, where Mediterranean water flows beneath Atlantic water, or in scenarios involving pollutant transport through water, it becomes necessary to adopt bilayer models for more accurate representation. To address such phenomena, several derivations of two-layer and multilayer shallow water models have been proposed, including the works cited in [1, 18].

This paper focuses on establishing the existence of global strong solutions for a one-dimensional bilayer shallow water model under the rigid-lid assumption. Previous studies on shallow water

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problems with the rigid-lid hypothesis include [15,16] for the single-layer case and [18] for the bilayer case. Here, we concentrate on the following model:

$$\partial_t h_1 + \partial_x(h_1 u_1) = 0, \tag{1.1}$$

$$\partial_t(h_1 u_1) + \partial_x(h_1 {u_1}^2) - \nu_1 \partial_x(h_1 \partial_x u_1) + h_1 \partial_x p = 0, \tag{1.2}$$

$$\partial_t h_2 + \partial_x(h_2 u_2) = 0, \tag{1.3}$$

$$\partial_t(h_2 u_2) + \partial_x(h_2 {u_2}^2) - \nu_2 \partial_x(h_2 \partial_x u_2) + h_2 \partial_x p = 0, \tag{1.4}$$

$$h_1 + h_2 = 1. \tag{1.5}$$

here, $(t,x) \in (0,T) \times \Omega$, with Ω being a one-dimensional periodic domain. The variables h_1 and h_2 represent the water heights for each layer, while u_1 and u_2 correspond to the velocities of the respective layers.

The pressure term p, which depends on h_1 and h_2, incorporates the exchange terms. Additionally, ν_1 and ν_2 denote the kinematic viscosities for the two layers.

From a theoretical perspective, numerous studies have focused on the existence of strong solutions for the shallow water and Navier-Stokes equations (see [7,8,12]). For instance, in [17], the authors demonstrated the existence of strong solutions for the one-dimensional compressible Navier-Stokes equations, assuming that the initial data is smooth and the initial density is bounded below by a positive constant. Building on the concepts introduced in [3,4,13,17], the work in [20] established the existence of strong solutions for a one-dimensional regularized bilayer model, where the initial energies associated with the model are:

$$\mathcal{E}_0 = \frac{1}{2}\int_\Omega h_{1_{0,\varepsilon}} |v_{1_{0,\varepsilon}}|^2 + \frac{g(1-r)}{2}\int_\Omega h_{2_{0,\varepsilon}} |v_{2_{0,\varepsilon}}|^2 + \frac{rg}{2}\int_\Omega |h_{1_{0,\varepsilon}} + h_{2_{0,\varepsilon}}|^2 \leq C\varepsilon^2 \leq C$$

and

$$\mathcal{F}_0 = \frac{1}{2}\int_\Omega \left| \nu_1 \frac{\partial_x \varphi_\varepsilon(h_{1_{0,\varepsilon}})}{\sqrt{h_{1_{0,\varepsilon}}}} \right|^2 + \frac{1}{2}\int_\Omega \left| \nu_2 \frac{\partial_x \varphi_\varepsilon(h_{2_{0,\varepsilon}})}{\sqrt{h_{2_{0,\varepsilon}}}} \right|^2 \leq C\varepsilon^2 \leq C.$$

As ε approaches 0, the resulting model corresponds to the stationary case. Recently, the authors in [19] analyzed an evolutionary model that resembles the one studied in [20] when ε tends to 0. In [10], the authors established the existence of global strong solutions for the Cauchy problem of a shallow water system in dimensions $N \geq 2$. Similarly, the work in [11] demonstrated the existence of global strong solutions for the compressible Navier-Stokes equations with a degenerate viscosity coefficient in one dimension. A key aspect of their proof involved controlling a new effective velocity (see [11]) in $L^\infty((0,T);L^\infty(\mathbb{R}))$, which allowed them to also control the inverse of the density, $1/\rho$, in the same space.

In [5], the authors proved the well-posedness of a system modeling two-layer shallow-water flow between two horizontal rigid plates. They assumed that the depth of the bottom layer, denoted h_1, satisfies $0 < h_1 < 1$ and used the relationship $h_1 + h_2 = 1$, where h_2 is the depth of the top layer. Building on the approach in [5], and under the rigid-lid assumption $h_1 + h_2 = 1$, we adopt the following hypothesis: there exists a constant $c > 0$ such that $0 < c \leq h_1 < 1$. The equations $(1.1)-(1.4)$ can then be reformulated as follows:

$$\partial_t h_1 + \partial_x(h_1 u_1) = 0, \tag{1.6}$$

$$\partial_t u_1 + u_1\partial_x u_1 - \nu_1\partial_x^2 u_1 - \nu_1\frac{\partial_x h_1}{h_1}\partial_x u_1 + \partial_x p = 0, \tag{1.7}$$

$$\partial_t h_1 + \partial_x((h_1-1)u_2) = 0, \tag{1.8}$$

$$\partial_t u_2 + u_2\partial_x u_2 - \nu_2\partial_x^2 u_2 - \nu_2\frac{\partial_x h_2}{h_2}\partial_x u_2 + \partial_x p = 0. \tag{1.9}$$

Thus the estimates obtained on h_1 will be valid for h_2 with $h_2 = 1 - h_1$.

This paper aims to prove the existence of global strong solutions for a one-dimensional bilayer shallow water model under the rigid-lid assumption, building upon the work in [19, 20]. Specifically, we establish appropriate regularity properties for the unknowns:

$$h_i \in L^\infty(0,T;L^\infty(\Omega)), \quad u_i \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^2(\Omega)), \quad p \in L^\infty(0,T;L^\infty(\Omega)).$$

These regularities are obtained through compactness arguments and by applying De Rham's theorem (refer to [2]).
The rest of this paper is structured as follows: Section 2 presents the initial and boundary conditions, followed by the main existence theorem. In the subsections of Section 2, we introduce the energy concepts (classical energy and mathematical entropy) and provide results that establish the regularity of the unknowns necessary to prove the stated theorem. Finally, in Section 3, we provide detailed proofs of the classical energy and mathematical entropy.

2. Main results

This section begins with the presentation of the initial data and the main results of this paper. The subsequent subsections will provide detailed evidence supporting these key findings. We assume that the initial data satisfies the following expressions and conditions:

$$h_{1_0} = h_{1|t=0}, \quad h_{2_0} = h_{2|t=0}, \quad u_{1_0} = u_{1|t=0} \quad \text{and } u_{2_0} = u_{2|t=0},$$

$$\begin{gathered} 0 < \underline{c}_{1_0} \leq h_{1_0} < 1, \quad 0 < \underline{c}_{2_0} \leq h_{2_0} < 1, \\ h_{1_0} \in H^1(\Omega), \quad u_{1_0} \in H^1(\Omega), \quad h_{2_0} \in H^1(\Omega), \quad u_{2_0} \in H^1(\Omega) \end{gathered} \tag{2.1}$$

where $\underline{c}_{1_0}$ and $\underline{c}_{2_0}$ are some positive constants. We further assume that the viscosities ν_1 and ν_2 verify the relation

$$\nu_2 > \nu_1 \tag{2.2}$$

and the following quantities are finished:

$$\frac{1}{2}\int_{\Omega}\Big[h_{1_0}|u_{1_0}|^2+h_{2_0}|u_{2_0}|^2\Big]dx\le C_1, \tag{2.3}$$

$$\frac{1}{2}\int_{\Omega}\Big[h_{1_0}|u_{1_0}+\partial_x\varphi(h_{1_0})|^2+h_{2_0}|u_{2_0}+\partial_x\varphi(h_{2_0})|^2\Big]dx\le C_2, \tag{2.4}$$

where C_1, C_2 are real constants and $\varphi(h_i)=\nu_i\log h_i, \quad i=\{1,2\}$.

Theorem 2.1. *The system* $(1.1)-(1.5)$ *admits global strong solutions corresponding to the initial data* $(2.1)-(2.4)$, *satisfying the conditions for* $i=\{1,2\}$.

$$\begin{aligned}
&h_i \quad \textit{is bounded in} \quad L^\infty(0,T;H^1(\Omega)),\\
&p \quad \textit{is bounded in} \quad L^2(0,T;H^1(\Omega)),\\
&u_i \quad \textit{is bounded in} \quad L^\infty(0,T;L^\infty(\Omega))\cap L^2(0,T;H^2(\Omega)),\\
&\partial_t u_i \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{2.5}$$

Remark 2.1. *Observe that as* h_1 *tends to* 0, h_2 *converges to* 1. *In this limit, equations* (1.1) *and* (1.4) *take the form:*

$$\partial_x u_i=0, \tag{2.6}$$

$$\partial_t u_i=0. \tag{2.7}$$

We deduce that the system $(2.6)-(2.7)$ *is a isentropic Euler system of the form*

$$\partial_t h+\partial_x u=0, \tag{2.8}$$

$$\partial_t u+\partial_x p(h)=0. \tag{2.9}$$

Global weak solutions for the conservation law $(2.8)-(2.9)$ *were constructed in [9], and the uniqueness of weak entropy solutions for small BV functions was later established in [6].*

In the following subsection, we will give some results that will help prove the previous theorem.

2.1. **Energies inequalities.** The energy equality associated with the system $(1.1)-(1.5)$ is given in the following proposition

Proposition 2.1. *Let* (h_1,h_2,u_1,u_2,p) *be a smooth solution of the system* $(1.1)-(1.5)$, *then the following classical equality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\Big[h_1|u_1|^2+h_2|u_2|^2\Big]dx+\nu_1\int_{\Omega}h_1|\partial_x u_1|^2+\nu_2\int_{\Omega}h_2|\partial_x u_2|^2dx=0. \tag{2.10}$$

From the previous result (2.10), we deduce the following estimates:

Corollary 2.1. *Let (h_1, h_2, u_1, u_2, p) be a solution of model $(1.1)-(1.5)$. We have the following uniform bounds:*

$$\sqrt{h_1}u_1 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\ \sqrt{h_2}u_2 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),$$
$$\sqrt{h_1}\partial_x u_1 \text{ is bounded in } L^2(0,T;L^2(\Omega)),\ \sqrt{h_2}\partial_x u_2 \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

The estimates obtained on h_1, h_2, u_1 and u_2 are insufficient, we need other estimates which we will find in the following results:

Proposition 2.2. *If (h_1, h_2, u_1, u_2, p) is a solution of $(1.1)-(1.3)$, the following equlity holds:*

$$\nu_1 \frac{d}{dt}\int_\Omega h_1|\partial_x \log h_1|^2 + \nu_2 \frac{d}{dt}\int_\Omega h_2|\partial_x \log h_2|^2 + \frac{d}{dt}\int_\Omega u_1\partial_x h_1 + \frac{d}{dt}\int_\Omega u_2\partial_x h_2$$
$$= \int_\Omega h_1(\partial_x u_1)^2 + \int_\Omega h_2(\partial_x u_2)^2. \tag{2.11}$$

This result allows us to have control over gradian of $\sqrt{h_i}$ and the gradian of $\partial_x h_i$ in $L^\infty(0,T;L^2(\Omega))$ for $i = \{1,2\}$.

Proposition 2.3. *Let (h_1, h_2, u_1, u_2, p) be a smooth solution of $(1.1)-(1.5)$, then the following mathematical BD entropy inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1 + \partial_x\varphi(h_1)|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega h_2|u_2 + \partial_x\varphi(h_2)|^2 dx$$
$$+\frac{\nu_2-\nu_1}{4}\int_\Omega |\partial_x p|^2 \leq .(\nu_2-\nu_1)\int_\Omega |\partial_x h_1|^2 \tag{2.12}$$

The previous result allows us to deduce the following estimates in the corollary

Corollary 2.2. *If (h_1, h_2, u_1, u_2, p) is a solution of model $(1.1)-(1.5)$ verifying the inequality given in (2.12). We obtain the following estimates:*

$$\partial_x h_1 \text{ is bounded in } L^2(0,T;L^2(\Omega)), \partial_x h_2 \text{ is bounded in } L^2(0,T;L^2(\Omega)),$$
$$\partial_x\sqrt{h_1} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \partial_x\sqrt{h_2} \text{ is bounded in } L^\infty(0,T;L^2(\Omega))$$

and

$$\partial_x p \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

Thanks to the **Corollary** 2.1, **Corollary** 2.2 and the equality $h_1 + h_2 = 1$ we deduce the following estimates on h_i and u_i for $i = \{1,2\}$.

Corollary 2.3. *Let (h_1, h_2, u_1, u_2, p) be a solution of model $(1.1)-(1.5)$. We have the following uniform bounds:*

$$h_1 \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)), h_2 \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),$$
$$u_1 \text{ is bounded in } L^2(0,T;L^\infty(\Omega)), u_2 \text{ is bounded in } L^2(0,T;L^\infty(\Omega)).$$

Now we are interested in the estimates on p.

2.2. **Estimation on p.**

Proposition 2.4. *If we assume that (h_1, h_2, u_1, u_2, p) is a solution of the system $(1.1)-(1.5)$ then we have:*

$$p \text{ is bounded in } L^\infty((0,T), L^\infty(\Omega)). \tag{2.13}$$

Proof. This proof borrows the ideas developed in [14, 18]. We introduce the following functional spaces

$$\mathcal{V} = \{\varphi : \quad \varphi \in \mathcal{D}(\Omega), \quad \partial_x \varphi = 0 \quad \text{in } \Omega\}.$$
$$\mathcal{H} = \{\phi : \quad \phi \in \mathcal{D}(\Omega), \int_\Omega \phi = 0\}$$

If we add up the equation (1.2) and (1.4), we have

$$\partial_t(h_1u_1) + \partial_t(h_2u_2) + \partial_x(h_1u_1^2) + \partial_x(h_2u_2^2) - \nu_1\partial_x(h_1\partial_xu_1)) - \nu_2\partial_x(h_1\partial_xu_1)) + \partial_xp = 0. \tag{2.14}$$

For $i = \{1, 2\}$, we have

$$\partial_t(h_iu_i) \text{ is bounded in } W^{-1,\infty}(0,T; L^\infty(\Omega)),$$
$$\partial_x(h_iu_i^2) \text{ is bounded in } W^{-1,\infty}(0,T; L^\infty(\Omega))$$

and

$$\partial_x(h_i\partial_xu_i) \text{ is bounded in } W^{-1,\infty}(0,T; L^2(\Omega)).$$

We then deduce the left term of the equation (2.14) is bounded in $W^{-1,\infty}(0,T; L^2(\Omega))$. Let $\varphi \in \mathcal{V}$, we multiply (2.14) by φ and integrate over Ω by taking into account that Ω is periodic to obtain

$$\int_\Omega \Big(\partial_t(h_1u_1) + \partial_t(h_2u_2) + \partial_x(h_1u_1^2) + \partial_x(h_2u_2^2) - \nu_1\partial_x(h_1\partial_xu_1) - \nu_2\partial_x(h_1\partial_xu_1)\Big)\varphi = 0 \tag{2.15}$$

So by De Rham's theorem, there exists a unique $p \in W^{-1;\infty}(0,T; L^2(\Omega))$ such that

$$\partial_t(h_1u_1) + \partial_t(h_2u_2) + \partial_x(h_1u_1^2) + \partial_x(h_2u_2^2) - \nu_1\partial_x(h_1\partial_xu_1)) - \nu_2\partial_x(h_1\partial_xu_1)) = \partial_xp,$$

and

$$\int_\Omega p\phi = 0 \quad \forall \phi \in \mathcal{H}.$$

Hence

$$p \in W^{-1,2}(0,T; L^2(\Omega)) \text{ and } \int_\Omega p\phi = 0 \quad \forall \phi \in \mathcal{H}.$$

Moreover if we add up the equations (1.1) and (1.3), we get

$$\partial_x(h_1u_1 + h_2u_2) = 0$$

and we add up the equations (1.2) and (1.4) and deriving from x, we have

$$-\partial_{x^2}p = \partial_x(\partial_xh_1u_1^2 + \partial_xh_2u_2^2) - \nu_1\partial_{x^2}(h\partial_xu_1)) - \nu_2\partial_{x^2}(h_2\partial_xu_2).$$

With the preceding regularities established on h_i and u_i we easily justify that the right-hand terms of the previous equation are all in $L^2(0,T; W^{-2,2}(\Omega))$. Then

$$\partial_{x^2}p \text{ is bounded in } L^2(0,T; W^{-2,2}(\Omega)).$$

Hence by regularity of the Laplacian we have

$$p \text{ is bounded in } L^2(0,T;L^2(\Omega)) \tag{2.16}$$

Thanks to the **Corollary**2.2 and injection of H^1 into L^∞ we deduce that

$$p \text{ is bounded in } L^2(0,T;L^\infty(\Omega)) \tag{2.17}$$

□

Remark 2.2. *With the estimates obtained on h_i and as p is defined as a function of h then we deduce that:*

$$p \textit{ is bounds in } L^\infty(0,T;L^\infty(\Omega)).$$

2.3. **Uniform bounds for the velocities.**

Proposition 2.5. *For (h_1,h_2,u_1,u_2,p) solution of the system $(1.1)-(1.5)$, we have the following estimates:*

$$u_1 \quad \textit{is bounded in} \quad L^2(0,T;H^2(\Omega)),\ \ \partial_t u_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$
$$u_2 \quad \textit{is bounded in} \quad L^2(0,T;H^2(\Omega)),\ \ \partial_t u_2 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)).$$

Proof. **Proposition 2.5**

We consider the momentum equation for $i=\{1,2\}$

$$\partial_t(h_iu_i)+\partial_x(h_i{u_i}^2)-\nu_i\partial_x(h_i\partial_xu_i)+h_i\partial_xp=0$$

We rewrite that as:

$$\begin{aligned}
&h_i\partial_tu_i+h_iu_i\partial_xu_i-\nu_i\partial_x(h_i\partial_xu_i)+h_i\partial_xp=0,\\
&h_i\partial_tu_i+h_iu_i\partial_xu_i-\nu_ih_i\partial_x^2u_i-\nu_i\partial_xh_i\partial_xu_i+h_i\partial_xp=0,\\
&\partial_tu_i+u_i\partial_xu_i-\nu_i\partial_x^2u_i-\nu_i\frac{\partial_xh_i}{h_i}\partial_xu_i+\partial_xp=0,
\end{aligned}$$

$$\partial_tu_i-\nu_i\partial_x^2u_i=-\partial_xp+(\nu_i\partial_x\log h_i-u_i)\partial_xu_i. \tag{2.18}$$

By virtue of **Corollary** 2.2, ∂_xp is bounded in $L^2\Big(0,T;L^2(\Omega)\Big)$. Drawing on the methods presented in [11] and [17], and employing Holder's inequality, the Gagliardo-Nirenberg inequality, and energy estimates, we obtain:

$$\begin{aligned}
&\|(\nu_i\partial_x\log h_i-u_i)\partial_xu_i\|_{L^2(0,T;L^2(\Omega))}\\
&\leq\|\nu_i\partial_x\log h_i-u_i\|_{L^\infty(0,T;L^2(\Omega))}\|\partial_xu_i\|_{L^2(0,T;L^\infty(\Omega))}\\
&\leq\|\nu_i\partial_x\log h_i-u_i\|_{L^\infty(0,T;L^2(\Omega))}\|\partial_xu_i\|^{\frac12}_{L^2(0,T;L^2(\Omega))}\|\partial_x^2u_i\|^{\frac12}_{L^2(0,T;L^2(\Omega))}\\
&\leq C\|\partial_x^2u_i\|^{\frac12}_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Using regularity results for parabolic equation of the form (2.18) gives for any $T\in(0,T_0)$:

$$\|\partial_tu_i\|_{L^2(0,T;L^2(\Omega))}+\|\partial_xu_i\|_{L^2((0,T;H^1(\Omega))}\leq C\|\partial_xu_i\|^{\frac12}_{L^2(0,T;H^1(\Omega))}+C,$$

with C depending on $\|u_{i_0}\|_{H^1}$ and by boostrap for any $T \in (0, T_0)$:

$$\|\partial_t u_i\|_{L^2((0,T),L^2(\Omega))} + \|u_i\|_{L^2((0,T),H^2(\Omega))} \le C(T).$$

□

3. Appendix

Proof. **Proposition 2.1**

We will multiply the equations (1.2) and (1.4) respectively by u_1 and u_2. We obtain the following equalities:

$$\int_\Omega \Big[(\partial_t h_1 u_1) + \partial_x(h_1 u_1^2)\Big] u_1 dx + \int_\Omega h_1 u_1 \partial_x p dx - \nu_1 \int_\Omega u_1 \partial_x(h_1 \partial_x u_1) dx = 0,$$

and

$$\int_\Omega \Big[(\partial_t h_2 u_2) + \partial_x(h_2 u_2^2)\Big] u_2 dx + \int_\Omega h_2 u_2 \partial_x p dx - \nu_2 \int_\Omega u_2 \partial_x(h_2 \partial_x u_2) dx = 0.$$

We rewrite the first two terms of each of the two previous equations as follows:

$$\int_\Omega \Big[(\partial_t h_1 u_1) + \partial_x(h_1 u_1^2)\Big] u_1 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega h_1 |u_1|^2 dx, \tag{3.1}$$

$$\int_\Omega \Big[(\partial_t h_2 u_2) + \partial_x(h_2 u_2^2)\Big] u_2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega h_2 |u_2|^2 dx. \tag{3.2}$$

We also observe that:

$$\int_\Omega h_1 u_1 \partial_x p dx + \int_\Omega h_2 u_2 \partial_x p dx = \int_\Omega p \partial_t (h_1 + h_2) dx. \tag{3.3}$$

Furthermore, we have:

$$-\nu_1 \int_\Omega u_1 \partial_x(h_1 \partial_x u_1) dx - \nu_2 \int_\Omega u_2 \partial_x(h_2 \partial_x u_2) = \nu_1 \int_\Omega h_1 |\partial_x u_1|^2 + \nu_2 \int_\Omega h_2 |\partial_x u_2|^2 dx. \tag{3.4}$$

Now we add the equations (3.1) − (3.4) to find the proclaimed equality. □

Proof. **Proposition 2.2** For $i = \{1, 2\}$, thanks to the mass equation, we have

$$\partial_t \partial_x h_i + \partial_x(h_i \partial_x u_i) + \partial_x(u_i \partial_x h_i) = 0 \tag{3.5}$$

Raplacing $\partial_x h_i$ by $h_i \partial_x \log h_i$, we get:

$$\partial_t(h_i \partial_x \log h_i) + \partial_x(h_i \partial_x u_i) + \partial_x(u_i h_i \partial_x \log h_i) = 0 \tag{3.6}$$

we multiply the previous equation by $\partial_x \log h_i$ to have:

$$\frac{1}{2}\partial_t[h_i|\partial_x \log h_i|^2] - \frac{1}{2}\frac{|\partial_x h_i|^2}{h_i^2}\partial_x(h_i u_i) + \partial_x^2 u_i \partial_x h_i + 2\partial_x u_i \frac{|\partial_x h_i|^2}{h_i} + \frac{u_i \partial_x^2 h_i \partial_x h_i}{h_i} = 0 \tag{3.7}$$

Let us multiply the momentum equation by $\partial_x \log h_i$ and simplify to have

$$(\partial_t u_i + u_i \partial_x u_i)\partial_x h_i - \nu_1 \partial_x^2 u_1 \partial_x h_i - \nu_1 \frac{|\partial_x h_i|^2}{h_1}\partial_x u_1 + \partial_x p \partial_x h_i = 0 \tag{3.8}$$

We multiply the equation (3.7) by ν_i add to the equation (3.8) and integrate to have

$$\frac{\nu_i}{2}\frac{d}{dt}\int_\Omega h_i|\partial_x \log h_i|^2 + \int_\Omega \partial_x h_i \partial_x p + \int_\Omega (\partial_t u_i + u_i\partial_x u_i)\partial_x h_i = 0,$$

so,

$$\frac{\nu_i}{2}\frac{d}{dt}\int_\Omega h_i|\partial_x \log h_i|^2 + \int_\Omega \partial_x h_i \partial_x p = -\int_\Omega (\partial_t u_i + u_i\partial_x u_i)\partial_x h_i$$

We use the mass equation to rewrite: $(\partial_t u + u\partial_x u)\partial_x h$ as

$$(\partial_t u_i + u_i\partial_x u_i)\partial_x h_i = \partial_t u_i \partial_x h_i + (-\partial_t h_i - h_i\partial_x u_i)\partial_x u_i.$$

So, we have

$$\frac{\nu_i}{2}\frac{d}{dt}\int_\Omega h_i|\partial_x \log h_i|^2 + \frac{d}{dt}\int_\Omega u_i\partial_x h_i + \int_\Omega \partial_x p\partial_x h_i = \frac{d}{dt}\int_\Omega h_i|\partial_x u_i|^2 \tag{3.9}$$

By summing the previous expression for $i = \{1,2\}$, we have

$$\frac{\nu_1}{2}\frac{d}{dt}\int_\Omega h_1|\partial_x \log h_1|^2 + \frac{\nu_2}{2}\frac{d}{dt}\int_\Omega h_2|\partial_x \log h_2|^2 + \frac{d}{dt}\int_\Omega u_1\partial_x h_1 + \frac{d}{dt}\int_\Omega u_2\partial_x h_2$$

$$= \int_\Omega h_1|\partial_x u_1|^2 + \int_\Omega h_2|\partial_x u_2|^2. \tag{3.10}$$

□

Proof. **Proposition 2.3**

The system (1.1) – (1.4) can be written as follows: for $i, j = 1,2$ with $i \neq j$

$$(S_i)\begin{cases} \partial_t h_i + \partial_x(h_i u_i) = 0, \\ \\ \partial_t(h_i u_i) + \partial_x(h_i {u_i}^2) - \nu_i\partial_x(h_i\partial_x u_i) + h_i\partial_x p = 0. \end{cases}$$

Following the idea proposed in [11], we set $v_i = u_i + \nu_i\partial_x \log h_i = u_i + \partial_x\varphi(h_i)$ and we can rewrite the system (S_i) as follows:

$$(S_i')\begin{cases} \partial_t h_i + \partial_x(h_i v_i) - \nu_i\partial_x^2 h_i = 0, \\ \\ h_i\partial_t(v_i) + h_i u_i\partial_x(v_i) + h_i\partial_x p = 0, \end{cases}$$

for $i, j = 1,2$.

We multiply the second equation of (S_i') by v_i and integrate on Ω, for $i = 1,2$.

We have for each layer:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1 + \partial_x\varphi(h_1)|^2 dx + \nu_1\int_\Omega \partial_x p\partial_x h_1 dx + \int_\Omega p\partial_t h_1 dx = 0, \tag{3.11}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_2|u_2+\partial_x\varphi(h_2)|^2dx+\nu_2\int_{\Omega}\partial_x p\partial_x h_2dx+\int_{\Omega} p\partial_t h_2dx=0. \tag{3.12}$$

We sum up the equations by performing a simple calculation to have:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_1|u_1+\partial_x\varphi(h_1)|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_2|u_2+\partial_x\varphi(h_2)|^2dx$$

$$+\nu_1\int_{\Omega}\partial_x p\partial_x h_1dx+\nu_2\int_{\Omega}\partial_x p\partial_x h_2dx=0.$$

Taking into account the equation $h_1+h_2=1$ to deduce that:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_1|u_1+\partial_x\varphi(h_1)|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_2|u_2+\partial_x\varphi(h_2)|^2dx$$

$$+\nu_1\int_{\Omega}\partial_x p\partial_x h_1dx-\nu_2\int_{\Omega}\partial_x p\partial_x h_1dx=0.$$

Using the equality: $ab=\frac{1}{2}\Big[(a+b)^2-a^2-b^2\Big]$ for the to last terms, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_1|u_1+\partial_x\varphi(h_1)|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_2|u_2+\partial_x\varphi(h_2)|^2dx+\frac{\nu_2-\nu_1}{2}\int_{\Omega}|\partial_x h_1|^2$$

$$+\frac{\nu_2-\nu_1}{2}\int_{\Omega}|\partial_x p|^2-.\frac{\nu_2-\nu_1}{2}\int_{\Omega}|\partial_x h_1+\partial_x p|^2=0$$

we deduce taking into account the inequality: $(a+b)^2\geq\frac{1}{2}a^2-\frac{3}{2}b^2$, we get:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_1|u_1+\partial_x\varphi(h_1)|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega} h_2|u_2+\partial_x\varphi(h_2)|^2dx$$

$$+\frac{\nu_2-\nu_1}{4}\int_{\Omega}|\partial_x p|^2\leq.(\nu_2-\nu_1)\int_{\Omega}|\partial_x h_1|^2=0. \tag{3.13}$$

□

4. Conclusion

This paper focuses on the theoretical study of a one-dimensional viscous bilayer shallow water model under the rigid-lid assumption. Using the estimates derived for the unknowns (h_i, u_i, and p), we demonstrate the existence of global strong solutions in time for this model. This work enhances and builds upon the results in [19]. Additionally, as part of ongoing research, we are exploring the existence of solutions in the two-dimensional case, for which stability was previously established in [*Discrete and Continuous Dynamical Systems Series B*, 18(1), (2011)].

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