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# The Rainbow Mean Index of Corona Product of Graphs

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**Abstract.** In a connected graph $G$ with at least three vertices, an edge coloring $c$ assigns positive integers to the edges. The chromatic mean of a vertex $v$ is determined by averaging the colors of all incident edges, provided that the result remains a positive integer. A coloring $c$ is a rainbow mean coloring if every vertex in $G$ has a unique chromatic mean. The rainbow mean index of $c$ is the highest chromatic mean assigned to any vertex, while the rainbow mean index of $G$ is the smallest possible maximum chromatic mean for all valid rainbow mean colorings. In this study, we calculate the rainbow mean index of corona product of $P_x \circ H$; $G \circ P_y$ and $P_x \circ P_y$, where $G$ and $H$ are regular graphs. In addition, we calculate the rainbow mean index of the join graph $G \vee K_1$, where $G$ is a regular graph.

## 1. Introduction

Let $V(G)$ and $E(G)$ be the vertex and edge set of a connected graph $G$. Concepts and terms not explained in this article are available in [1]. The number of edges incident to $v$ in $G$ is known as degree of $v$ and it is denoted by $d(v)$. If $d(v) = \ell \ \forall \ v \in V(G)$, then $G$ is known as *$\ell$-regular.* A cycle $C_x$ in $G$ is a *spanning cycle* if $V(C_x) = V(G)$. For $x \geq 3$, $P_x$, $F_x$, $K_x$, $K_{x,x}$ and $O_x$ denotes the path, fan, complete, complete bipartite and null graphs, respectively.

An edge coloring $c : E(G) \to \mathbb{N}$ assigns positive integer values to the edges of $G$ (where the color of the adjacent edges may be the same) is named mean coloring of $G$ if the chromatic mean $cm(v)$ of $v \in V(G)$, calculated by $cm(v) = \frac{\sum_{e \in E_v} c(e)}{d(v)}$, $E_v$ denotes the number of edges connected to $v$, is a positive integer. If $cm(v) \ \forall \ v \in V(G)$ are distinct, then $c$ is named Rainbow Mean Coloring, briefly RMC of $G$. For an RMC $c$, the largest $cm(v)$ among the vertices is named Rainbow Mean Index, briefly RMI of $c$, represented by $\chi_{rm}(c)$. The RMI of $G$, represented as $\chi_{rm}(G)$ is calculated by

$$\chi_{rm}(G) = \min\{\chi_{rm}(c) : c \text{ is a RMC of } G\}.$$

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Let $x \in \mathbb{N}$ and $L$ be a subset of $\{1,2,\ldots,\lfloor \frac{x}{2} \rfloor\}$. The graph $C(x;L)$ with $V(C(x;L)) = \{v_0, v_1, v_2, \ldots, v_{x-1}\}$ and $E(C(x;L)) = \{v_i v_{i+\ell} : i \in \{0,1,2,\ldots,x-1\}, \ell \in L\}$ is known as circulant graph.

The *corona product* of two graphs $G$ and $H$ [2], represented by $G \circ H$ is constructed by taking a single copy of $G$ and $|V(G)|$ copies of $H$. Each vertex in $G$ is then connected to every vertex in its corresponding copy of $H$.

The *join graph* of two graphs $G$ and $H$ [2], represented by $G \vee H$ is constructed by each vertex in $G$ adjacent to every vertex in $H$.

The concept of RMI was first studied by Chartrand et al. [3] and they have proposed the RMI conjecture, which states that for every connected graph $G$ of order at least 3, then $|V(G)| \leq \chi_{rm}(G) \leq |V(G)| + 2$.

**Observation 1.1.** *For any connected graph $G$ with $x$ vertices, $\chi_{rm}(G) \geq x$.*

**Theorem 1.1.** *[3] Let $H$ be a connected graph with $y \geq 6$ vertices, $y = 4\tau + 2$ where $\tau \geq 1$ and $d(v)$ are odd for all $v \in V(H)$. Then $\chi_{rm}(H) \geq y + 1$.*

Also, they have calculated the RMI of path, cycle, complete and star graph. Additionally, in 2021 the same authors [4] calculated the RMI of the Path graph.

In 2020, Hallas et al. [5] discussed the RMI of the cartesian product of two graphs; complete bipartite graph and $n$-dimensional hypercube. Moreover, in 2020 the same authors [6] determined the RMI of double star, cubic caterpillars of even order, and the subdivision of stars.

In 2022, Anantharaman et al. [7] calculated the RMI of the regular; wheel; caterpillar and fan graph. Also, they have calculated the RMI of wreath product of two graphs.

**Theorem 1.2.** *[7] For $x \geq 4$, $\chi_{rm}(F_x) = x + 1$.*

In 2023, Garciano et al. [8] calculated the RMI of particular family of caterpillars.

In 2024, Maheswari and Rajasekaran [9] discussed the RMI of the cartesian product of two graphs; Chain of cycle; Join of $n$ wheel and transformation of path graph.

Rainbow mean index can be applied to network optimization, scheduling and resource allocation, data clustering and classification, design of fault-tolerant systems, cryptography and secure communication, etc.

We are motivated by the above results and contributed some results in corona product graphs.

## 2. Main Results

**Lemma 2.1.** *For $x \geq 2$,*

$$\chi_{rm}(P_x \circ P_2) = \begin{cases} 7 \text{ if } x = 2, \\ 3x \text{ otherwise.} \end{cases}$$

*Proof.* Let $V(P_x \circ P_2) = V(P_x) \cup \{v_j^i : 1 \leq i \leq x, 1 \leq j \leq 2\}$ and $E(P_x \circ P_2) = E(P_x) \cup \{u_i v_j^i : i \in \{1,2,\ldots,x\}, j \in \{1,2\}\} \cup \{v_1^i v_2^i : i \in \{1,2,\ldots,x\}\}$. Define $c' : E(P_x \circ P_2) \to \mathbb{N}$ as follows:

If $x = 2$, then $\chi_{rm}(P_2 \circ P_2) = 7$ (refer FIGURE 1).

If $x \geq 3$, then color the edges of $P_x \circ P_2$ as:

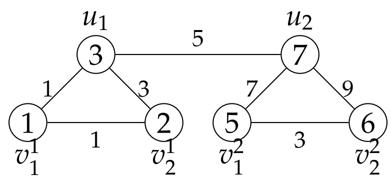


FIGURE 1. $\chi_{rm}(P_2 \circ P_2) = 7$.

**Case 1.** $x$ is odd,

For $1 \leq i \leq x-1$ and $1 \leq j \leq 2$,

$c'(u_i v_j^i) = 3i + 2j - 4$ and $c'(v_1^i v_2^i) = 3i - 2$;

$c'(u_x v_j^x) = 2j + 3(x-1)$ if $j = 1, 2$;

$c'(v_1^x v_2^x) = 3(x-1)$;

Next, color the edges of $P_x$ as:

For $1 \leq i \leq x-1$,

$$c'(u_i u_{i+1}) = \begin{cases} 3i + 2 \text{ if odd } i; \\ 3(i+1) \text{ if even } i; \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_x \circ P_2$ is given by:

For $i \in \{1, 2, \ldots, x\}$ and $j = 1, 2$, $cm(v_j^i) = j + 3i - 3$ and $cm(u_i) = 3i$;

**Case 2.** $x$ is even,

For $j = 1, 2$,

$c'(u_{x-1} v_j^{x-1}) = 3x + 2j - 4$;

$c'(u_x v_j^x) = 3x + 2j - 10$;

$c'(v_1^{x-1} v_2^{x-1)} = c'(v_1^x v_2^x) = 3x - 2$;

$c'(u_{x-1} u_x) = 3x + 5$;

and color the remaining edges of $P_x \circ P_2$ as in *Case 1.*

Therefore, the chromatic mean of the vertices of $P_x \circ P_2$ is given by:

For $j = 1, 2$,

$cm(v_j^{x-1}) = j + 3(x-1)$;

$cm(v_j^x) = j + 3(x-2)$;

$cm(u_{x-1}) = 3x$; $cm(u_x) = 3(x-1)$; and the chromatic mean of the remaining vertices of $P_x \circ P_2$ follows as in *Case 1.*

Hence, in both cases $\chi_{rm}(P_x \circ P_2) = 3x$ (refer FIGURE 2, $\chi_{rm}(P_5 \circ P_2) = 15$). □

**Lemma 2.2.** *For $x \geq 2$, $\chi_{rm}(P_x \circ P_3) = 4x$.*

*Proof.* Let $V(P_x \circ P_3) = V(P_x) \cup \{v_j^i : 1 \leq i \leq x, 1 \leq j \leq 3\}$ and $E(P_x \circ P_3) = E(P_x) \cup \{u_i v_j^i : i \in \{1, 2, \ldots, x\}, j \in \{1, 2, 3\}\} \cup \{v_j^i v_{j+1}^i : i \in \{1, 2, \ldots, x\}, j \in \{1, 2\}\}$. Define $c' : E(P_x \circ P_3) \to \mathbb{N}$ as:

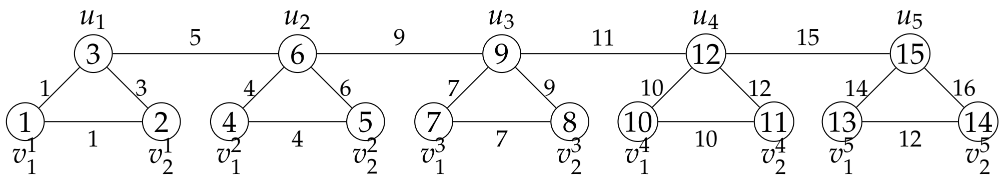

FIGURE 2. $\chi_{rm}(P_5 \circ P_2) = 15$.

For $1 \le i \le x-1$,

$$c'(u_i v_j^i) = j + 4(i-1) \text{ if } 1 \le j \le 3;$$
$$c'(v_j^i v_{j+1}^i) = 4i + 2j - 5 \text{ if } 1 \le j \le 2.$$

If $x$ is odd, then
$c'(u_x v_j^x) = j + 4x - 1$ if $1 \le j \le 2$;
$c'(u_x v_3^x) = c'(v_2^x v_3^x) = 4x - 1$; $c'(v_1^x v_2^x) = 2(2x-3)$.
If $x$ is even, then
$c'(u_x v_j^x) = j + 4x - 2$ if $1 \le j \le 2$;
$c'(u_x v_3^x) = c'(v_2^x v_3^x) = 4x - 1$; $c'(v_1^x v_2^x) = 4x - 5$.
Next, color the edges of $P_x$ in $P_x \circ P_3$ as:
For $1 \le i \le x-1$,

$$c'(u_i u_{i+1}) = \begin{cases} 4i + 6 \text{ if odd } i, \\ 4(i+1) \text{ if even } i. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_x \circ P_3$ is given by:

$$cm(v_j^i) = j + 4i - 4 \text{ if } i \in \{1, 2, \dots, x\} \text{ and } j = 1, 2, 3;$$
$$cm(u_1) = 4;$$
$$cm(u_i) = 4i \text{ if } 2 \le i \le x-1;$$
$$cm(u_x) = 4x.$$

Hence $\chi_{rm}(P_x \circ P_3) = 4x$ (refer to FIGURE 3. $\chi_{rm}(P_4 \circ P_3) = 16$). □

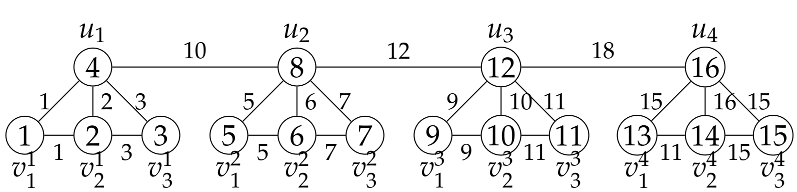

FIGURE 3. $\chi_{rm}(P_4 \circ P_3) = 16$.

**Lemma 2.3.** *For $x \geq 2$ and $y \geq 4$,*

$$\chi_{rm}(P_x \circ P_y) = \begin{cases} 2y+3 \text{ if } x=2 \text{ and } y=4\tau\ ,\ y=4\tau+2,\ \tau \geq 1, \\ xy+x \text{ otherwise.} \end{cases}$$

*Proof.* Let $V(P_x \circ P_y) = V(P_x) \cup \{v_j^i : 1 \leq i \leq x, 1 \leq j \leq y\}$ and $E(P_x \circ P_y) = E(P_x) \cup \{u_i v_j^i : i \in \{1,2,\ldots,x\}, j \in \{1,2,\ldots,y\}\} \cup \{v_j^i v_{j+1}^i : i \in \{1,2,\ldots,x\}, j \in \{1,2,\ldots,y\}\}$.

Define $c' : E(P_x \circ P_y) \to \mathbb{N}$ as follows:
Assign colors to the edges $u_1 v_j^1$ $(1 \leq j \leq y)$ by $c$ as defined in the proof of [7, Theorem 1.6].
If $x=2$, then color the edges of $P_2 \circ P_y$ in the following manner:
**Case 1.** $y = 4\tau$ and $y = 4\tau+2$, $\tau \geq 1$.

$$\begin{aligned} c'(u_2 v_j^2) &= c(u_1 v_j^1) + \frac{3(y+2)}{2} \text{ if } j=1,2; \\ c'(u_2 v_j^2) &= c(u_1 v_j^1) + y + 2 \text{ if } j \in \{3,4,\ldots,y\}; \\ c'(v_1^2 v_2^2) &= \frac{y+4}{2}; \\ c'(v_j^2 v_{j+1}^2) &= c(v_j^1 v_{j+1}^1) + 2 + y \text{ if } 2 \leq j \leq y-1; \end{aligned}$$

$$c'(u_1 u_2) = \begin{cases} y \text{ if } y = 4\tau; \\ y+1 \text{ if } y = 4\tau+2. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_2 \circ P_y$ is given by:
$cm(v_j^2) = cm(v_j^1) + y + 2$ if $j \in \{1,2,\ldots,y\}$;

$$cm(u_2) = \begin{cases} 2(y+1) \text{ if } y = 4\tau; \\ 2y+3 \text{ if } y = 4\tau+2. \end{cases}$$

Hence, $\chi_{rm}(P_2 \circ P_y) = 2y+3$.
**Case 2.** $y = 4\tau+1$ and $y = 4\tau+3$, $\tau \geq 1$.

$$\begin{aligned} c'(u_2 v_j^2) &= c(u_1 v_j^1) + \frac{3(y+1)}{2} \text{ if } j=1,2; \\ c'(u_2 v_j^2) &= c(u_1 v_j^1) + y + 1 \text{ if } j \in \{3,4,\ldots,y\}; \\ c'(v_1^2 v_2^2) &= \frac{y+3}{2}; \\ c'(v_j^2 v_{j+1}^2) &= c(v_j^1 v_{j+1}^1) + 1 + y \text{ if } j \in \{2,3,\ldots,y-1\}; \end{aligned}$$

$$c'(u_1 u_2) = \begin{cases} y \text{ if } y = 4\tau+1; \\ y+1 \text{ if } y = 4\tau+3. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_2 \circ P_y$ is given by:
$cm(v_j^2) = cm(v_j^1) + y + 1$ if $1 \le j \le y$;

$$cm(u_2) = \begin{cases} 1 + 2y \text{ if } y = 4\tau + 1; \\ 2y + 2 \text{ if } y = 4\tau + 3. \end{cases}$$

Hence, $\chi_{rm}(P_2 \circ P_y) = x(y+1)$.

**Case 3.** $x \ge 3$.

**Case 3.1.** $x$ is odd.

For $i \in \{2, 3, \ldots, x\}$ and $j \in \{1, 2, \ldots, y\}$,

$$c'(u_i v_j^i) = c(u_1 v_j^1) + (y+1)(i-1);$$
$$c'(v_j^i v_{j+1}^i) = c(v_j^1 v_{j+1}^1) + (y+1)(i-1);$$

For $i \in \{1, 3, 5, \ldots, x-2\}$ and $\tau \ge 1$,

$$c'(u_i u_{i+1}) = \begin{cases} i(y+1) - 1 \text{ if } y = 4\tau \text{ and } y = 4\tau + 1; \\ i(y+1) \text{ if } y = 4\tau + 2 \text{ and } y = 4\tau + 3; \end{cases}$$

For $i \in \{2, 4, 6, \ldots, x-1\}$ and $\tau \ge 1$,

$$c'(u_i u_{i+1}) = \begin{cases} y(i+1) + i \text{ if } y = 4\tau \text{ and } y = 4\tau + 1; \\ (y+1)(i+1) \text{ if } y = 4\tau + 2 \text{ and } y = 4\tau + 3. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_x \circ P_y$ is given by:
For $1 \le i \le x$ and $1 \le j \le y$, $cm(v_j^i) = cm(v_j^1) + yi - y + i - 1$.
For $y = 4\tau, y = 4\tau + 1$ and $\tau \ge 1$,

$$cm(u_1) = y;$$
$$cm(u_i) = i(y+1) - 1 \text{ if } i \in \{2, 3, \ldots, x-1\};$$
$$cm(u_x) = x(y+1) - 1.$$

For $y = 4\tau + 2$ and $y = 4\tau + 3$, $\tau \ge 1$

$$cm(u_1) = y + 1;$$
$$cm(u_i) = i(y+1) \text{ if } i \in \{2, 3, \ldots, x-1\};$$
$$cm(u_x) = x(y+1).$$

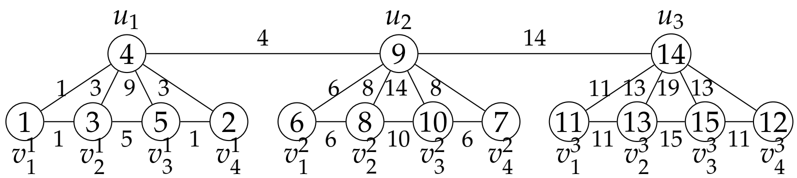


FIGURE 4. $\chi_{rm}(P_3 \circ P_4) = 15$.

Hence $rm(P_x \circ P_y) = x(y+1)$ (refer to FIGURE 4. $\chi_{rm}(P_3 \circ P_4) = 15$).

**Case 3.2.** $x$ is even and $\tau \geq 1$.

$$c'(u_{x-1}v_j^{x-1}) = c(u_1v_j^1) + (x-1)(y+1) \text{ if } j \in \{1,2,\ldots,y\};$$
$$c'(v_j^{x-1}v_{j+1}^{x-1}) = c(v_j^1v_{j+1}^1) + yx + x - y - 1 \text{ if } j \in \{1,2,\ldots,y-1\};$$
$$c'(u_xv_j^x) = c(u_1v_j^1) + yx - 3y + x - 3 \text{ if } j = 1,2;$$
$$c'(u_xv_j^x) = c(u_1v_j^1) + xy - 2y + x - 2 \text{ if } 3 \leq j \leq y;$$
$$c'(v_1^xv_2^x) = c(v_1^1v_2^1) + xy - y + x - 1;$$
$$c'(v_j^xv_{j+1}^x) = c(v_j^1v_{j+1}^1) + xy - 2y + x - 2 \text{ if } 2 \leq j \leq y-1.$$

Then,

$$c'(u_{x-1}u_x) = \begin{cases} x(y+1) + y \text{ if } y = 4\tau \text{ and } y = 4\tau+1; \\ (x+1)(y+1) \text{ if } y = 4\tau+2 \text{ and } y = 4\tau+3; \end{cases}$$

color the remaining edges of $P_x \circ P_y$ same as in Case 1.

Therefore, the chromatic mean of the vertices of $P_x \circ P_y$ is given by:

$$cm(u_{x-1}) = \frac{y^2x + 3xy - y - 2 + 2x}{y+2} \text{ if } y = 4\tau \text{ and } y = 4\tau+1;$$
$$cm(u_{x-1}) = x(y+1) \text{ if } y = 4\tau+2 \text{ and } y = 4\tau+3;$$
$$cm(u_x) = xy - y + x - 2 \text{ if } y = 4\tau \text{ and } y = 4\tau+1;$$
$$cm(u_x) = (y+1)(x-1) \text{ if } y = 4\tau+2 \text{ and } y = 4\tau+3.$$

Therefore, the chromatic mean of the remaining vertices of $P_x \circ P_y$ same as in *Case 1*.

Hence $\chi_{rm}(P_x \circ P_y) = x(y+1)$. □

From lemmas 2.1, 2.2, and 2.3, we conclude that:

**Theorem 2.1.** *For $x \geq 2$ and $y \geq 2$,*

$$\chi_{rm}(P_x \circ P_y) = \begin{cases} 2y+3 \textit{ if } x = 2 \textit{ and } y = 4\tau,\ y = 4\tau+2,\ \tau \geq 1, \\ x(y+1) \textit{ otherwise.} \end{cases}$$

In the following lemmas 2.4, 2.5, 2.6, Theorem 2.2 and Theorem 2.3, the graph $G$ is a $\ell$ regular graph that contains a hamiltonian cycle with $V(G) = \{u_i : i \in \{1,2,\ldots,x\}\}$ and let $V(C_x) = \{u_1, u_2, \ldots, u_x\}$ be a hamiltonian cycle in $G$.

**Lemma 2.4.** *Let $G$ be an $\ell$ regular graph that contains a hamiltonian cycle with $x$ vertices. For $x \geq 3$,*

$$\chi_{rm}(G \circ P_2) = \begin{cases} 1 + 3x \text{ if } x = 4\tau + 2, \ \tau \geq 1 \text{ and } \ell \text{ is odd,} \\ 3x \text{ otherwise.} \end{cases}$$

*Proof.* Let $V(P_2) = \{v_j : j = 1, 2\}$. Clearly, $V(G \circ P_2) = V(G) \cup \{v_j^i : i \in \{1, 2, \ldots, x\}, j \in \{1, 2\}\}$. Define $c' : E(G \circ P_2) \to \mathbb{N}$ as:

**Case 1.** $x = 4\tau, 4\tau + 1$ where $\tau \geq 1$.

For $i \in \{1, 2, \ldots, \lceil \frac{x}{2} \rceil\}$,

$c'(u_i v_1^i) = 6i - 5$; $c'(u_i v_2^i) = 6i - 3$; $c'(v_1^i v_2^i) = 6i - 5$;

for $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x$,

$c'(u_i v_1^i) = 6(x - i) + 4$; $c'(u_i v_2^i) = 6(x - i + 1)$; $c'(v_1^i v_2^i) = 6(x - i) + 4$.

Color the edges of the hamiltonian cycle as:

For $1 \leq i \leq \lceil \frac{x}{2} \rceil$,

$$c'(u_i u_{i+1}) = \begin{cases} 4 + 3\ell(i - 1) \text{ if odd } i; \\ 4 + 3\ell i \text{ if even } i; \end{cases}$$

$c'(u_i u_{i+1}) = 4 + 3\ell(x - i)$ if $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x - 1$,

$c'(u_x u_1) = 4$, and assign the color 3 to all edges remaining in $G \circ P_2$.

Therefore, the chromatic mean of the vertices of $G \circ P_2$ is given by:

For $1 \leq j \leq 2$,

$$cm(v_j^i) = j + 3(2i - 2) \text{ if } 1 \leq i \leq \lceil \frac{x}{2} \rceil;$$

$$cm(v_j^i) = j + 3(2x - 2i + 1) \text{ if } \left\lceil \frac{x}{2} \right\rceil + 2 \leq i \leq x;$$

$$\begin{aligned} cm(u_i) &= \frac{1}{\ell + 2} \left[4 + 3\ell(i - 2) + 4 + 3\ell i + 3(\ell - 2) + 1 + 6(i - 1) + 3 + 6(i - 1)\right] \\ &= 3(2i - 1) \text{ if } 1 \leq i \leq \left\lceil \frac{x}{2} \right\rceil; \end{aligned}$$

$$cm(u_i) = 6(x - i + 1) \quad \text{if} \quad \left\lceil \frac{x}{2} \right\rceil + 1 \leq i \leq x \quad (\text{refer to FIGURE 5. } \chi_{rm}(K_{3,3} \circ P_2) = 19).$$

**Case 2.** $x = 4\tau + 2$ where $\tau \geq 1$.

**Case 2.1** $\ell$ is even,

$$c'(u_{\frac{x}{2}+1} v_1^{\frac{x}{2}+1}) = \frac{6x - 4 + 3\ell}{2};$$

$$'(u_{\frac{x}{2}+1} v_2^{\frac{x}{2}+1}) = \frac{6x + 3\ell}{2};$$

$$c'(v_1^{\frac{x}{2}+1} v_2^{\frac{x}{2}+1}) = \frac{6x - 3\ell - 4}{2};$$

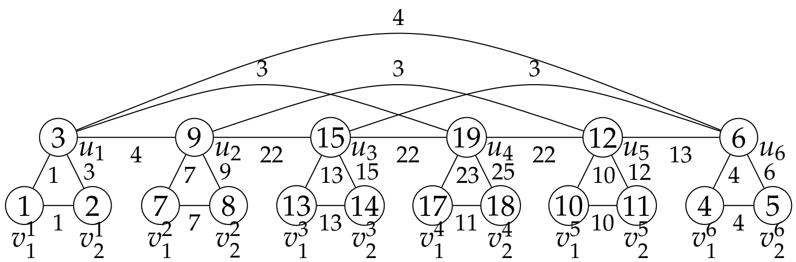


FIGURE 5. $\chi_{rm}(K_{3,3} \circ P_2) = 19$.

and color the remaining edges and vertices of $G \circ P_2$ as in *Case 1.*

**Case 2.2** $\ell$ is odd,

$$c'(u_{\frac{x}{2}+1}v_1^{\frac{x}{2}+1}) = 3x + 2\ell - 1;$$
$$'(u_{\frac{x}{2}+1}v_2^{\frac{x}{2}+1}) = 3x + 2\ell + 1;$$
$$c'(v_1^{\frac{x}{2}+1}v_2^{\frac{x}{2}+1}) = 3x - 1 - 2\ell;$$

and color the remaining edges of $G \circ P_2$ as in *Case 1.*

Therefore, the chromatic mean of the vertices of $G \circ P_2$ is given by:

$cm\left(u_{\frac{x}{2}+1}\right) = 3x + 1;$

$cm(v_j^{\frac{x}{2}+1}) = j + 3x - 2$; and the remaining chromatic mean of the vertices of $G \circ P_2$ is described as in *Case 1.*

**Case 3.** $x = 4\tau + 3$ where $\tau \geq 1$.

$$c'(u_{\lceil \frac{x}{2} \rceil}v_1^{\lceil \frac{x}{2} \rceil}) = \frac{6x - 4 + 3\ell}{2},$$
$$c'(u_{\lceil \frac{x}{2} \rceil}v_2^{\lceil \frac{x}{2} \rceil}) = \frac{6x + 3\ell}{2};$$
$$c'(v_1^{\lceil \frac{x}{2} \rceil}v_2^{\lceil \frac{x}{2} \rceil}) = \frac{6x - 4 - 3\ell}{2};$$
$$c'(u_{\frac{x+1}{2}}u_{\frac{x+3}{2}}) = \frac{8 + 3\ell(x-1)}{2};$$

and color the remaining edges and vertices of $G \circ P_2$ is described as in *Case 1.*

Hence,

$$\chi_{rm}(G \circ P_2) = \begin{cases} 1 + 3x \text{ if } x = 4\tau + 2, \ \tau \geq 1 \text{ and } \ell \text{ is odd,} \\ 3x \text{ otherwise.} \end{cases}$$

□

**Lemma 2.5.** *Let G be a $\ell$ regular graph that contains a hamiltonian cycle with x vertices. For $x \geq 3$, $\chi_{rm}(G \circ P_3) = 4x$.*

*Proof.* Let $V(P_3) = \{v_j : j = 1, 2, 3\}$. Clearly, $V(G \circ P_3) = V(G) \cup \{v_j^i : 1 \leq i \leq x, j = 1, 2, 3\}$. Define $c' : E(G \circ P_3) \to \mathbb{N}$ is given by:

**Case 1.** $x = 4\tau,\ 4\tau + 1$ where $\tau \geq 1$,
For $1 \leq i \leq \lceil \frac{x}{2} \rceil$,
$c'(u_i v_j^i) = j + 8(i-1)$ if $1 \leq j \leq 3$ and $c'(v_j^i v_{j+1}^i) = 2j + 8i - 9$ if $j = 1, 2$;
For $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x$,
$c'(u_i v_j^i) = j + 8(x-i) + 4$ if $1 \leq j \leq 3$;
$c'(v_j v_{j+1}) = 2j + 8(x-i) + 3$ if $j = 1, 2$;
Color the edges of the hamiltonian cycle as:
For $1 \leq i \leq \lceil \frac{x}{2} \rceil$,

$$c'(u_i u_{i+1}) = \begin{cases} 7 + 4\ell(i-1) \text{ if odd } i; \\ 7 + 4\ell i \text{ if even } i; \end{cases}$$

$c'(u_i u_{i+1}) = 4\ell x + 7 - i4\ell$ if $i \in \{\lceil \frac{x}{2} \rceil + 1, \lceil \frac{x}{2} \rceil + 2, \ldots, x-1\}$; $c'(u_x u_1) = 7$, and give a color 4 to all left over edges in $G \circ P_3$.
Therefore, the chromatic mean of the vertices of $G \circ P_3$ is given by:
For $j = 1, 2, 3$,

$$cm(v_j^i) = j + 4(2i-2) \text{ if } 1 \leq i \leq \left\lceil \frac{x}{2} \right\rceil;$$
$$cm(v_j^i) = j + 4(2x - 2i + 1) \text{ if } \left\lceil \frac{x}{2} \right\rceil + 1 \leq i \leq x;$$
$$cm(u_i) = 8i - 4 \text{ if } 1 \leq i \leq \left\lceil \frac{x}{2} \right\rceil;$$
$$cm(u_i) = 8(x - i + 1) \text{ if } \left\lceil \frac{x}{2} \right\rceil + 2 \leq i \leq x.$$

**Case 2.** $x = 4\tau + 2$ where $\tau \geq 1$,

$$c'(u_{\frac{x}{2}+1} v_j^{\frac{x}{2}+1}) = j + 4x - 4 + 2\ell \text{ if } j = 1, 2;$$
$$c'(u_{\frac{x}{2}+1} v_3^{\frac{x}{2}+1}) = c'(v_2^{\frac{x}{2}+1} v_3^{\frac{x}{2}+1}) = 4x - 1;$$
$$c'(v_1^{\frac{x}{2}+1} v_2^{\frac{x}{2}+1}) = 4x - 3 - 2\ell;$$

and color the remaining edges and vertices of $G \circ P_3$ is described as in *Case 1.*
**Case 3.** $x = 4\tau + 3$ where $\tau \geq 1$,

$$c'(u_{\lceil \frac{x}{2} \rceil} v_j^{\lceil \frac{x}{2} \rceil}) = j + 4x - 4 + 2\ell \text{ if } j = 1, 2;$$
$$c'(u_{\lceil \frac{x}{2} \rceil} v_3^{\lceil \frac{x}{2} \rceil}) = c'(v_2^{\lceil \frac{x}{2} \rceil} v_3^{\lceil \frac{x}{2} \rceil}) = 4x - 1;$$
$$c'(v_1^{\lceil \frac{x}{2} \rceil} v_2^{\lceil \frac{x}{2} \rceil}) = 4x - 3 - 2\ell;$$
$$c'(u_{\frac{x+1}{2}} u_{\frac{x+3}{2}}) = 7 + 2\ell(x-1);$$

and color the remaining edges and vertices of $G \circ P_3$ is described as in *Case 1.*
Hence $\chi_{rm}(G \circ P_3) = 4x$. □

**Lemma 2.6.** *Let $G$ be a $\ell$ regular graph that contains a hamiltonian cycle with $x$ vertices, $x \geq 3$ and $P_y$ be a path with $y \geq 4$ vertices. Then*

$$\chi_{rm}(G \circ P_y) = \begin{cases} x(y+1)+1 \text{ if } x = 4\tau+2,\ y = 4\tau,\ 4\tau+2,\ \tau \geq 1 \text{ and } \ell \text{ is odd}, \\ x(y+1) \text{ otherwise.} \end{cases}$$

*Proof.* Let $V(P_y) = \{v_j : j \in \{1,2,\dots,y\}\}$. Clearly, $V(G \circ P_y) = \{u_i : i \in \{1,2,\dots,x\}\} \cup \{v_j^i : 1 \leq i \leq x, 1 \leq j \leq y\}$. Define $c' : E(G \circ P_y) \to \mathbb{N}$ as follows:
Fix the colors to the edges $u_1 v_j^i$ by $c$ as defined in [7, Theorem 1.2].
**Case 1.** $x = 4\tau,\ 4\tau+1$ where $\tau \geq 1$,
For $2 \leq i \leq \lceil \frac{x}{2} \rceil$ and $1 \leq j \leq y$,

$$c'(u_i v_j^i) = c(u_1 v_j^1) + (y+1)(2i-2);$$
$$c'(v_j^i v_{j+1}^i) = c(v_j^1 v_{j+1}^1) + (y+1)(2i-2);$$

For $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x$ and ,

$$c'(u_i v_j^i) = c(u_1 v_j^1) + (y+1)(2x-2i+1) \text{ if } j \in \{1,2,\dots,y\};$$
$$c'(v_j^i v_{j+1}^i) = c(v_j^1 v_{j+1}^1) + (y+1)(2x-2i+1) \text{ if } j \in \{1,2,\dots,y-1\};$$

Color the edges of the hamiltonian cycle as follows:
**Case 1.1.** $y = 4\tau,\ y = 4\tau+1,\ \tau \geq 1$,
For $1 \leq i \leq \lceil \frac{x}{2} \rceil$,

$$c'(u_i u_{i+1}) = \begin{cases} y + \ell(i-1)(y+1) \text{ if odd } i; \\ y + \ell i y + i\ell \text{ if even } i; \end{cases}$$

$c'(u_i u_{i+1}) = y + (xy - yi + x - i)\ell$ if $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x-1$,
$c'(u_x u_1) = y$; and assign the color $y$ to all edges remaining in $G \circ P_y$.
**Case 1.2** $y = 4\tau+2,\ y = 4\tau+3,\ \tau \geq 1$,
For $i \in \{1,2,\dots,\lceil \frac{x}{2} \rceil\}$;

$$c'(u_i u_{i+1}) = \begin{cases} (y+1)(\ell i - \ell + 1) \text{ if odd } i; \\ (\ell i + 1)(y+1) \text{ if even } i; \end{cases}$$

$c'(u_i u_{i+1}) = (y+1)(\ell x - i\ell + 1)$ if $\lceil \frac{x}{2} \rceil + 1 \leq i \leq x-1$; $c'(u_x u_1) = 1 + y$.
Assign the color $y+1$ to all edges remaining in $G \circ P_y$.
Therefore, the chromatic mean of the vertices of $G \circ P_y$ is given by:
For $1 \leq i \leq \left\lceil \frac{x}{2} \right\rceil$, $1 \leq j \leq y$ and $\tau \geq 1$,

$$cm(v_j^i) = cm(v_j^1) + (y+1)(2i-2);$$
$$cm(u_i) = (2i-1)(y+1) - 1 \text{ if } y = 4\tau \text{ and } y = 4\tau+1;$$
$$cm(u_i) = 2iy - y + 2i - 1 \text{ if } y = 4\tau+2 \text{ and } y = 4\tau+3;$$

For $i \in \{\lceil \frac{x}{2} \rceil + 1, \lceil \frac{x}{2} \rceil + 2, \lceil \frac{x}{2} \rceil + 3, \ldots, x\}$,

$$cm(v_j^i) = cm(v_j^1) + (2x - 2i + 1)(y + 1) \text{ if } 1 \leq j \leq y;$$
$$cm(u_i) = 2(y + 1)(x - i) + 2y + 1 \text{ if } y = 4\tau \text{ and } y = 4\tau + 1;$$
$$cm(u_i) = 2(y + 1)(x - i + 1) \text{ if } y = 4\tau + 2 \text{ and } y = 4\tau + 3.$$

**Case 2.** $x = 4\tau + 2$ where $\tau \geq 1$,

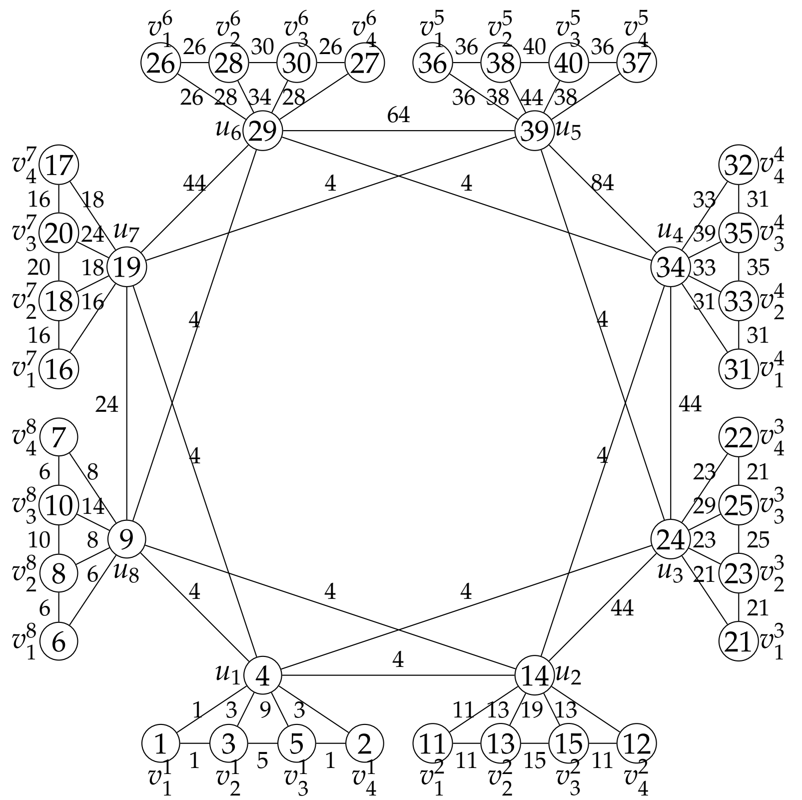


FIGURE 6. $\chi_{rm}(C_8; \{1, 2\} \circ P_4) = 19$.

**Case 2.1** $y = 4\tau,\ 4\tau + 2,\ \tau \geq 1$ and $\ell$ is odd.

$$c'(u_{\frac{x}{2}+1} v_j^{\frac{x}{2}+1}) = c(u_1 v_j^1) + \frac{(y + 1)(2x - 2 + \ell) + 2 + \ell}{2} \text{ if } j = 1, 2;$$
$$c'(u_{\frac{x}{2}+1} v_j^{\frac{x}{2}+1}) = c(u_1 v_j^1) + (y + 1)(x - 1) + 1 \text{ if } 3 \leq j \leq y;$$
$$c'(v_1^{\frac{x}{2}+1} v_2^{\frac{x}{2}+1}) = c(v_1^1 v_2^1) + \frac{(y + 1)(2x - 2 - \ell) + 2 - \ell}{2};$$
$$C'(v_j^{\frac{x}{2}+1} v_{j+1}^{\frac{x}{2}+1}) = c(v_j^1 v_{j+1}^1) + (y + 1)(x - 1) + 1;$$

and color the remaining edges of $G \circ P_y$ as in *Case 1.*

Therefore, the chromatic mean of the vertices of $G \circ P_y$ is given by:

$cm(v_j^{\frac{x}{2}+1}) = cm(v_j^1) + x(y + 1) - y$ if $1 \leq j \leq y$;

$$cm(u_{\frac{x}{2}+1}) = \begin{cases} x(y+1) \text{ if } y = 4\tau, \\ x(y+1)+1 \text{ if } y = 4\tau+2; \end{cases}$$

and the remaining chromatic mean of the vertices of $G \circ P_y$ is described as in *Case 1.*

**Case 2.2** $(y+1)\ell$ is even.

$$c'(u_{\frac{x}{2}+1}v_j^{\frac{x}{2}+1}) = c(u_1v_j^1) + \frac{(y+1)(2x-2+\ell)}{2} \text{ if } j = 1,2;$$
$$c'(v_1^{\frac{x}{2}+1}v_2^{\frac{x}{2}+1}) = \frac{2+(y+1)(2x-2-\ell)}{2};$$

and color the remaining edges and vertices of $G \circ P_y$ is described as in *Case 1.*

**Case 3.** $x = 4\tau + 3$ where $\tau \geq 1$,

$$c'(u_{\lceil\frac{x}{2}\rceil}v_j^{\lceil\frac{x}{2}\rceil}) = c(u_1v_j^1) + \frac{(1+y)(2x+\ell-2)}{2} \text{ if } j = 1,2;$$
$$c'(v_1^{\lceil\frac{x}{2}\rceil}v_2^{\lceil\frac{x}{2}\rceil}) = c(v_1^1v_2^1) + \frac{(y+1)(2x-\ell-2)}{2};$$

$$c'(u_{\frac{x+1}{2}}u_{\frac{x+3}{2}}) = \begin{cases} \frac{\ell(y+1)(x-1)+2y}{2} \text{ if } x = 4\tau,\ 4\tau+1,\ \tau \geq 1, \\ \frac{(y+1)(\ell x+2-\ell)}{2} \text{ if } x = 4\tau+2,\ 4\tau+3,\ \tau \geq 1; \end{cases}$$

and color the remaining edges and vertices of $G \circ P_y$ is described as in *Case 1.*

Hence

$$\chi_{rm}(G \circ P_y) = \begin{cases} x(y+1)+1 \text{ if } x = 4\tau+2,\ y = 4\tau,\ 4\tau+2,\ \tau \geq 1 \text{ and } \ell \text{ is odd}, \\ x(y+1) \text{ otherwise.} \end{cases}$$

□

From Lemmas 2.4, 2.5, and 2.6, we conclude that:

**Theorem 2.2.** *Let $G$ be $\ell$ regular graph that contains a Hamiltonian cycle and $|V(G)| = x$, $x \geq 4$ and $P_y$ denote a path with $y$ vertices. For $y \geq 4$,*

$$\chi_{rm}(G \circ P_y) = \begin{cases} x(y+1)+1 \textit{ if } x = 4\tau+2,\ y = 4\tau,\ 4\tau+2,\ \tau \geq 1 \textit{ and} \\ \quad \ell \textit{ is odd}, \\ x(y+1) \textit{ otherwise.} \end{cases}$$

**Theorem 2.3.** *Let $G$ be a regular graph that contains a hamiltonian cycle with $x$ vertices. For $x \geq 3$,*

$$\chi_{rm}(G \vee K_1) = \begin{cases} x+2 \textit{ if } x = 4\tau+1,\ \tau \geq 1, \\ x+1 \textit{ otherwise.} \end{cases}$$

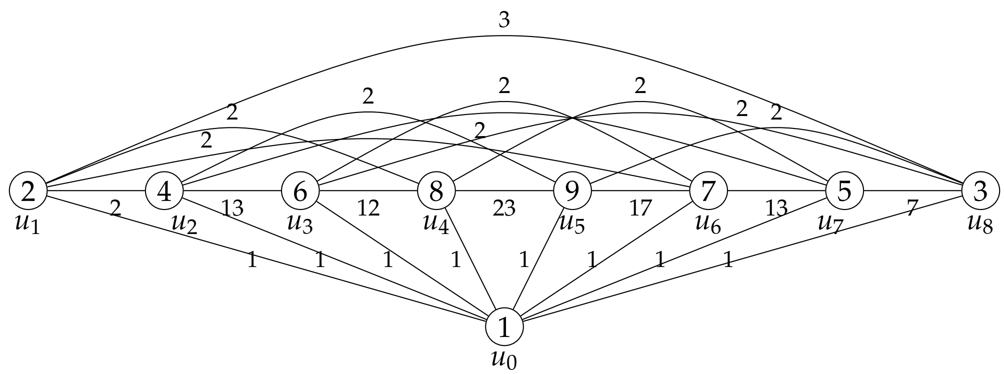


FIGURE 7. $\chi_{rm}(C_8;\{1,3\} \circ K_1) = 9$.

*Proof.* Let $V(G \vee K_1) = \{u_i : 1 \le i \le x\} \cup \{u_0\}$ and $E(G \vee K_1) = E(G) \cup \{u_i u_0 : 1 \le i \le x\}$. Define $c' : E(G \vee K_1) \to \mathbb{N}$ as:
If $x = 3$, then $G \vee K_1 \cong K_4$. In [3], the authors proved that $\chi_{rm}(K_4) = 4$.
**Case 1.** $x = 4\tau,\ \tau \ge 1$.

$$c'(u_0u_i) = 1 \text{ if } 1 \le i \le x,$$
$$c'(u_iu_{i+1}) = \ell i - \ell + i + 1 \text{ if } i \in \{1,3,5,\dots,\frac{x-2}{2}\},$$
$$c'(u_iu_{i+1}) = 3 + (\ell+1)i \text{ if } i \in \{2,4,\dots,\frac{x}{2}\},$$
$$c'(u_iu_{i+1}) = 2 + (\ell+1)(x-i) \text{ if } i \in \{\frac{x+2}{2},\frac{x+6}{2},\dots,x-1\},$$
$$c'(u_iu_{i+1}) = 3 + (\ell+1)(x-i) \text{ if } i \in \{\frac{x+4}{2},\frac{x+8}{2},\dots,x-2\},$$
$$c'(u_xu_1) = 3,$$

assign the color 2 to all edges remaining in $G \vee K_1$.
Therefore, the chromatic mean of the vertices of $G \vee K_1$ is given by:

$$cm(u_0) = \frac{x}{x} = 1;$$
$$cm(u_i) = 2i \text{ if } i \in \{1,2,\dots,\frac{x}{2}\};$$
$$cm(u_i) = 3 + 2(x-i) \text{ if } \frac{x+2}{2} \le i \le x.$$

Hence $\chi_{rm}(G \vee K_1) = x + 1$ (refer to FIGURE 6. $\chi_{rm}(K_{4,4} \circ K_1) = 9$. )
**Case 2.** $x = 4\tau + 2,\ \tau \ge 1$.
$c'(u_0u_i) = 2i - 1$ if $1 \le i \le 4$, $c'(u_0u_i) = 4$ if $5 \le i \le x-2$,
$c'(u_0u_{x-1}) = 6$, $c'(u_0u_x) = 2$,

$$c'(u_iu_{i+1}) = \begin{cases} 1 + \ell(i-1) \text{ if } i = 1,3, \\ 1 + i\ell \text{ if } i = 2,4, \end{cases}$$

For $5 \le i \le \frac{x+2}{2}$,

$$c'(u_iu_{i+1}) = \begin{cases} 1 + i(\ell+1) - \ell \text{ if odd } i, \\ i(\ell+1) - 3 \text{ if even } i; \end{cases}$$

For $\frac{x+4}{2} \le i \le x-1$,

$$c'(u_iu_{i+1}) = \begin{cases} (\ell+1)(x-i) + 2 \text{ if odd } i, \\ (\ell+1)(x-i) + 2\ell - 1 \text{ if even } i; \end{cases}$$

$c'(u_xu_1) = 1$; and assign the color 1 to all edges remaining in $G \vee K_1$.
Therefore, the chromatic mean of the vertices of $G \vee K_1$ is given by:

$$cm(u_0) = 4,$$
$$cm(u_i) = 2i - 1 \text{ if } 1 \le i \le \frac{x+2}{2};$$
$$cm(u_i) = 2(x - i + 2) \text{ if } \frac{x+4}{2} \le i \le x-1;$$
$$cm(u_x) = 2.$$

Hence $\chi_{rm}(G \vee K_1) = x + 1$.
**Case 3.** $x = 4\tau + 3,\ \tau \ge 1$.

$$c'(u_0u_1) = 1;\ c'(u_0u_2) = 5;\ c'(u_0u_3) = 9,$$
$$c'(u_0u_i) = 5 \text{ if } 4 \le i \le x-4;$$
$$c'(u_0u_i) = 2(x-i+1) \text{ if } x-3 \le i \le x;$$
$$c'(u_1u_2) = 1,\ c'(u_2u_3) = 2\ell - 1;$$

For $3 \le i \le \frac{x-3}{2}$,

$$c'(u_iu_{i+1}) = \begin{cases} i(\ell+1) + 1 \text{ if even } i; \\ \ell(i+1) + i - 2 \text{ if odd } i; \end{cases}$$

For $\frac{x+1}{2} \le i \le x-4$,

$$c'(u_iu_{i+1}) = \begin{cases} (\ell+1)(x-i) + 1 \text{ if even } i, \\ x(\ell+1) - i\ell - 3 \text{ if odd } i; \end{cases}$$

$c'(u_iu_{i+1}) = \ell(x-i) + 1$ if $x - 3 \le i \le x$, and assign the color 1 to all edges remaining in $G \vee K_1$.
Therefore, the chromatic mean of the vertices of $G \vee K_1$ is given by:
$cm(u_0) = 5$;
$cm(u_i) = 2i - 1$ if $i = 1, 2$;
$cm(u_i) = 2i + 1$ if $3 \le i \le \frac{x-1}{2}$;
$cm(u_i) = 2x - 2i + 2$ if $\frac{x+1}{2} \le i \le x$.
Hence $\chi_{rm}(G \vee K_1) = x + 1$.

**Case 4.** $x = 4\tau + 1,\ \tau \geq 1$.

$c'(u_0u_i) = 1$ if $i \in \{1, 2, \ldots, x\}$,

$$c'(u_iu_{i+1}) = \begin{cases} i(\ell+1) - \ell + 3 \text{ if } i \in \{1,3,5,\ldots,\frac{x+1}{2}\}, \\ 4 + (\ell+1)i \text{ if } i \in \{2,4,\ldots,\frac{x-1}{2},\} \\ 4 + (\ell+1)(x-i) \text{ if } i \in \{\frac{x+3}{2},\frac{x+5}{2},\ldots,x-1\}. \end{cases}$$

$c'(u_nu_1) = 4$, assign the color 3 to all edges remaining in $G \vee K_1$.

Therefore, the chromatic mean of the vertices of $G \vee K_1$ is given by:

$cm(u_0) = \frac{x}{x} = 1$;

$cm(u_i) = 2i + 1$ if $i \in \{1, 2, \ldots, \frac{x+1}{2}\}$;

$cm(u_i) = 2x - 2i + 4$ if $i \in \{\frac{x+3}{2}, \frac{x+5}{2}, \ldots, x\}$.

Hence $\chi_{rm}(G \vee K_1) = x + 2$. □

**Corollary 2.1.** *[7] For $y \geq 4$,*

$$\chi_{rm}(C_y \vee K_1) = \begin{cases} y + 2 \textit{ if } y = 4\tau + 1 \textit{ and } \tau \geq 1, \\ y + 1 \textit{ otherwise.} \end{cases}$$

**Theorem 2.4.** *Let $H$ be a regular graph that contains a hamiltonian cycle with $y$ vertices, $y \geq 4$ and $P_x$ denote a path with $x \geq 3$ vertices. Then*

$$\chi_{rm}(P_x \circ H) = \begin{cases} xy + x + 1 \textit{ if } x \textit{ is odd and } y = 4\tau + 1,\ \tau \geq 1, \\ xy + x \textit{ otherwise.} \end{cases}$$

*Proof.* Let$V(P_m) = \{u_1, u_2, \ldots, u_x\}$ and $V(H) = \{v_1, v_2, \ldots, v_y\}$. Clearly $V(P_x \circ H) = V(P_x) \cup \{u_iv_j^i : i \in \{1, 2, \ldots, x\},\ j \in \{1, 2, \ldots, y\}\}$ and let $H$ be $\ell$ regular graph. Consider $H^1, H^2, H^3, \ldots, H^x$ be $x$ disjoint isomorphic copies of $H$.

Define $c'' : E(P_x \circ H) \to \mathbb{N}$ as follows:

Assign colors to the edges $u_1v_j^1$ using the coloring $c'$, as shown in the proof of Theorem 2.3.

**Case 1.** $x$ is odd and $y \neq 4\tau + 1,\ \tau \geq 1$.

For $i \in \{2, \ldots, x\}$ and $1 \leq j, j' \leq y$,

$c''(v_j^iv_{j'}^i) = c'(v_j^1v_{j'}^1) + iy - y + i - 1$;

$c''(u_iv_j^i) = c'(u_1v_j^1) + iy + i - y - 1$;

For $i \in \{1, 3, 5, \ldots, x - 2\}$,

$$c''(u_iu_{i+1}) = \begin{cases} y(i-1) + 1 \text{ if } y = 4\tau; \\ i(y+1) - y + 3 \text{ if } y = 4\tau + 2; \\ i(y+1) - y + 4 \text{ if } y = 4\tau + 3; \end{cases}$$

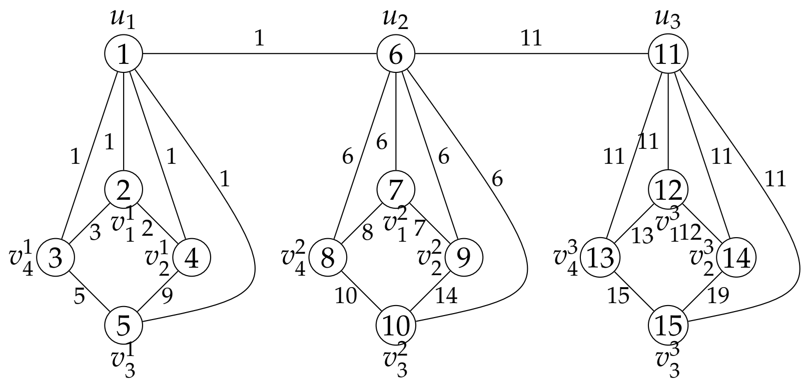


FIGURE 8. $\chi_{rm}(P_3 \circ C_4) = 15$.

For $i \in \{2, 4, 6, \ldots, x-1\}$,

$$c''(u_i u_{i+1}) = \begin{cases} 1 + i(y+1) \text{ if } y = 4\tau; \\ 4 + i(y+1) \text{ if } y = 4\tau + 2; \\ 5 + i(y+1) \text{ if } y = 4\tau + 3. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_x \circ H$ is given by:
$cm(v_j^i) = cm(v_j^1) + iy - y + i - 1$ if $1 \leq i \leq x, 1 \leq j \leq y$;
for $i \in \{1, 2, \ldots, x\}$,

$$cm(u_i) = \begin{cases} i(y+1) - y \text{ if } y = 4\tau; \\ i(y+1) - y + 3 \text{ if } y = 4\tau + 2; \\ i(y+1) - y + 4 \text{ if } y = 4\tau + 3. \end{cases}$$

Hence $\chi_{rm}(P_x \circ H) = x(y+1)$ (refer to FIGURE 8. $\chi_{rm}(P_3 \circ C_4) = 15$. )
**Case 2.** $x$ is even and $y \neq 4\tau + 1,\ \tau \geq 1$.
For $1 \leq j, j' \leq y$,

$$c''(v_j^{x-1} v_{j'}^{x-1}) = c'(v_j^1 v_{j'}^1) + (x-1)(y+1);$$
$$c''(u_{x-1} v_j^{x-1}) = c'(u_1 v_j^1) + (x-1)(y+1);$$

For $j = 1, 2$,

$$c''(u_x v_j^x) = c'(u_1 v_j^1) + (y+1)(x-3);$$
$$c''(v_1^x v_2^x) = c'(v_1^1 v_2^1) + (y+1)(x-1);$$
$$c''(u_x v_j^x) = c'(u_1 v_j^1) + (y+1)(x-2) \text{ if } 3 \leq j \leq y;$$
$$c''(v_j^x v_{j'}^x) = c'(v_j^1 v_{j'}^1) + (y+1)(x-2) \text{ if } \{E(H^x) \setminus v_1^x v_2^x : 1 \leq j, j' \leq y\};$$

$$c''(u_{x-1}u_x) = \begin{cases} x(y+1)+1 \text{ if } y = 4\tau; \\ x(y+1)+4 \text{ if } y = 4\tau+2; \\ x(y+1)+5 \text{ if } y = 4\tau+3; \end{cases}$$

and color the remaining all edges of $P_x \circ H$ same as in *Case 1.*
Therefore, the chromatic mean of the vertices of $P_x \circ H$ is given by:
for $1 \le j \le y$,
$cm(v_j^{x-1}) = cm(v_j^1) + (y+1)(x-1)$;
$cm(v_j^x) = cm(v_j^1) + xy - 2y + x - 2$;

$$cm(u_{x-1}) = \begin{cases} x(y+1) - y \text{ if } y = 4\tau; \\ x(y+1) - y + 3 \text{ if } y = 4\tau+2; \\ x(y+1) - y + 4 \text{ if } y = 4\tau+3; \end{cases}$$

$$cm(u_x) = \begin{cases} x(y+1) - 2y - 1 \text{ if } y = 4\tau; \\ x(y+1) - 2y + 2 \text{ if } y = 4\tau+2; \\ x(y+1) - 2y + 3 \text{ if } y = 4\tau+3; \end{cases}$$

The remaining chromatic mean of the vertices of $P_x \circ H$ follows as in *Case 1.*
Hence $\chi_{rm}(P_x \circ H) = xy + x$.
**Case 3.** $y = 4\tau + 1,\ \tau \ge 1$.
Define $c' : E(G \circ H) \to \mathbb{N}$ as follows:
Let $V(C_y) = \{v_1, v_2, v_3, \ldots, v_y\}$ be a hamiltonian cycle in $H^1$.
First, color the edges of hamiltonian cycle $H^1$ as follows:

$$c'(v_j^1 v_{j+1}^1) = \begin{cases} 1 + \ell(j-1) \text{ if } j \in \{1, 3, 5, \ldots, \frac{y+1}{2}\}, \\ 1 + \ell j \text{ if } j \in \{2, 4, 6, \ldots, \frac{y-1}{2}\}, \\ 1 + \ell(y-j) \text{ if } j \in \{\frac{y+3}{2}, \frac{y+5}{2}, \ldots, y-1\}; \end{cases}$$

$c'(v_y^1 v_1^1) = c'(v_1^1 v_2^1)$, give a color 1 to all the edges remaining in $H^1$.
Therefore, the chromatic mean of the vertices of $H^1$ is given by:
$cm(v_j^1) = 2j - 1$ if $j \in \{1, 2, 3, \ldots, \frac{y+1}{2}\}$;
$cm(v_j^1) = 2(y - j + 1)$ if $j \in \{\frac{y+3}{2}, \frac{y+5}{2}, \ldots, y\}$;
Next, color the edges of $x - 1$ disjoint isomorphic copies of $H^1$ as follows:
**Case 3.1.** $x$ is odd.
For $i \in \{2, 3, \ldots, x-1\}$ and $1 \le j, j' \le y$,

$$c'(v_j^i v_{j'}^i) = c'(v_j^1 v_{j'}^1) + (y+1)(i-1);$$
$$cm(v_j^i) = cm(v_j^1) + (y+1)(i-1);$$
$$c'(u_i v_j^i) = cm(v_j^i) \text{ if } i \in \{1, 2, \ldots, x-1\};$$

$$c'(v_j^x v_k^x) = c'(v_j^1 v_k^1) + x(y+1) - y;$$
$$cm(v_j^x) = cm(v_j^1) + x(y+1) - y.$$
$$c'(u_x v_j^x) = cm(v_j^x) + \frac{y(y+1)+2}{4} \text{ if } j \in \{1,2\};$$
$$c'(v_1^x v_2^x) = 1 + \frac{(y+1)(4x-4-y)+2}{4};$$
$$c'(u_x v_j^x) = cm(v_j^x) \text{ if } 3 \le j \le y;$$

Color the edges $u_i u_{i+1}$ as follows:

$$c'(u_i u_{i+1}) = \begin{cases} \frac{(y+1)(y+2i)}{2} & \text{if } i \in \{1,3,5,\ldots,x-2\}; \\ \frac{(y+1)(2i+2)}{2} & \text{if } i \in \{2,4,6,\ldots,x-1\}. \end{cases}$$

Therefore, the chromatic mean of the vertices of $P_x \circ H$ is given by:

$$cm(u_i) = i(y+1) \text{ if } 1 \le i \le x-1;$$
$$cm(u_x) = x(y+1) + 1.$$

Hence $\chi_{rm}(P_x \circ H) = x(y+1) + 1$.

**Case 3.2.** $x$ is even.

For $1 \le j, k \le y$,

$$c''(u_{x-1} v_j^{x-1}) = cm(v_j^1) + (y+1)(x-1);$$
$$c''(v_j^{x-1} v_{j'}^{x-1}) = c(v_j^1 v_{j'}^1) + xy - y + x - 1;$$
$$c''(u_x v_j^x) = cm(v_j^1) + (y+1)(x-3) \text{ if } j = 1,2;$$
$$c''(v_1^x v_2^x) = c(v_1^1 v_2^1) + (y+1)(x-1);$$
$$c''(u_x v_j^x) = cm(v_j^1) + (y+1)(x-2) \text{ if } j \in \{3,4,\ldots,y\};$$
$$c''(v_j^x v_{j'}^x) = c(v_j^1 v_{j'}^1) + (y+1)(x-2) \text{ if } \{E(H^x) \setminus v_1^x v_2^x : 1 \le j, j' \le y\};$$
$$c''(u_{x-1} u_x) = \frac{(y+1)(y+2x+2)}{2};$$

and color the remaining edges of $P_x \circ H$ follows as in Case 3.1.

Therefore, the chromatic mean of the vertices of $P_x \circ H$ is given by:

for $1 \le j \le y$,

$$cm(v_j^{x-1}) = cm(v_j^1) + (x-1)(y+1);$$
$$cm(v_j^x) = cm(v_j^1) + (x-2)(y+1);$$
$$cm(u_{x-1}) = x(y+1);\ cm(u_x) = (x-1)(y+1);$$

and the chromatic mean of the remaining vertices of $P_x \circ H$ follows as in Case 3.1.

Hence $\chi_{rm}(P_x \circ H) = x(y+1)$. □

## 3. Conclusion

As rainbow mean coloring is a recent development in graph coloring, numerous graph families still have undetermined rainbow mean coloring. In this study, we analyze the RMI of the corona product and join graph of some particular classes of graphs such as: $P_x \circ H$; $P_x \circ P_y$; $G \circ P_y$ and $G \vee K1$. The results presented in this paper support the conjecture stated in [3]. In future, we investigate the RMI of other product of graphs.

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