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On the Existence and Stability of a Hybrid Delay Itô−Differential Equation with Stochastic Feedback Control

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Abstract. In this paper, the main objective is to prove the existence of solutions $\mathcal{U} \in \mathbb{C}([0,\varpi], L_2(Q))$ for a hybrid delay Itô-differential equation with stochastic feedback control, a problem that incorporates delay effects introducing memory-like behavior into the system and leading to intricate dynamics. Additionally, the uniqueness of the solutions with sufficient conditions are provided. Furthermore, the solutions with the continuous dependence on initial data and certain functions, as well as the concept of Hyers-Ulam stability, are analyzed. These findings serve as a foundational framework from which well-established results in the literature naturally emerge. The concluding section is dedicated to applying these results to specific examples, illustrating the uniqueness and existence of solutions for hybrid delay Itô-differential equations with stochastic feedback control. These examples not only validate the theoretical findings but also enhance understanding by offering practical insights into the study of such equations.

1. Introduction

Stochastic differential equations have attracted significant attention because they can mathematically describe the behavior of systems with random inputs or parameters. This is particularly useful in fields such as finance, where they are used in market dynamics to model inherent uncertainty and randomness. In physics and biology, researchers use stochastic integral equations to analyze how random fluctuations affect the behavior of complex systems with random elements. Engineers also use stochastic integral equations to understand and anticipate dynamic systems under random disturbances and design powerful and flexible systems (see [1] - [3]).

One of the essential points of attention in the theoretical basis of stochastic analysis is discussing the existence of a unique solution for stochastic integral equations. This process requires an accurate understanding of many branches in mathematics: probability theory, functional analysis,

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measurement theory, and topology. Researchers use appropriate mathematical techniques, such as fixed-point theory, to advance in this field; the authors prove the existence of one solution or more for functional stochastic integral equations in Banach Algebra using Darbo's fixed-point theorem related to values of non-compactness [4]. The Schauder fixed-point theorem is applied in many research studies; for example, El-Sayed and H-Fouad [5], [6] used it to show the existence of a mean square continuous solution for the coupled systems of stochastic and random differential equations, subject to a nonlinear nonlocal stochastic integral condition. Also, El-borai and Youssef [7] by using it proved the uniqueness and existence of the solution for a nonlocal functional stochastic differential equation in the space of all second-order integrable stochastic processes under sufficient conditions.

Control theory is one of the branches of applied mathematics interested in studying the fundamental principles in the analysis and design of control systems. It has many applications across various domains aimed at influencing the behavior of an object to achieve a desired result [8], [9]. Feedback control is one such topic, meaning that state measurements can be used to determine a control action to achieve a desired outcome. It is widely used in everyday scenarios, from basic home thermostats that maintain certain temperatures to sophisticated devices that keep communications satellites in position. Feedback control also occurs naturally, such as in the regulation of blood sugar levels in the body. One added benefit of feedback control is that by examining the output of a system, unstable processes can be stabilized.

Mao [10], [11] had made great progress in the study of stochastic differential delay equations with feedback control, he was able to transform unstable stochastic differential problem into a mean-square exponentially stable solution, and considered delay feedback controls for the exponential mean-square stabilization of hybrid stochastic differential equations.

In addition, some researchers tended to study the dependence continuous of solutions on the control variable, such as, El-Sayed et al. [12]- [16] proved that the solution of quadratic nonlinear integral equations, hybrid delay functional integral equations and the cubic functional equations subject to feedback control in the real half axis are dependent continuously on the control variable. The hybrid problems (differential and integral) and the nonlocal boundary value problems have been considered by some authors (see, for example [17]- [20].

In our work, we let $(\mathcal{Q},\mathcal{F},\wp)$ be a complete probability space occurring during the time interval $[0,\varpi]$, $\wp$ is a probability measure and $\mathcal{F}$ is a $\sigma-$algebra of events defined on a sample space $\mathcal{Q}$.

Let $\mathcal{U} \in \mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$ where $\mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$ be the space of all second order mean square (m.s) continuous stochastic processes on $[0,\varpi]$. The norm of $\mathcal{U}$ is given by

$$\|\mathcal{U}(t)\|_2 = \sqrt{\mathbb{E}(\mathcal{U}^2(t))},\ \|\mathcal{U}\|_{\mathbb{C}} = \sup_{t\in[0,\varpi]} \|\mathcal{U}(t)\|_2.$$

Here, we study the existence of mean square continuous solutions $\mathcal{U} \in \mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$ of the feedback problem of the hybrid delay It$\hat{o}-$differential equation

$$d\Big(\frac{\mathcal{U}(t)-\mathfrak{a}(t)}{Q(t,\mathcal{U}(t))}\Big) = F(t,\mathcal{U}(\vartheta(t)))d\mathsf{W}_1(t),\ \mathcal{U}(0)=\mathfrak{a}(0),\ t\in(0,\varpi] \tag{1.1}$$

subject to the stochastic feedback control

$$\mathfrak{a}(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta),\ t\in(0,\varpi], \tag{1.2}$$

where $\mathsf{W}_1(t), \mathsf{W}_2(t)$ are two independent standard Brownian motions on a complete probability space $(\mathcal{Q},\mathcal{F},\wp)$ [2].

2. Quadratic Itô-integral Equation

Consider the feedback problem of the hybrid delay Itô−differential problem (1.1)-(1.2) under the following hypotheses:

$(A1)$ $\vartheta : [0,\varpi] \to [0,\varpi]$ be a continuous function provided that $\vartheta(t) \le t$.

$(A2)$ $F : [0,\varpi]\times \mathsf{L}_2(\mathcal{Q}) \to \mathsf{L}_2(\mathcal{Q})$ is continuous in $t\in[0,\varpi]$, $\forall\ \mathcal{U}\in \mathsf{L}_2(\mathcal{Q})$, continuous in $\mathcal{U}\in \mathsf{L}_2(\mathcal{Q}), \forall t\in[0,\varpi]$ and there exists a positive constant β_1 and a second order stochastic process $\alpha_1 \in \mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$, continuous in $\mathcal{U}\in \mathsf{L}_2(\mathcal{Q}), \forall t\in[0,\varpi]$ such that

$$\|F(t,\mathcal{U})\|_2 \le \|\alpha_1\|_{\mathbb{C}} + \beta_1\|\mathcal{U}(t)\|_2.$$

$(A3)$ $G : [0,\varpi]\times \mathsf{L}_2(\mathcal{Q}) \to \mathsf{L}_2(\mathcal{Q})$ is continuous in $t\in[0,\varpi]$, $\forall\ \mathcal{U}\in \mathsf{L}_2(\mathcal{Q})$ continuous in $\mathcal{U}\in \mathsf{L}_2(\mathcal{Q}), \forall t\in[0,\varpi]$ and there exists a positive constant β_2 and a second order stochastic process $\alpha_2 \in \mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$ such that

$$\|G(t,\mathcal{U})\|_2 \le \|\alpha_2\|_{\mathbb{C}} + \beta_2\|\mathcal{U}(t)\|_2.$$

$(A4)$ $Q : [0,\varpi]\times \mathsf{L}_2(\mathcal{Q}) \to \mathbb{R}\setminus\{0\}$ is continuous in t for every $\mathcal{U}\in \mathsf{L}_2(\mathcal{Q})$, continuous in $\mathcal{U}\in \mathsf{L}_2(\mathcal{Q}), \forall t\in[0,\varpi]$ and Lipschitz condition is satisfied for every $\mathcal{U}_1,\ \mathcal{U}_2\in \mathsf{L}_2(\mathcal{Q})$ as

$$|Q(t,\mathcal{U}_1)-Q(t,\mathcal{U}_2)| \le \beta_3\ \|\mathcal{U}_1(t)-\mathcal{U}_2(t)\|_2,\ \text{for every } t\in[0,\varpi],\ \beta_3\in\mathbb{R}^+.$$

From this hypotheses we can deduce that

$$|Q(t,\mathcal{U})| \le \alpha_3 + \beta_3\ \|\mathcal{U}(t)\|_2,\ \alpha_3 = \sup_{t\in[0,\varpi]} |Q(t,0)|.$$

$(A5)$ $\mu = (B\sqrt{\varpi}-1)^2 - 4BA\sqrt{\varpi} > 0,\ B\sqrt{\varpi}(1+2A) + \sqrt{\mu} < 1$, such that $A = \max\{\|\alpha_1\|_{\mathbb{C}},\ \|\alpha_2\|_{\mathbb{C}}, \alpha_3\},\ B = \max\{\beta_1,\beta_2,\beta_3\}$

Lemma 2.1. *The problem* (1.1)-(1.2) *equivalent to the quadratic Itô−stochastic integral equation*

$$\mathcal{U}(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t,\mathcal{U}(t))\int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \tag{2.1}$$

Proof. Using Itô− formula [21], [22], we deduce that the problem (1.1)-(1.2) equivalent to the stochastic integral equation

$$\frac{\mathcal{U}(t)-\mathfrak{a}(t)}{Q(t,\mathcal{U}(t))} = \int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta).$$

substituting the value of $\mathfrak{a}(t)$ we obtain the Itô-stochastic integral equation (1.1). □

In order to achieve our main goal, studying the existence of solutions $\mathcal{U} \in \mathbb{C}([0,\varpi], \mathsf{L}_2(Q))$ of the nonlinear quadratic stochastic Itô−integral equation (2.1) which involving two independent Brownian motions W_1 and W_2, we define the set Θ by

$$\begin{aligned} \Theta &= \{\mathcal{U} \in \mathbb{C}([0,\varpi], \mathsf{L}_2(Q)) : \|\mathcal{U}(t)\|_2 \leq r\} \subset \mathbb{C}([0,\varpi], \mathsf{L}_2(Q)), \\ & r = \frac{1 - B\sqrt{\varpi}(1+2A) - \sqrt{\mu}}{2B^2\sqrt{\varpi}}. \end{aligned}$$

It is clear that Θ is a nonempty, closed, bounded and convex set.

Following, we define the mapping $\mathcal{F}$ which is used to discuss the existence theory

$$\mathcal{F}\mathcal{U}(t) = \int_t^{\varpi} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t, \mathcal{U}(t)) \int_0^t F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta). \tag{2.2}$$

Theorem 2.1. *Let the hypotheses $(A1) - (A5)$ be satisfied, then there exists at least one solution $\mathcal{U} \in \mathbb{C}([0,\varpi], \mathsf{L}_2(Q))$ of the the nonlinear quadratic stochastic Itô−integral equation involving two independent Brownian motions* (2.1).

Proof. Let $\mathcal{U} \in \Theta$, then we have

$$\begin{aligned} \|\mathcal{F}\mathcal{U}(t)\|_2 &\leq \|\int_t^{\varpi} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta)\|_2 + |Q(t, \mathcal{U}(t))| \, \|\int_0^t F(\zeta, \mathcal{V}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\|_2 \\ &\leq \sqrt{\int_t^{\varpi} \|G(\zeta, \mathcal{U}(\vartheta(\zeta)))\|_2^2 d\zeta} + |Q(t, \mathcal{U}(t))| \sqrt{\int_0^t \|F(\zeta, \mathcal{U}(\vartheta(\zeta)))\|_2^2 d\zeta} \\ &\leq \sqrt{\int_t^{\varpi} (\|\alpha_2\|_C + \beta_2 r)^2 d\zeta} + (\alpha_3 + \beta_3 r) \sqrt{\int_0^t (\|\alpha_1\|_C + \beta_1 r)^2 d\zeta} \\ &\leq (A + Br)\sqrt{\varpi} + (A + Br)^2\sqrt{\varpi} = r. \end{aligned}$$

By assumption $(A5)$, this implies that the class $\{\mathcal{F}\mathcal{U}\}$ is uniformly bounded and $\mathcal{F}\mathcal{U} : \Theta \to \Theta$.

Let $\mathcal{U} \in \Theta$ and define

$$\varphi(\delta) = \sup\{|Q(t_2, \mathcal{U}(t_1)) - Q(t_1, \mathcal{U}(t_1))| : t_1, t_2 \in [0,\varpi], t_1 < t_2, |t_2 - t_1| < \delta, \|\mathcal{U}(t)\|_2 \leq r\},$$

Then from the uniformly continuity of $\mathcal{U}$ and Q, we get $\varphi(\delta) \to 0$ as $\delta \to 0$ is independent of $\mathcal{U} \in \Theta$.

Now, let $\mathcal{U} \in \Theta,\ t_1, t_2 \in [0, \varpi],\ t_1 < t_2$ such that $|t_2 - t_1| < \delta$,

$$\begin{aligned}
\mathcal{F}\mathcal{U}(t_2) - \mathcal{F}\mathcal{U}(t_1) &= \int_{t_2}^{\varpi} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t_2, \mathcal{U}(t_2)) \int_0^{t_2} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\
&- \int_{t_1}^{\varpi} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) - Q(t_1, \mathcal{U}(t_1)) \int_0^{t_1} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\
&= \int_{t_2}^{t_1} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t_2, \mathcal{U}(t_2))) \int_{t_1}^{t_2} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\
&+ (Q(t_2, \mathcal{U}(t_2)) - Q(t_1, \mathcal{U}(t_1))) \int_0^{t_1} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)
\end{aligned}$$

But

$$\begin{aligned}
|Q(t_2, \mathcal{U}(t_2)) - Q(t_1, \mathcal{U}(t_1))| &= |Q(t_2, \mathcal{U}(t_2)) - Q(t_2, \mathcal{U}(t_1)) + Q(t_2, \mathcal{U}(t_1)) - Q(t_1, \mathcal{U}(t_1))| \\
&\le \beta_3 \|\mathcal{U}(t_2))) - \mathcal{U}(t_1))\|_2 + \varphi(\delta) \\
&\le \beta_3\, \epsilon + \varphi(\delta),
\end{aligned}$$

since $\mathcal{U} \in \Theta$, then $\|\mathcal{U}(t_2))) - \mathcal{U}(t_1))\|_2 \le \epsilon \to 0$ as $\delta \to 0$, then we have

$$\begin{aligned}
\|\mathcal{F}\mathcal{U}(t_2) - \mathcal{F}\mathcal{U}(t_1)\|_2 &\le \| \int_{t_2}^{t_1} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta)\|_2 + |Q(t_2, \mathcal{U}(t_2)))|\, \| \int_{t_1}^{t_2} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\|_2 \\
&+ |Q(t_2, \mathcal{U}(t_2)) - Q(t_1, \mathcal{U}(t_1))|\, \| \int_0^{t_1} F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\|_2 \\
&\le \sqrt{t_2 - t_1}(A + Br) + |Q(t_2, \mathcal{U}(t_2))|\, \sqrt{t_2 - t_1}(A + Br) \\
&+ (\beta_3\, \epsilon) + \varphi(\delta))(A + Br)\, \sqrt{\varpi}
\end{aligned}$$

This implies that

$$\|\mathcal{F}\mathcal{U}(t_2) - \mathcal{F}\mathcal{U}(t_1)\|_2 \to 0 \ \text{ as } \delta \to 0,$$

and the class of function $\{\mathcal{F}\mathcal{U}\}$ is equicontinuous. Therefore the closure of $\{\mathcal{F}\mathcal{U}\}$ is a compact subset of $\mathbb{C}$ (Arzelà-Ascoli theorem [3]).

Now, consider $\mathcal{U}_n \in \Theta$ being such that $L.i.m_{n\to\infty}\mathcal{U}_n = \mathcal{U}\ w.p.1$ where $L.i.m$ denotes the limit of the continuous second order process in the mean square sense ([21]- [23]), when we apply Lebesgue dominated theorem [24], we can obtain

$$\begin{aligned}
L.i.m_{n\to\infty}\mathcal{F}\mathcal{U}_n &= L.i.m_{n\to\infty}\{ \int_t^{\varpi} G(\zeta, \mathcal{U}_n(\vartheta(\zeta))))d\mathsf{W}_2(\zeta) \\
&+ Q(t, \mathcal{U}_n(t)) \int_0^t F(\zeta, \mathcal{U}_n(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\} \\
&= \int_t^{\varpi} G(\zeta, L.i.m_{n\to\infty}\mathcal{U}_n(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) \\
&+ Q(t, L.i.m_{n\to\infty}\mathcal{U}_n(t)) \int_0^t F(\zeta, L.i.m_{n\to\infty}\mathcal{U}_n(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\}
\end{aligned}$$

$$= \int_t^{\varpi} G(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t,\mathcal{U}(t))\int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) = \mathcal{F}\mathcal{U}.$$

This implies that the operator $\mathcal{F}:\Theta\to\Theta$ is continuous, here applying Schauder fixed point theorem [3] and [24], there exists at least one solution $\mathcal{U}\in\mathbb{C}$ of the nonlinear quadratic stochastic Itô−integral equation (2.1). □

3. Uniqueness of The Solution

To discuss the uniqueness of solution of the problem (1.1)-(1.2), let us replace hypotheses $(A2),(A3)$ by

$(A2^*)$ $F:[0,\varpi]\times\mathsf{L}_2(Q)\to\mathsf{L}_2(Q)$ is continuous in $t\in[0,\varpi]$, $\forall\,\mathcal{U}\in\mathsf{L}_2(Q)$ and satisfies Lipschitz condition

$$\|F(t,\mathcal{U}(t))-F(t,\mathcal{V}(t))\|_2\le\beta_1\|\mathcal{U}(t)-\mathcal{V}(t)\|_2. \tag{3.1}$$

$(A3^*)$ $G:[0,\varpi]\times\mathsf{L}_2(Q)\to\mathsf{L}_2(Q)$ is continuous in $t\in[0,\varpi]$, $\forall\,\mathcal{U}\in\mathsf{L}_2(Q)$ and satisfies Lipschitz condition

$$\|G(t,\mathcal{U}(t))-G(t,\mathcal{V}(t))\|_2\le\beta_2\|\mathcal{U}(t)-\mathcal{V}(t)\|_2. \tag{3.2}$$

Theorem 3.1. *Let the hypotheses $(A1),(A2^*)-(A3^*)$ and $(A4)-(A5)$ be satisfied, then the solution $\mathcal{U}\in\mathbb{C}([0,\varpi],\mathsf{L}_2(Q))$ of the problem* (1.1) *and* (1.2) *is unique.*

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two solutions of the integral equation (2.1)

$$\begin{aligned}
\mathcal{U}_1(t)-\mathcal{U}_2(t) &= \int_t^{\varpi} G(\zeta,\mathcal{U}_1(\vartheta(\zeta)))d\mathsf{W}_2(\zeta)-\int_t^{\varpi} G(\zeta,\mathcal{U}_2(\vartheta(\zeta)))d\mathsf{W}_2(\zeta)\\
&+ Q(t,\mathcal{U}_1(t))\int_0^t F(\zeta,\mathcal{U}_1(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)-Q(t,\mathcal{U}_2(t))\int_0^t F(\zeta,\mathcal{U}_2(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\\
&= \int_t^{\varpi}[G(\zeta,\mathcal{U}_1(\vartheta(\zeta)))-G(\zeta,\mathcal{U}_2(\vartheta(\zeta)))]d\mathsf{W}_2(\zeta)\\
&+ Q(t,\mathcal{U}_1(t))\int_0^t[F(\zeta,\mathcal{U}_1(\vartheta(\zeta)))-F(\zeta,\mathcal{U}_2(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta)\\
&+ [Q(t,\mathcal{U}_1(t))-Q(t,\mathcal{U}_2(t))]\int_0^t F(\zeta,\mathcal{U}_2(\vartheta(\zeta)))d\mathsf{W}_1(\zeta),
\end{aligned}$$

then we can get

$$\begin{aligned}
\|\mathcal{U}_1(t)-\mathcal{U}_2(t)\|_2 &\le \left\|\int_t^{\varpi}[G(\zeta,\mathcal{U}_1(\vartheta(\zeta)))-G(\zeta,\mathcal{U}_2(\vartheta(\zeta)))]d\mathsf{W}_2(\zeta)\right\|_2\\
&+ |Q(t,\mathcal{U}_1(t))|\left\|\int_0^t[F(\zeta,\mathcal{U}_1(\vartheta(\zeta)))-F(\zeta,\mathcal{U}_2(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta)\right\|_2\\
&+ |Q(t,\mathcal{U}_1(t))-Q(t,\mathcal{U}_2(t))|\left\|\int_0^t F(\zeta,\mathcal{U}_2(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\right\|_2\\
&\le B\|\mathcal{U}_1-\mathcal{U}_2\|_{\mathbb{C}}\sqrt{\varpi}+(A+B\|\mathcal{U}_1\|_{\mathbb{C}})(B\|\mathcal{U}_1-\mathcal{U}_2\|_{\mathbb{C}})\sqrt{\varpi}\\
&+ B\|\mathcal{U}_1-\mathcal{U}_2\|_{\mathbb{C}}(A+B\|\mathcal{U}_2\|_{\mathbb{C}})\sqrt{\varpi}
\end{aligned}$$

$$\leq \quad B\sqrt{\varpi}(1+2A+2Br)\|\mathcal{U}_1-\mathcal{U}_2\|_C.$$

This gives that

$$[1-B\sqrt{\varpi}(1+2A+2Br)]\|\mathcal{U}_1-\mathcal{U}_2\|_C = \sqrt{\mu}\|\mathcal{U}_1-\mathcal{U}_2\|_C \leq 0.$$

Then

$$\|\mathcal{U}_1-\mathcal{U}_2\|_C = 0$$

and $\mathcal{U}_1 = \mathcal{U}_2$ which proves the uniqueness of the solution to the problem (1.1), (1.2). □

4. Continuous Dependence

Stability is one of the essential characteristics of the quality of the solution of any problem, and if we can prove that the solution of the problem (1.1)-(1.2) dependents continuously on some of the parameters of the problem, we guarantee stability of the solution on these parameters. Here, we set the definition of the continuous dependence of the solution. So, the following theorems prove that the solution of the problem (1.1)–(1.2) dependents continuously on three functions G, Q and F.

Definition 4.1. *The solution $\mathcal{U} \in \mathbb{C}([0,\varpi], L_2(Q))$ of the problem* (1.1)–(1.2) *depends continuously on the random function G if for all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that*

$$\|G(t,\mathcal{U}(t)) - G^*(t,\mathcal{U}(t))\|_2 \leq \delta_1 \ \text{ implies that } \ \| \mathcal{U} - \mathcal{U}^* \|_C \leq \varepsilon.$$

where $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = \int_t^{\varpi} G^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))dW_2(\zeta) + Q(t,\mathcal{U}^*(t))\int_0^t F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))dW_1(\zeta).$$

Theorem 4.1. *The unique solution of the problem* (1.1)- (1.2) *is continuous dependent on the random function G.*

Proof. Let $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = \int_t^{\varpi} G^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))dW_2(\zeta) + Q(t,\mathcal{U}^*(t))\int_0^t F(\zeta,\mathcal{V}^*(\vartheta(\zeta)))dW_1(\zeta),$$

such that $\|G(t,\mathcal{U}(t)) - G^*(t,\mathcal{U}(t))\|_2 \leq \delta_1$. Then we have

$$\begin{aligned}
\mathcal{U}(t)-\mathcal{U}^*(t) &= \int_t^{\varpi}[G(\zeta,\mathcal{U}(\vartheta(\zeta))) - G^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]dW_2(\zeta)\\
&+ Q(t,\mathcal{U}(t))\int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))dW_1(\zeta) - Q(t,\mathcal{U}^*(t))\int_0^t F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))dW_1(\zeta)\\
&= \int_t^{\varpi}[G(\zeta,\mathcal{U}(\vartheta(\zeta))) - G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]dW_2(\zeta)\\
&+ \int_t^{\varpi}[G(\zeta,\mathcal{U}^*(\vartheta(\zeta))) - G^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]dW_2(\zeta)\\
&+ [Q(t,\mathcal{U}(t)) - Q(t,\mathcal{U}^*(t))]\int_0^t F(\zeta,\mathcal{U}^*\vartheta(\zeta)))dW_1(\zeta)
\end{aligned}$$

$$+ \quad Q(t,\mathcal{U}^*(t)) \int_0^t [F(\zeta,\mathcal{U}(\vartheta(\zeta))) - F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))] d\mathsf{W}_1(\zeta),$$

then

$$\begin{aligned} \|\mathcal{U}(t) - \mathcal{U}^*(t)\|_2 &\leq (\delta_1 + B\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}}) \sqrt{\varpi} + B\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} (A + Br) \sqrt{\varpi} \\ &+ (A + Br)B\|\mathcal{V} - \mathcal{V}^*\|_{\mathbb{C}} \sqrt{\varpi}, \end{aligned}$$

and we can obtain that

$$[1 - B\sqrt{\varpi}(1 - 2A + 2Br)]\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} \leq \delta_1 \sqrt{\varpi},$$

which completes our result

$$\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} \leq \frac{\delta_1 \sqrt{\varpi}}{\sqrt{\mu}} = \varepsilon.$$

□

Definition 4.2. *The solution* $\mathcal{U} \in \mathbb{C}([0,\varpi], \mathsf{L}_2(\mathcal{Q}))$ *of the problem* (1.1)–(1.2) *depends continuously on the function* Q *if for all* $\varepsilon > 0$*, there exists* $\delta_2 > 0$ *such that*

$$|Q(t,\mathcal{U}(t)) - Q^*(t,\mathcal{U}(t))| \leq \delta_2 \text{ implies that } \| \mathcal{U} - \mathcal{U}^* \|_{\mathbb{C}} \leq \varepsilon.$$

where $\mathcal{U}^*$ *be the solution of*

$$\mathcal{U}^*(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q^*(t,\mathcal{U}^*(t)) \int_0^t F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_1(\zeta).$$

Theorem 4.2. *The unique solution of the problem* (1.1)- (1.2) *is continuous dependent on the random function* Q*.*

Proof. Let $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q^*(t,\mathcal{U}^*(t)) \int_0^t F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_1(\zeta),$$

such that $\|Q(t,\mathcal{U}(t)) - Q^*(t,\mathcal{U}(t))\|_2 \leq \delta_2$. That implies

$$\begin{aligned} \mathcal{U}(t) - \mathcal{U}^*(t) &= \int_t^{\varpi} [G(\zeta,\mathcal{U}(\vartheta(\zeta))) - G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))] d\mathsf{W}_2(\zeta) \\ &+ [Q(t,\mathcal{U}(t)) - Q(t,\mathcal{U}^*(t))] \int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\ &+ [Q(t,\mathcal{U}^*(t)) - Q^*(t,\mathcal{U}^*(t))] \int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\ &+ Q^*(t,\mathcal{U}^*(t)) \int_0^t [F(\zeta,\mathcal{U}(\vartheta(\zeta))) - F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))] d\mathsf{W}_1(\zeta), \end{aligned}$$

then

$$\begin{aligned} \|\mathcal{U}(t) - \mathcal{U}^*(t)\|_2 &\leq B\sqrt{\varpi}\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} + B\sqrt{\varpi}(A + Br)\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} \\ &+ \delta_2 \sqrt{\varpi}(A + Br) + B(A + Br)\sqrt{\varpi}\|\mathcal{U} - \mathcal{U}^*\|_{\mathbb{C}} \end{aligned}$$

arriving at

$$[1-B\sqrt{\varpi}(1-2A+2Br)]\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \le \delta_2\sqrt{\varpi}(A+Br),$$

resulting in

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \le \frac{\delta_2\sqrt{\varpi}(A+Br)}{\sqrt{\mu}} = \varepsilon,$$

thus completing the proof. □

Definition 4.3. *The solution $\mathcal{U}\in\mathbb{C}([0,\varpi],L_2(Q))$ of the problem* (1.1)–(1.2) *depends continuously on the function F if for all $\varepsilon>0$, there exists $\delta_3>0$ such that*

$$|F(t,\mathcal{U}(t))-F^*(t,\mathcal{U}(t))| \le \delta_3 \ \text{implies that} \ \|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \le \varepsilon.$$

where $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t,\mathcal{U}^*(t))\int_0^t F^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_1(\zeta).$$

Theorem 4.3. *The unique solution of the problem* (1.1)- (1.2) *is continuous dependent on the random function F.*

Proof. Let $\mathcal{U}^*$ be the solution of

$$\mathcal{U}^*(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) + Q(t,\mathcal{U}^*(t))\int_0^t F^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))d\mathsf{W}_1(\zeta),$$

such that $\|F(t,\mathcal{U}(t))-F^*(t,\mathcal{U}(t))\|_2 \le \delta_3$. Then we get

$$\begin{aligned}\mathcal{U}(t)-\mathcal{U}^*(t) &= \int_t^{\varpi}[G(\zeta,\mathcal{U}(\vartheta(\zeta)))-G(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]d\mathsf{W}_2(\zeta)\\ &+ Q(t,\mathcal{U}(t))\int_0^t[F(\zeta,\mathcal{U}(\vartheta(\zeta)))-F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta)\\ &+ Q(t,\mathcal{V}(t))\int_0^t[F(\zeta,\mathcal{U}^*(\vartheta(\zeta)))-F^*(\zeta,\mathcal{U}^*(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta)\\ &+ [Q(t,\mathcal{U}(t))-Q(t,\mathcal{U}^*(t))]\int_0^t F^*(\zeta,\mathcal{U}^*\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\end{aligned}$$

then

$$\begin{aligned}\|\mathcal{U}(t)-\mathcal{U}^*(t)\|_2 &\le B\sqrt{\varpi}\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} + B\sqrt{\varpi}(A+Br)\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}}\\ &+ \delta_3\sqrt{\varpi}(A+Br) + B(A+Br)\sqrt{\varpi}\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}}\end{aligned}$$

we get to

$$[1-B\sqrt{\varpi}(1-2A+2Br)]\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \le \delta_3\sqrt{\varpi}(A+Br),$$

resulting in

$$\|\mathcal{U}-\mathcal{U}^*\|_{\mathbb{C}} \le \frac{\delta_3\sqrt{\varpi}(A+Br)}{\sqrt{\mu}} = \varepsilon,$$

and the proof is done. □

5. Hyers-Ulam Stability

One of the most important ways to study the stability of the problems of differential equations is Hyers-Ulam stability (see [25]- [27]). Below we present our definition and theorem to discuss Hyers-Ulam stability in relation to our problem (1.1)-(1.2).

Definition 5.1. *Let the solution $\mathcal{U} \in \mathbb{C}([0,\varpi], L_2(Q))$ of the problem* (1.1)-(1.2) *exists, then the problem* (1.1)-(1.2) *is Hyers-Ulam stable if for every $\kappa > 0$, $\exists\, \delta(\kappa)$ such that for any $\delta-$approximate solution $\mathcal{U}_s$ of the problem* (1.1)-(1.2) *satisfies,*

$$\left\|\mathcal{U}_s - \int_t^{\varpi} G(\zeta, \mathcal{U}_s(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) - Q(t, \mathcal{U}_s(t)) \int_0^t F(\zeta, \mathcal{U}_s(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\right\|_2 \leq \delta \tag{5.1}$$

implies $\| \mathcal{U}_s - \mathcal{U} \|_{\mathbb{C}} < \kappa$.

Theorem 5.1. *Let the hypotheses of Theorem 2.1 be satisfied. Then the problem* (1.1)-(1.2) *is Hyers-Ulam stable.*

Proof. Let $\mathcal{U}_s$ and $\mathcal{U}$ be the approximate and exact solutions of the problem (1.1)-(1.2), then

$$\begin{aligned}
\mathcal{U}_s(t) - \mathcal{U}(t) &= \mathcal{U}_s(t) - \int_t^{\varpi} G(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) - Q(t, \mathcal{U}(t)) \int_0^t F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\
&= \mathcal{U}_s - \int_t^{\varpi} G(\zeta, \mathcal{U}_s(\vartheta(\zeta)))d\mathsf{W}_2(\zeta) - Q(t, \mathcal{U}_s(t)) \int_0^t F(\zeta, \mathcal{U}_s(\vartheta(\zeta)))d\mathsf{W}_1(\zeta) \\
&+ \int_t^{\varpi} [G(\zeta, \mathcal{U}_s(\vartheta(\zeta))) - G(\zeta, \mathcal{U}(\vartheta(\zeta)))]d\mathsf{W}_2(\zeta) \\
&+ Q(t, \mathcal{U}_s(t)) \int_0^t [F(\zeta, \mathcal{U}_s(\vartheta(\zeta))) - F(\zeta, \mathcal{U}(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta) \\
&+ [Q(t, \mathcal{U}_s(t)) - Q(t, \mathcal{U}(t))] \int_0^t F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta),
\end{aligned}$$

using (5.1), then we can get

$$\begin{aligned}
\|\mathcal{U}_s(t) - \mathcal{U}(t)\|_2 &\leq \delta + \|\int_t^{\varpi} [G(\zeta, \mathcal{U}_s(\vartheta(\zeta))) - G(\zeta, \mathcal{U}(\vartheta(\zeta)))]d\mathsf{W}_2(\zeta)\|_2 \\
&+ |Q(t, \mathcal{U}_s(t))|\, \|\int_0^t [F(\zeta, \mathcal{U}_s(\vartheta(\zeta))) - F(\zeta, \mathcal{U}(\vartheta(\zeta)))]d\mathsf{W}_1(\zeta)\|_2 \\
&+ |Q(t, \mathcal{U}_s(t)) - Q(t, \mathcal{U}(t))|\, \|\int_0^t F(\zeta, \mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta)\|_2 \\
&\leq \delta + B\|\mathcal{U}_s - \mathcal{U}\|_C \sqrt{\varpi} + (A + B\|\mathcal{U}_s\|_{\mathbb{C}})(B\|\mathcal{U}_s - \mathcal{U}\|_{\mathbb{C}}) \sqrt{\varpi} \\
&+ B\|\mathcal{U}_s - \mathcal{U}\|_{\mathbb{C}}(A + B\|\mathcal{U}\|_{\mathbb{C}}) \sqrt{\varpi} \\
&\leq \delta + B \sqrt{\varpi}(1 + 2A + 2Br)\|\mathcal{U}_s - \mathcal{U}\|_{\mathbb{C}}.
\end{aligned}$$

This implies that

$$[1 - B \sqrt{\varpi}(1 + 2A + 2Br)]\|\mathcal{U}_s - \mathcal{U}\|_{\mathbb{C}} = \sqrt{\mu}\|\mathcal{U}_s - \mathcal{U}\|_{\mathbb{C}} \leq \delta.$$

Then

$$\|\mathcal{U}_s - \mathcal{U}\|_C \le \frac{\delta}{\sqrt{\mu}} = \kappa$$

and the problem (1.1)-(1.2) is Hyers-Ulam stable. □

6. Examples

(I) Let $Q(t,\mathcal{U}(t)) = 1$. Then our results can be applied to the delay Itô−differential equation

$$d(\mathcal{U}(t) - \mathfrak{a}(t)) = F(t,\mathcal{U}(\vartheta(t)))d\mathsf{W}_1(t),\ \mathcal{U}(0) = \mathfrak{a}(0),\ t \in (0,\varpi] \tag{6.1}$$

with stochastic feedback control

$$\mathfrak{a}(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}(\zeta))d\mathsf{W}_2(\zeta)$$

with the solution

$$\mathcal{U}(t) = \int_t^{\varpi} G(\zeta,\mathcal{U}(\zeta))d\mathsf{W}_2(\zeta) + \int_0^t F(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_1(\zeta). \tag{6.2}$$

(II) Let $Q(t,\mathcal{U}(t)) = 1$, $\varpi = 1$, and $\vartheta = t^{\alpha}$, $\alpha \ge 1$. Then our results can be applied to the delay Itô−differential equation

$$d(\mathcal{U}(t) - \mathfrak{a}(t)) = F(t,\mathcal{U}(t^{\alpha}))d\mathsf{W}_1(t),\ \mathcal{U}(0) = \mathfrak{a}(0),\ t \in (0,1] \tag{6.3}$$

with stochastic feedback control

$$\mathfrak{a}(t) = \int_t^{1} G(\zeta,\mathcal{U}(\zeta^{\alpha}))d\mathsf{W}_2(\zeta)$$

with the solution

$$\mathcal{U}(t) = \int_t^{1} G(\zeta,\mathcal{U}(\zeta^{\alpha}))d\mathsf{W}_2(\zeta) + \int_0^t F(\zeta,\mathcal{U}(\zeta^{\alpha}))d\mathsf{W}_1(\zeta). \tag{6.4}$$

(III) Study the hybrid Itô−differential equation

$$d\Big(\frac{\mathcal{U}(t) - \mathfrak{a}(t)}{\frac{t^3+\|\mathcal{U}(t)\|_2}{20(1+\|\mathcal{U}(t)\|_2)}}\Big) = \frac{t\,\mathcal{U}(t^{\alpha})}{16(1+\|\mathcal{U}(t^{\alpha})\|_2)}d\mathsf{W}_1(t),\ \alpha \ge 1,\ t \in (0,\frac{1}{4}] \tag{6.5}$$

with stochastic feedback control

$$\mathfrak{a}(t) = \int_t^{\frac{1}{4}} \frac{e^{-\zeta}\mathcal{U}(\zeta^{\alpha})}{30+\zeta^2}d\mathsf{W}_2(\zeta), \tag{6.6}$$

where

$$\begin{aligned}
\|F(t,\mathcal{U}(\vartheta(t)))\|_2 &= \frac{|t|}{16}\Big\|\frac{\mathcal{U}(t^{\alpha})}{1+\|\mathcal{U}(t^{\alpha})\|_2}\Big\|_2 \le \frac{1}{64} + \frac{\|\mathcal{U}(t)\|_2}{64}, \\
\|G(t,\mathcal{U}(\vartheta(t)))\|_2 &= \Big|\frac{e^{-t}}{30+t^2}\Big|\,\|\mathcal{U}(t^{\alpha})\|_2 \le \frac{\|\mathcal{U}(t)\|_2}{30}, \\
|Q(t,\mathcal{U}(t))| &= \Big|\frac{t^3+\|\mathcal{U}(t)\|_2}{20(1+\|\mathcal{U}(t)\|_2)}\Big| \le \frac{1}{20}\Big[\frac{1}{64} + \|\mathcal{U}(t)\|_2\Big].
\end{aligned}$$

Easily, the problem (6.5)- (6.6) satisfies all the hypotheses $(A1)-(A5)$ of Theorem 2.1 with $A=\frac{1}{64}$, $B=\frac{1}{20}$, $\mu=0.9490625$, we see that the problem (6.5)- (6.6) on $[0,\frac{1}{4}]$ has at least one solution. In addition, the functions F, Q and G satisfy the Lipschitz conditions $(A2^{*}),(A3^{*})$ and $(A4)$ then the solution is unique.

7. Conclusions

In this study, we investigated the uniqueness and stability of solutions $\mathcal{U}\in\mathbb{C}([0,\varpi],\mathsf{L}_2(\mathcal{Q}))$ for hybrid delay It$\hat{a}-$differential equations

$$d\Big(\frac{\mathcal{U}(t)-\mathfrak{a}(t)}{Q(t,\mathcal{U}(t))}\Big)=F(t,\mathcal{U}(\vartheta(t)))d\mathsf{W}_1(t),\ \mathcal{U}(0)=\mathfrak{a}(0),\ t\in(0,\varpi]$$

subject to the stochastic feedback control

$$\mathfrak{a}(t)=\int_t^{\varpi}G(\zeta,\mathcal{U}(\vartheta(\zeta)))d\mathsf{W}_2(\zeta),\ t\in(0,\varpi].$$

By incorporating delay effects, we demonstrated sufficient conditions when the solution is unique and explored the continuous dependence of the solutions on initial data and specific functions such as G; Q and F. Additionally, the concept of Hyers-Ulam stability was analyzed to establish a robust theoretical foundation. Our results provide a comprehensive framework that bridges theoretical findings with practical applications in different fields, such as engineering, ecology, physics, biology and others. To illustrate the applicability of these results, we presented several examples and spatial cases that validate the existence and uniqueness of solutions, further enhancing the understanding of such problems. These examples not only highlight the relevance of our findings but also offer a clear pathway for extending this work to broader contexts. Future research may focus on refining these techniques or applying them to more complex problems with additional stochastic influences.

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