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On the Discrete Extension and Neighborhood Assignments

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Abstract. Let S be any proper subset of a topological space X. We can introduce a finer topology ρ^S on X by designating all singletons in $X \setminus S$ as open subsets. Each point in S retains the same open neighborhoods as in the original topology. We define a function $\jmath$ as a neighborhood assignment or operator if it maps elements from X to the topology of X, associates pairs of ordered disjoint closed subsets to the topology of X, or links pairs (x, U), where U is an open neighborhood of x, to the space X. The space X is termed monotonically normal if there exists an M- operator on X that satisfies specific criteria. Our findings reveal that if X possesses the property of being monotonically normal, then for any proper subset S of X, the discrete extension space X^S is also monotonically normal. Furthermore, we demonstrate that for a given topological space X and any finite subset $S \subset X$, the discrete extension X^S achieves monotonic normality if either $S \subset F_1$ or all elements of S lie outside F_1 for every ordered pair (F_1, F_2) of disjoint closed subsets. Our exploration also examines the interplay between this type of extension and the concept of D- spaces. Notably, we establish that if X is a topological space and S is a compact proper subset of X such that X^S is discretely complete, then X^S qualifies as a D- space provided that $X \setminus S$ is locally finite.

1. Introduction

Van Douwen and Pfeffer first introduced a type of neighborhood assignment in [1]. This concept pertains to a specific kind of space known as a $D-space$. For a given topological space (X, ρ), we define X to be a D- space if, for any neighborhood assignment $\jmath : X \to \rho$ with the property that $x \in \jmath(x)$, there exists a closed discrete subspace M of X such that $\jmath(M) = X$. It was established in [1] that all compact spaces are D- spaces. Additionally, all strong Σ- spaces and Moore spaces qualify as D- spaces; for more details, see [2] and [3].

In [4], van Mill proposed another type of space, called a dually $\mathcal{R}$ space, which also relies on the same neighborhood assignment. A space X is termed dually $\mathcal{R}$ if, for any neighborhood assignment

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$\jmath$, there exists a subspace $M \subset X$ with the property $\mathcal{R}$ such that $\jmath(M) = X$. A particular subclass of dually $\mathcal{R}$ spaces is the dually discrete spaces, which are discussed in detail in [5], [6], and [7].

In this paper, we demonstrate that if X is a topological space and $S \subset X$ is compact, and if X^S is also compact, then X must be compact, and thus X is a D- space. We provide a counterexample showing that X being compact does not necessarily imply that X^S is compact. Furthermore, we show that if X is any space and $S \subset X$ is compact such that X^S is discretely complete, then X^S is a D- space, provided that $X \setminus S$ is locally finite.

With regard to the notion of a dually discrete space, we explain that any dual operator on X gives rise to a dual operator on any discrete extension of X. The concept of monotonically normal spaces was first introduced in [9] and is based on a different type of neighborhood assignment. This new operator assigns an open subset to each pair of disjoint closed subsets while respecting certain conditions. For more details and recent work on monotonically normal spaces, see [10].

We explore this notion in the context of discrete extension spaces. In this paper, we show that if X is a monotonically normal topological space, then for any nonempty proper subset $S \subset X$, X^S is also monotonically normal. However, the converse is not always true, as illustrated by a counterexample. Another result we present is that if X is a space and $S = \{x_1, x_2, \ldots, x_n\} \subset X$, then X^S is monotonically normal if, for any pair of closed subsets (F_1, F_2), either $S \subset F_1$, or if $x \in S$, then $x \notin F_1$.

Following the conventions in [16], we define a space X to be normal if it satisfies the first and fourth separation axioms. Moreover, we note that all compact spaces are Hausdorff unless stated otherwise. A space X is considered discretely complete if all infinite discrete subspaces of X have complete accumulation points in X, as discussed in [17]. Throughout this paper, all spaces are assumed to be T_1 unless stated otherwise.

2. Discrete Extension and Assignments

Definition 2.1. Let (X, ρ) be a topological space, and let S be any nonempty proper subset of X. We define a new topology on X as follows:

$$\rho^{(S)} = \{V \cup D : V \in \rho \text{ and } D \subseteq X \setminus S\}.$$

The resulting topology $(X, \rho^{(S)})$, which is referred to as X^S, is known as the discrete extension of (X, ρ) [8].

Remark 2.1. *In general, for any space X and a proper subset S, the discrete extension topology ρ^S on X is strictly finer than ρ. However, in some cases, these two topologies can coincide. To illustrate this, consider the following example.*

Example 2.1. *Let $X = ([0, \omega_2] \times [0, \omega_1]) \setminus \{\langle \omega_2, \omega_1 \rangle\}$, where ω_1 is the first uncountable ordinal, and ω_2 is the successor ordinal of ω_1. The topology ρ is generated by considering all points in the rectangle $[0, \omega_2) \times [0, \omega_1)$ as open, along with the sets defined as $Q_\alpha(\beta) = \{\langle \beta, \gamma \rangle : \alpha < \gamma \leq \omega_1\}$ and $Q'_\alpha(\beta) = \{\langle \gamma, \beta \rangle : \alpha < \gamma \leq \omega_2\}$.*

Now, let us define $S = \bigcup_{\alpha,\beta}(Q_\alpha(\beta) \cup Q'_\alpha(\beta))$. It is evident that the topologies ρ and ρ^S coincide.

Remark 2.2. *In the discrete topology X^S, all singletons in $X \setminus S$ are open. This can be demonstrated by expressing $\{x\} \subset X \setminus S$ as $\{x\} = \{x\} \cup \emptyset$.*

Definition 2.2. Given a T_1 topological space X. We say that X is a monotonically normal space if there exists a function $\jmath$ such that for any pair (H, H') of disjoint closed subsets of X assigning an open subset $\jmath(H, H')$, where the following hold:

(1) $H \subset \jmath(H, H') \subset \overline{\jmath(H, H')} \subset X \setminus H'$;
(2) Let T, T' be closed subsets of X such that $T \cap T' = \emptyset$, where $H \subset T$ and $T' \subset H'$, then $\jmath(H, H') \subset \jmath(T, T')$ [15] .

Remark 2.3. *An equivalent definition of monotonically normal spaces was provided in [9]. Given a T_1 space (X, ρ) and $\chi^\rho = \{(x, V) : x \in V$ and $V \in \rho\}$, we say that X is monotonically normal if there exists a function $\jmath$ such that for any $(x, V) \in \chi^\rho$, it assigns an open neighborhood $\jmath(x, V) = V^\bullet$ with $x \in V^\bullet$ and $\overline{V^\bullet} \subset V$. The following conditions must hold for all $(x, V), (d, D) \in \chi^\rho$:*

(1) *$V^\bullet \subset V$;*
(2) *If $V^\bullet \cap D^\bullet \neq \emptyset$, then either $x \in D$ or $d \in V$.*

We will refer to the function $\jmath$ as an M- operator. In this paper, we will use both definitions of monotonically normal spaces interchangeably throughout.

Remark 2.4. *Let us note that if X is monotonically normal, then it is also normal. Consider any two disjoint closed subsets F_1 and F_2 of X. Since there exists an M- operator $\jmath$, we have $F_1 \subset \jmath(F_1, F_2) \subset \overline{\jmath(F_1, F_2)} \subset X \setminus F_2$. Therefore, $\jmath(F_1, F_2)$ and $X \setminus \overline{\jmath(F_1, F_2)}$ are open sets that contain F_1 and F_2, respectively. It is clear that $\jmath(F_1, F_2) \cap (X \setminus \overline{\jmath(F_1, F_2)}) = \emptyset$.*

Theorem 2.3. *Let X be a monotonically normal topological space. Then, for any non-empty proper subset $S \subset X$, X^S is also monotonically normal.*

Proof. Denote the topology on X by ρ. It is clear that $\rho^{(S)}$ is finer than ρ. We define $\chi^\rho = \{(x, V) : x \in V$ and $V \in \rho^{(S)}\}$.

Next, we introduce an operator $\jmath^S$ on $(X, \rho^{(S)})$ as follows: If $x \in X \setminus S$, then $\jmath^S(x, V) = \{x\}$. If $x \in S$, we can express an open neighborhood V of x as $V = C_1 \cup C_2$, where $C_1 \subseteq X \setminus S$ and $C_2 \subseteq S$ such that $C_2 \in \rho^{(S)}$. In this case, we have $\jmath^S(x, V) = \jmath(x, C_2) = C_2^\bullet$.

It is evident that $\jmath^S$ is an M- operator. Therefore, $(X, \rho^{(S)})$ is a monotonically normal space. □

Corollary 2.4. *If X is a monotonically normal topological space, then X^S is normal for any $S \subset X$.*

Remark 2.5. *It was proved in [10] that if a topological space X is monotonically normal, then for any neighborhood assignment ς for X, it is possible to find a discrete subspace K of X such that*

$$X = \left(\bigcup\{\varsigma(k) : k \in K\}\right) \cup \overline{K}$$

and the set $\overline{K} \setminus \bigcup\{\varsigma(k) : k \in K\}$ is left separated. From this, we can derive the following result.

Proposition 2.5. *Let X be a monotonically normal space. For any subset $S \subset X$ and any neighborhood assignment ς for X^S, there exists a discrete subspace K of X such that $X = (\bigcup\{\varsigma(k) : k \in K\}) \cup \overline{K}$ and $\overline{K} \setminus \bigcup\{\varsigma(k) : k \in K\}$ is left separated.*

Proof. This follows from the fact that the discrete extension X^S is monotonically normal. □

Lemma 2.6. *Let X be a topological space and S a proper subset of X. Then, any subset U of X that does not include S as a subset is open in X^S.*

Proof. We express U as a union of singletons, each of which is open. Therefore, U is open. □

Theorem 2.7. *Let X be a topological space and let $S = \{x_1, x_2, \dots, x_n\} \subset X$. Then, X^S is monotonically normal if for any pair of disjoint closed subsets (F_1, F_2), it is true that either $S \subset F_1$ or if $x \in S$, then $x \notin F_1$.*

Proof. Consider any two disjoint closed subsets of X, denoted as F_1 and F_2. It is important to note that any set that is disjoint from S is considered open, according to the definition of the discrete extension. We define an operator as follows:

$$\jmath^S(F_1, F_2) = \begin{cases} F_1 & \text{if } S \not\subset F_1 \\ (F_1 \setminus S) \cup (X \setminus F_2) & \text{if } S \subset F_1 \end{cases}$$

Next, we examine three cases:

(1) If $S \not\subset F_1$ and $S \not\subset F_2$, then both F_1 and F_2 are open. We can deduce the following:
   (a) $F_1 \subset \jmath^S(F_1, F_2) = F_1 \subset \overline{F_1} = F_1 \subset X \setminus F_2$.
   (b) Let (N_1, N_2) be a pair of disjoint closed subsets of X:
      (i) If $F_1 \subset N_1$, $N_2 \subset F_2$, and N_1 does not contain S as a subset, then it follows that $\jmath^S(F_1, F_2) \subset \jmath^S(N_1, N_2)$.
      (ii) Given the assumption $S \not\subset N_2$ since $N_2 \subset F_2$, we must consider the case where $S \subset N_1$. In this situation, we have

$$N_1 \subset \jmath^S(N_1, N_2) = (N_1 \setminus S) \cup (X \setminus N_2) \subset \overline{(N_1 \setminus S) \cup (X \setminus N_2)} \subset \overline{(X \setminus N_2)} = (X \setminus N_2) \subset X \setminus N_2.$$

It is clear that $\jmath^S(F_1, F_2) \subset \jmath^S(N_1, N_2)$. This clarifies the relationships and properties of the defined operator $\jmath^S$ with respect to the sets involved.

(2) Let $S \subset F_1$. Then:
   (a) $F_1 \subset \jmath^S(F_1, F_2) = (F_1 \setminus S) \cup (X \setminus F_2) \subset \overline{(F_1 \setminus S) \cup (X \setminus F_2)} \subset \overline{(X \setminus F_2)} = (X \setminus F_2) \subset X \setminus F_2$;
   (b) Let (N_1, N_2) be a pair of disjoint closed subsets of X such that $F_1 \subset N_1$ and $N_2 \subset F_1$, then $S \subset N_1$. Hence, we take $\jmath^S(N_1, N_2) = (N_1 \setminus S) \cup (X \setminus N_2) \subset \overline{(N_1 \setminus S) \cup (X \setminus N_2)} \subset \overline{(X \setminus N_2)} = (X \setminus N_2) \subset X \setminus N_2$. Therefore, $\jmath^S(F_1, F_2) \subset \jmath^S(N_1, N_2)$.

(3) If $S \subset F_2$. Then:
   (a) $F_1 \subset \jmath^S(F_1, F_2) = F_1 \subset \overline{F_1} = F_1 \subset X \setminus F_2$;
   (b) Let (N_1, N_2) be a pair of disjoint closed subsets of X such that $F_1 \subset N_1$ and $N_2 \subset F_2$:

(i) If $S \subset N_1$, then $\jmath^S(N_1, N_2) = (N_1 \setminus S) \cup (X \setminus N_2) \subset \overline{(N_1 \setminus S) \cup (X \setminus N_2)} \subset \overline{(X \setminus N_2)} = (X \setminus N_2) \subset X \setminus N_2$. Therefore, $\jmath^S(F_1, F_2) \subset \jmath^S(N_1, N_2)$;

(ii) If $S \not\subset N_1$, then $N_1 \subset \jmath^S(N_1, N_2) = N_1 \subset \overline{N_1} = N_1 \subset X \setminus N_2$. Hence, $\jmath^S(F_1, F_2) \subset \jmath^S(N_1, N_2)$.

□

Corollary 2.8. *Let X be a topological space and let $S = \{x_1\}$ for some $x_1 \in X$. Then, X^S is monotonically normal.*

Proof. By applying Theorem 2.7, we can take $S = \{x_1\}$. It is evident that for any two closed subsets F_1 and F_2 such that $F_1 \cap F_2 = \emptyset$, there are only two possible scenarios. The first scenario is that $S \subset F_1$, while the second scenario is that $x_1 \notin F_1$. Therefore, we conclude that X^S is monotonically normal. □

Example 2.2. *In this example, we demonstrate that if X is a topological space and S is a proper subset of X such that X^S is monotonically normal, then X does not necessarily have to be monotonically normal. In [11], it was established that there exists a countable regular space X that is not monotonically normal. Subsequently, in [12], X was identified as an $\aleph_0$-space, as described in [14]. It was shown that this space is also not monotonically normal. Now consider a point $S = \{x\} \subset X$. The discrete extension X^S is a monotonically normal space; however, X itself is not monotonically normal.*

Remark 2.6. *It was shown in [1] that all compact spaces are D- spaces.*

Theorem 2.9. *Let X be a topological space and let $S \subset X$ be a compact subset. If X^S is compact, then X is a D- space.*

Proof. We need to demonstrate that the space X is compact, which will establish that it is a D-space. Since X^S is compact and according to the definition of a discrete extension, we know that $X \setminus S$ must be finite. Let $X \setminus S = \{s_1, s_2, \ldots, s_n\}$. Given that the set S is compact, we can conclude that the union $(X \setminus S) \cup S = X$ is also compact. Therefore, we have shown that X is a D- space. □

Theorem 2.10. *Let X be a topological space, and let $S \subset X$ be compact such that X^S is discretely complete. Then, X^S is a D- space if $X \setminus S$ is locally finite.*

Proof. To demonstrate that X^S is compact, we first observe that $X \setminus S = D$, which is a discrete subspace of X^S. Let's assume that D is an infinite subset. By the properties of discrete spaces, D must have a complete accumulation point, denoted as x. This point cannot belong to D because, for any $x \in D$, the singleton set $\{x\}$ serves as an open neighborhood of x that intersects D only finitely.

Thus, we conclude that x must be in S. However, any open neighborhood U of x will intersect D. This leads to a contradiction since it suggests that x is also included in the closure of D, denoted as $\overline{D}$, while we know that $\overline{D} = D$ due to the local finiteness of D.

Therefore, we deduce that D must be finite. Consequently, since $D \cup S = X$ is compact, it follows that X^S is compact as well. Hence, we can conclude that X^S is a D- space. □

Example 2.3. *It is not always true that if X is compact, then X^S is compact for some proper subset $S \subset X$. To illustrate this, consider $\mathbb{R}$ with the usual topology, and let $X = I = [0,1] \subset \mathbb{R}$. Note that I is a compact subset. Now take $S = [0, 1/2] \subset X$. It is clear that X^S is not compact.*

Remark 2.7. *Let φ be a neighborhood assignment for the space X. We will call φ a dual operator on X if it satisfies the condition of dually discrete spaces. Any neighborhood assignment for a dually discrete space is a dual operator.*

Proposition 2.11. *Let φ be a dual operator on a space X such that $\varphi(Y) = X$, where Y is a discrete subspace of X. Then, for any subset $S \subset X$, the operator $\varphi' = \varphi|_Y \cup id_{X \setminus Y}$ serves as a dual operator for X^S.*

Proof. Since Y is a discrete subspace of X, it remains a discrete subspace of X^S. It is evident that φ' defines a function from X to the topology on X^S. Therefore, $\varphi'(Y) = \varphi(Y) \cup \emptyset = X$. □

3. Conclusion

The main objective of this paper is to examine the relationship between two concepts in topology: neighborhood assignments and the discrete extension of a topological space. Topologists have previously investigated various topological properties; some of these properties, such as countable tightness, have been shown to be invariant under discrete extensions. This leads us to an important question: do properties related to neighborhood assignments still hold when we perform such an extension?

We found that if X is monotonically normal, then its discrete extension will also be monotonically normal. However, we discovered a negative result for the converse: a discrete extension of a space X being monotonically normal does not necessarily imply that X itself, with its original topology, is monotonically normal.

Another question we explored is under what conditions the discrete extension of X will be monotonically normal. We demonstrate that by adding specific conditions on S, the extension X^S can be forced to be monotonically normal. Additionally, we examined the relationship between D- spaces and discrete extensions, leading to some noteworthy results.

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