Int. J. Anal. Appl. (2025), 23:136

International Journal of Analysis and Applications

Exploring the Role of [d, e]-Lindelöf Spaces: Theoretical Insights and Practical Implications

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Abstract. Several recent notions have expanded the field of topological generalized structures. Notably, among these generalizations, [d,e]-compactness spaces have emerged as particularly significant. The concept of [d,e]-Lindelöfness topology, serving as corresponding generalizations of [d,e]-compactness topology is introduced. The emphasis in this research is on exploring separation axioms and limit points in [d,e]-Lindelöfness spaces through the use of [d,e]-open covers, aiming to make contributions in this area. Description is provided for these concepts, and their behavior is examined in relation to the perfect functions and infinite products. The definitions that are introduced align with their counterparts in topological spaces. The thesis delves into the sufficient conditions and in general, elucidates their fundamental characteristics.

1. Introduction

Topological spaces offer a mathematical framework for examining the properties of sets and their interconnections within a given topology. Numerous researchers have shown interest in studying general topology, such as Engelking [12]; Fletcher et al. [13]. Various concepts such as Lindelöfness, [d,e]-compactness, semi-compactness, local compactness, local Lindelöfness, weak compactness, pseudo-compactness, countable compactness, weak Lindelöfness, and their variations play crucial roles in comprehending the topological structure of spaces. This research delves into the intricate

Received: Mar. 12, 2025.

2020 *Mathematics Subject Classification.* 18F60, 54D20, 54C10.

Key words and phrases. topological spaces; [d,e]-compact; perfect function; Lindelöf spaces; open covers.

https://doi.org/10.28924/2291-8639-23-2025-136

ISSN: 2291-8639

connections and deeper implications of these concepts, shedding light on the complex nature of topological spaces. Lindelöf spaces stand out as fundamental and intriguing within the field of topological spaces, showcasing unique attributes in topology studies. They offer valuable insights into the structure of open covers. Introduced by Alexandroff and Urysohn [1], the concept of Lindelöfness states that a space is Lindelöf if every open cover of it has a countable sub cover. The concept of compactness is a cornerstone in mathematics, driving mathematicians to investigate numerous generalizations. Among these, semi-compactness has gained attention. The study of [d,e]-compactness dates back to the groundbreaking studies by Alexandrov and Urysohn [2]. Since then, many mathematicians have advanced both the theory and its extensions. Even with the considerable research dedicated to [d,e]-compactness, it continues to be a dynamic and progressing field of study. This hypothesis holds pivotal significance in the current thesis, which aims to explore the interconnections between cardinal numbers and topology. It encompasses fascinating extensions and alterations, such as pseudo-compactness and real compactness.

The study of generalized topological properties has evolved significantly since Alexandroff and Urysohn's foundational work on compactness and Lindelöf spaces in 1929. Frolík's weakly Lindelöf spaces [14] and Barr et al.'s characterization of productively Lindelöf spaces via Alster spaces [11] established the foundation for modern cardinality-based frameworks. The concept of $[d,e]$-compactness, formalized by Smirnov (1950) using open cover cardinality criteria, provides a unifying lens for classical properties: compactness corresponds to $[\aleph_0, \infty]$-compactness; countable compactness to $[\aleph_0, \aleph_1]$-compactness and Lindelöf function to $[\aleph_1, \infty]$-compactness.

Mishchenko [16] demonstrated that $[d,e]$-compactness and its variation $[d,e]$-compactness* are not equivalent for regular cardinals in spaces like R^*. Arhangel'skii's [4] cardinality bounds for G_δ-point Lindelöf spaces and Tall's [19] exploration of indestructible Lindelöfness under force are two recent advancements. Mohammad [17] expands these ideas through μ-β-Lindelöf sets, while Hdeib et al. [15, 18] have expanded the theory of $[d,e]$-perfect functions and pairwise Lindelöf mappings. Alster [3] demonstrate that the interplay between separation axioms, product spaces, and cardinal invariants (e.g., ω^ω structure) is still fundamental to this discourse. This study advances the theory of $[d,e]$-Lindelöfness by addressing unresolved concerns related to separation axiom maintenance, infinite product behavior, and functional characterizations in generalized topological spaces.

Cover structures have become increasingly valuable as a tool in topological analysis in recent years. Hence, it is crucial to embark on additional research focusing on covers and their characterization by the Lindelöf concept. To propel the investigation of [d,e]-compact spaces and their extension forward, the researcher introduces an innovative concept called [d,e]-Lindelöf spaces, utilizing a newly developed specific type of cover known as [d,e]-cover. This cover is characterized by [d,e]-sets, denoted as [d,e], and the researcher extends its application to include a novel space known as [d,e]-Lindelöf space. The researcher discusses various properties and theories about these concepts.

This paper aims to Introduce the notion of [d,e]-Lindelöf space using a specific type of cover, establish equivalence conditions between [d,e]-compactness and [d,e]-Lindelöfness alongside other topological spaces, and characterize the productivity of [d,e]-Lindelöf spaces, Lindelöf space, and other spaces under certain additional conditions.

This research Presents the main findings related to compact and Lindelöf spaces, discussing fundamental operations and various properties. A new definition of [d,e]-Lindelöf spaces is introduced, along with an exploration of the characteristics of products involving [d,e]-Lindelöf spaces and potential areas for future research. Furthermore, this study establishes the foundation for connecting the concepts and conclusions provided here with other related ideas and studies such as [5–10].

2. An Alternative Perspective On [d, e]-Lindelöfness In Topological Spaces

In this section, a comprehensive understanding of the fundamental concepts related to [d,e]-Lindelöf spaces is developed. The key notions of topological spaces, which are central to the study of topology, are introduced. Various examples, properties, and the significance of [d,e]-sets in describing the framework of these spaces are explored. Additionally, this chapter delves into basic concepts such as Lindelöfness, compactness, and [d,e]-compactness. These concepts are crucial for defining and analyzing different topological spaces, providing valuable tools for understanding their geometric properties and structural aspects.

This chapter also examines the connections between theoretical concepts and their relationships with other topological spaces. To enhance understanding, this chapter includes clear explanations, notable examples, and innovative theorems. By the end of this chapter, reader is expected to have a strong grasp of fundamental ideas in topology, setting a solid foundation for further exploration of more advanced topics in the field.

We introduce new definitions related to [d,e]–Lindelöf spaces. These definitions are explored and their properties and implications are examined within the broader context of topology.

Definition 2.1. *A space W is called [d,e]-Lindelöf for countable infinite cardinal numbers d, e and c if, for any open cover $\mathcal{U}$ of W with $|\mathcal{U}| \leq e$, there exists a subcover with cardinality greater than d. If W is [d,e]-Lindelöf for all $d \geq c$, then it is referred to as $[c, \infty]$-Lindelöf.*

Definition 2.2. *A space W is termed d-open if for every point $w \in W$, there exists an open set H containing w such that the cardinality of $H \setminus W$ is less than d. A space W is d-closed if the complement $H \setminus W$ is d-open.*

Definition 2.3. *A family $\mathcal{E} = \{E_\alpha : \alpha \in \Delta\}$ is said to have the d-intersection property if the intersection of any d members of $\mathcal{E}$ results in a non-empty set.*

Definition 2.4. *A subset Z of a space W is known as an E_d-subset if it can be represented as the union of closed sets, with the total number of these sets being less than d.*

Definition 2.5. *A function $\pi : (W, \sigma) \to (Z, \tau)$ is referred to as a strongly (or weakly) [d,e]-function if for every open cover $\widetilde{\mathcal{H}} = \{H_\alpha : \alpha \in \Delta\}$ of W with $|\Delta| < e$, there exists an open cover $\widetilde{\mathcal{G}} = \{G_\alpha : \alpha \in \gamma\}$ of Z*

with $|\gamma| < e$, such that $\pi^{-1}(\widetilde{\mathcal{G}}) \subseteq \bigcup\{H_\alpha : \alpha \in \Delta_1, \Delta_1 \subset \Delta, \Delta_1 < e, \forall H \in \widetilde{\mathcal{H}}\}$. If π is a strongly (or weakly) [d,e]-function for all $d > e$, then it is called an $[e, \infty]$-function.

Definition 2.6. *A function $\pi : (W, \sigma) \to (Z, \tau)$ is termed d-closed if it maps closed sets in W to e-closed sets in Z.*

Definition 2.7. *A function $\pi : (W, \sigma) \to (Z, \tau)$ is called a [d,e]-Lindelöf perfect function if it is d-closed and, for each z in Z, the preimage $\pi^{-1}(z)$ is [d,e]-Lindelöf. Functions that are $(\chi_\circ, \infty)$-perfect are exactly the perfect functions.*

Definition 2.8. *A function $\pi : (W, \sigma) \to (Z, \tau)$ is referred to as a [d,e]-Lindelöf-perfect function if it is closed and the preimage $\pi^{-1}(z)$ is [d,e]-Lindelöf for every $z \in Z$. Functions that are (χ_1, ∞)-perfect correspond exactly to d-perfect functions.*

Definition 2.9. *A subset G of a topological space (W, α) is called $C[d, e]$-closed if for every [d,e]-Lindelöf F_n-subset C of (W, α), the intersection $G \cap C$ is closed in C.*

Definition 2.10. *A Hausdorff space (W, α) is referred to as a $C[d, e]$-space if every $C[d, e]$-closed subset of (W, α) is d-closed in C. A $C(\chi_1, \infty)$-space is exactly the same as a C-space.*

Definition 2.11. *A topological space (W, α) is termed $K[d, e]$ if every [d,e]-Lindelöf subset of (W, α) is d-closed.*

Definition 2.12. *A function $\pi : (W, \alpha) \to (Z, \tau)$ is called a [d,e]-Lindelöf function if for every [d,e]-Lindelöf F_n-subset I of (Z, τ), the preimage $\pi^{-1}(I)$ is also [d,e]-Lindelöf.*

Definition 2.13. *A function $\pi : (W, \alpha) \to (Z, \beta)$ is called a finite covering if, for every point $y \in (Z, \beta)$, there exists an open neighborhood V of y in (Z, β) such that $\pi^{-1}(V)$ is a finite disjoint union with each part being homeomorphic to V under π.*

Definition 2.14. *A subset K of a topological space (W, α) is called sequentially open if every sequence in (W, α) that converges to a point in K eventually lies within K.*

Definition 2.15. *A topological space (Z, β) is called a sequential space if every sequentially open subset of (Z, β) is also an open set.*

2.1. **Theories, Ideas and Examples about [d,e]- Lindelöf Spaces.** In this section, various theories, ideas, and examples related to [d,e]–Lindelöf spaces are explored. This section delves into key concepts, presents important theorems, and provides illustrative examples to enhance understanding of these spaces and their properties.

Theorem 2.1. *If $\pi : (W, \alpha) \to (Z, \beta)$ is a perfect function and (W, α) is [d,e]-Lindelöf, then (Z, β) also has the [d,e]-Lindelöf property.*

Proof. Let $\widetilde{\mathcal{H}} = \{H_\alpha : \alpha \in \Gamma\}$ be an open cover of (Z, β) where each $H_\alpha \in \beta$ and $|\Gamma| < e$. Since π is a perfect function, then the preimage $\pi^{-1}\{H_\alpha : \alpha \in \Gamma\}$ forms an open cover of W with cardinality $< e$. However, since W is a [d,e]-Lindelöf space, then there exists a subset $\Gamma_1 \subset \Gamma$ such that $\pi^{-1}\{H_\alpha : \alpha \in \Gamma_1\}$ forms a subcover of W with $|\Gamma_1| < d$. Thus, $\widetilde{\mathcal{H}} = \{H_\alpha : \alpha \in \Gamma\}$ forms a subcover of (Z, β) with cardinality less than d. Consequently, (Z, β) is [d,e]-Lindelöf. □

Theorem 2.2. *States that if d and e are countably infinite cardinal numbers such that $\sum_{c<d} e^c = e$, where c represents countably infinite cardinal numbers less than d, and if $\pi : (W, \alpha) \to (Z, \beta)$ is a [d,e]-perfect function, then π is also a strong [d,e]-function.*

Proof. Let $\widetilde{\mathcal{H}} = \{H_\alpha : \alpha \in \Gamma\}$ be an open cover of W, with $|\Gamma| < e$. Since π is [d,e]-perfect function, then for all $z \in Z$, $\pi^{-1}(z)$ is [d,e]-Lindelöf, such that

$$\pi^{-1}(z) \subseteq \bigcup \{H_\alpha : \alpha \in \Gamma_z, \Gamma_z \subset \Gamma, |\Gamma_z| < d\}$$

Let $O_z = Z \setminus \pi(W \setminus \bigcup_{\alpha \in \Gamma_z} H_\alpha)$. Since π is d-closed, O_z is d-open, for all $z \in Z$ there exists an open cover $\widetilde{\mathcal{L}} = \{L_\phi : \phi \in \Delta\}$ of Z, such that $|L \setminus O_z| < d$. Now, $\widetilde{\mathcal{L}} = [L \setminus O_z] \cup [L \cap O_z]$. Therefore, $\pi^{-1}(L) \subseteq \bigcup \{H_\alpha : \alpha \in \Gamma_1, \Gamma_1 \subset \Gamma, |\Gamma_1| < d\}$.

Let $L_{\Gamma_1} = \bigcup \{L_\circ, \pi^{-1}(L_\circ) \subseteq \bigcup \{H_\alpha : \alpha \in \Gamma_1, \Gamma_1 \subset \Gamma, |\Gamma_1| < d\}$. Since $\sum_{c<d} e^c = e$, where c is a countable infinite cardinal $c < d$. Let $\widetilde{\mathcal{L}} = \{L_{\Gamma_1} : \Gamma_1 \subset \Gamma, |\Gamma_1| < d\}$. Then $\widetilde{\mathcal{L}}$ is an open cover of Z with $|\widetilde{\mathcal{L}}| < e$. And $\pi^{-1}(L_{\Gamma_1})$ contains the union of members of $\widetilde{\mathcal{L}}$ with cardinality $< d$. □

Example 2.1. *Let N be the set of natural numbers. Let α_u, α_s denote the usual and Sorgenfrey topology on N respectively. Suppose $\pi : (N, \alpha_s) \to (N, \alpha_u)$, then π is (χ_1, ∞)-function. However, π is not χ_1-closed because the image of the interval $[0, 1]$ is closed in (N, α_s) is not χ_1-closed in (N, α_u).*

Theorem 2.3. *If $\pi : (W, \alpha) \to (Z, \beta)$ is a strong (d,e)-onto function, then if (Z, β) is [d,e]-Lindelöf, (W, α) must also be [d,e]-Lindelöf.*

Proof. Suppose $\widetilde{\mathcal{H}} = \{H_\alpha : \alpha \in \Gamma\}$ be an open cover of (W, α), $|\alpha| \le e$. Since π is strong-[d,e]-function, then there exists an open cover $\check{\mathcal{L}} = \{L_\phi : \phi \in \Delta\}$ of Z with $|\Delta| \le e$ such that

$$\pi^{-1}(L) \subseteq \bigcup \{H_\alpha : \alpha \in \Gamma_1, \Gamma_1 \subset \Gamma, |\Gamma_1| < d\}.$$

But Z is [d,e]-Lindelöf, so there exists $\phi_1 \subset \Delta$ with $|\phi_1| < d$ such that

$$Z = \bigcup_{B \in \phi_1} Z_B.$$

Hence, $W = \bigcup_{B \in \phi_1} \pi^{-1}(L_\phi)$. So $\pi^{-1}(L_\phi)$ contains less than d members of $\widetilde{\mathcal{H}}$. Thus W is [d,e]-Lindelöf. □

Remark 2.1. *If $\pi : (W, \alpha) \to (Z, \beta)$ is a weak [d,e]-onto function, then (W, α) will be [d,e]-Lindelöf provided that (Z, β) is [d,e]-Lindelöf.*

Corollary 2.1. *Let $\pi : (W, \alpha) \to (Z, \beta)$ be a strong [d,e]-Lindelöf function. Then π is [d,e]-Lindelöf function.*

Example 2.2. *Consider N as the set of all natural numbers. Let α_{dis} represents the discrete topology on N and α_{cof} represent the cofinite topology on N. The function $\pi : (W, \alpha_{dis}) \to (Z, \alpha_{cof})$ is (χ_1, χ_1)-Lindelöf, but it is not a strong (χ_1, χ_1)-function.*

Theorem 2.4. *Let π be a function from (W, α) to (Z, β) and let (Z, β) be $C(d, e)$-closed, $K(d, e)$-space. Then the following are equivalent:*

(1) *F is a [d,e]-Lindelöf function.*
(2) *F is a [d,e]-perfect function.*
(3) *F is a strong [d,e]-function.*

Proof. (1) → (2) Suppose that π is [d,e]-Lindelöf function. It is enough to show that π is d-closed. Let $C \subset (W, \alpha)$ be closed and C be any [d,e]-Lindelöf, F_n subset of (Z, β), therefore $F \cap \pi^{-1}(C)$ is [d,e]-Lindelöf. Hence $\pi(F) \cap C$ is [d,e]–Lindelöf. Since (Z, β) is $C(d, e)$-closed, $K(d, e)$-space, $\pi(F)$ is e-closed.

(2) → (3) by Theorem 6.2.2

(3) → (1) by Corollary 6.2.1 □

Corollary 2.2. *Let $\pi : (W, \alpha) \to (Z, \beta)$ is strong [d,e]-Lindelöf, then π is [d,e]-Lindelöf function.*

3. Several Theoretical Concepts on [d, e]-Lindelöfness Spaces.

In this section, the primary relationship among these roles is demonstrated, and an additional conclusion on the topological properties of the [d,e]-Lindelöfness space is provided.

Definition 3.1. *A function $g : W \to Z$ is called a Lindelöf perfect function if it is continuous, closed, and for each z in Z, the preimage $g(z)^{-1}$ is Lindelöf.*

Definition 3.2. *A function is termed a Lindelöf function if, for every Lindelöf subset k of z, the inverse image $f(k)^{-1}$ is Lindelöf.*

Definition 3.3. *A space W is called weakly-m-$w_\circ$ Lindelöf if every open cover $\mathcal{V}$, with $|\mathcal{V}| \leq d$, there is a countable sub family $\mathcal{H}$ of $\mathcal{V}$ such that $W \in cl \cup (\{h : h \in \mathcal{H}\})$.*

Definition 3.4. *A collection $\mathcal{S}$ of subsets of a space N is referred to as a point m (or locally m) family if for each point in N (or a suitable neighborhood of each point in N), there exists an intersection with more than m members of $\mathcal{S}$.*

Definition 3.5. *A space S is deemed m-expandable (or almost m-expandable) if, for every locally-$w_\circ$ collection $\{g_q : q \in Q\}$ where $|Q| \leq m$ for subsets of S, there exists an open locally-$w_\circ$ (or point-$w_\circ$) collection $\{G_q : q \in Q\}$ for S such that, for each q in Q, g_q is a subset of G_q.*

Definition 3.6. *A continuously [d,e]-Lindelöf function is referred to as [d,e]-perfect and $[d, \infty)$-with-perfect function.*

Definition 3.7. *A function $g : W \to Z$ is called $[\omega_1,m]$-pseudo function if for every open cover $\mathcal{U}$ of W, $|\mathcal{U}| \le m$, then there is an open cover $\mathcal{V}$ of Z, with $|\mathcal{U}| \le m$ such that for each u in $\cup f^{-1}(u) \subset cl(\bigcup_{i=1}^{k} U_i)$.*

Theorem 3.1. *Given countable infinite cardinals d and e, where $\sum_{c<d} e^c = e$, the function $g : W \to Z$ is a strong [d,e]-function if and only if it is a weak [d,e]-function.*

Proof. If g is a strong [d,e]-function, it is automatically a weak [d,e]-function. Conversely, suppose g is a weak [d,e]-function, and $\widetilde{\mathcal{V}} = \{V_q : q \in Q\}$ be an open cover of W with $|Q| < e$. Suppose that there exists an open cover $\mathcal{V} = \{V_q : q \in Q\}$ with $|Q| < e$. Then there is an open cover $\mathcal{H} = \{h_i : i \in I\}$ of Z such that $g^{-1}(h)$ is contained in the union $\bigcup_{i=1}^{k}\{H_q : q \in Q_H\}$, where $|Q_H| < d$, for each h in $\mathcal{H}$. Now, consider each subset F of Q with a cardinality greater than d. Define H_F as the union of $\{H_i : g^{-1}(H)\}$ and $\{V_q : q \in F\}$. Since $\sum_{c<d} e^c = e$, it follows that the set $W = \{V_F : F \in Q, |F| \le d\}$ has the cardinality $|W|$, where $g^{-1}(H_F)$ is the union of more than d members of $\mathcal{V}$. Consequently, g is identified as a strong [d,e]-function. □

Corollary 3.1. *Any function on [d,e]-Lindelöf space onto arbitrary space is strong [d,e]-function.*

Corollary 3.2. *Consider a closed function $g : W \to Z$, such that $g^{-1}(z)$ is [d,e]-Lindelöf for each z in Z. Then, g is characterized as a weak [d,e]-function*

Corollary 3.3. *A weak [d,e]-function in strong form may not necessarily be a closed function.*

Corollary 3.4. *If $g : W \to W$ is a strong [d,e]-function, then W is [d,e]-Lindelöf if and only if W is [d,e]-Lindelöf under the function g.*

Consider countable infinite cardinals d and e, satisfying $\sum_{c<d} e^c = e$. Then the following Theorem are valid:

Theorem 3.2. *In a [d,e]-Lindelöf space, any locally-d family has fewer than d elements.*

Proof. Consider a locally-d family $\mathcal{D}$. If $|\mathcal{D}| < d$, then there is nothing to prove. Suppose $|\mathcal{D}| \ge d$. Now, let A have a subfamily $\mathcal{B}$ of cardinality d, $\mathcal{B} = \{D_i : i \in q\}$ such that $|Q| = d$. Define $j_d(Q) = \{Q_1 \subset Q : |Q_1| < d\}$. It is easy to observe that $|j_d(q)| \le \sum_{c<d} d^c \le \sum_{c<d} e^c = e$. As $\mathcal{B}$ is a locally-d family, there exists an open cover $\mathcal{M}$ of W, where each member of m covers more than d family members of $\mathcal{B}$ for each q in $j_d(q)$. Define $R(Q_1) = \{m \in \mathcal{M} : m \in \cup\{D_i : i \in Q_1\} \cup \{W \setminus D_i : i \in q\}\}$. It is evident that $\mathcal{R} = \{R(Q_1) : Q_1 \in j_d(Q)\}$ forms an open cover of W, and $|\mathcal{R}| \le e$. It is asserted that no subcover of $\mathcal{R}$ has cardinality greater than d. Let $\mathcal{V}$ be a subcover of $\mathcal{R}$ with cardinality greater than d. Due to the construction of members in $\mathcal{R}$, it is clear that $\mathcal{V}$ covers more than d members of $\mathcal{B}$, contradicting the fact that $\mathcal{V}$ is a cover of $\mathcal{B}$ with cardinality greater than d. This, in turn, contradicts the fact that $\mathcal{V}$ is a cover of W, implying that W is not an [d,e]-Lindelöf space. Hence, the result follows. □

Theorem 3.3. *Any weak [d,e]-function maps every locally-d family with a cardinality of at most e to a locally-d family.*

Proof. Consider A locally-d-family of subsets of space W, denoted as $\mathcal{D} = \{D_i : i \in I\}$, where $|I| \leq d$. Let g be a weak [d,e]-function from W onto the space Z. Define

$$j_d(I) = \{Q \subset i : |Q| < d\},$$

then $|j_d(I)| \leq \sum_{c<d} e^c = e$. Since A is a locally-d family, there exists an open cover of W with a cardinality of at most d. However, as g is a weak [d,e]-function, it implies that there exists an open cover $\mathcal{V}$ of Z such that the inverse image of each member of $\mathcal{V}$ contains more than d members of $\mathcal{R}$. Now, it is easy to observe that each member of $\mathcal{V}$ intersects $< d$ members of $g(\mathcal{D})$. Hence the result follows with the same method of proof, and the researcher obtain the following corollaries: □

Corollary 3.5. *Suppose $g : W \to Z$ is a strong (weak) [d, e]-perfect function, and let k be a closed [d,e]-compact ($[d, \infty)$-Lindelöf) subset of Z. Then, $g(k)$ is [d,e]-Lindelöf.*

Corollary 3.6. *Consider $g : W \to Z$ as a weak [d,e]-perfect function, where z is T_1. Then, $g^{-1}(z)$ is [d,e]-Lindelöf for every z in Z.*

Corollary 3.7. *Suppose $g : W \to Z$ is a $[\omega_1, \infty)$-function, where z is a T_2-space. Then, g is a Lindelöf function.*

4. Characteristics of Products Employing [d,e]-Lindelöf Spaces.

In this section, the characteristics of products involving [d,e]-Lindelöf spaces shall be examined. One could think of the next theorem as a pillar of the [d,e]-Lindelöfness theory.

Theorem 4.1. *Consider a T_1-space Z such that each point has a neighborhood base of cardinality $\leq e$, where e is a countable infinite cardinal. Then, W is [ω_1,d]-Lindelöf if and only if the projection map $j_z : W \times W \to Z$ is a weak [ω_1,d]-function.*

Proof. Assume W is [ω_1,d]-Lindelöf. Consider $\mathcal{V}$ as an open cover of $W \times Z$ with a cardinality of at most z. Let $\{H(i) : i \in T_z\}$, where $|T_z| < d$, be a neighborhood base at any point in Z. Consider $z \in Z$ to be fixed. For each ω in W, there exists a neighborhood $K_i(X, \mathcal{V})$ such that $K_i(\omega, \mathcal{V}) \times H_i(z)$ is contained in $\mathcal{V}$, for some i in T_z and some v in $\mathcal{V}$. But

$$K_i(\mathcal{V}) = \cup\{K_i(\omega, \mathcal{V}) : K_i(X, \mathcal{V}) \times H_i(z) \subset \mathcal{V}\}$$

for all v in $\mathcal{V}$ then $\{\, K_i(\mathcal{V}) \mid H \in \mathcal{V}, i \in I_z\}$ is open cover of W of cardinality $< d$. Hence, it possesses a countable subcover denoted as $K_{i_1}(V_1)$. Consequently

$$j_z^{-1}\left(\bigcap_{l=1}^{\infty} H_{i_l}(z)\right) \subset \bigcup_{l=1}^{\infty} K_{i_l}(V_l) \times H_{tj}(z) \subset \bigcup_{l=1}^{\infty} V_l.$$

Now, it is straightforward to demonstrate that j_z is a weak [ω_1,e]-function using the same method of proof. □

Proposition 4.1. *Consider a space W, where each point has a neighborhood base of cardinality < e, and e is a countable infinite cardinal. In such a case, the projection map $j_z : W \times W \to Z$ is a weak [ω_1,e]-pseudo Lindelöf perfect function.*

Lemma 4.1. *Suppose d and e are countable infinite cardinals such that $\sum_{c<d} e^c = e$, and let W and Z be [d,e]-Lindelöf spaces. Then, the projection map $j_z : W \times W \to Z$ is a strong [d,e]-Lindelöf perfect function if and only if for each z in Z, there exists a neighborhood U_z of Z such that $W \times cl_z(U_z)$ is [d,e]-Lindelöf.*

Proof. Assuming j_z is a strong [d,e]-Lindelöf perfect function, it follows, by Corollary 4.4.7 that $W \times Z$ is [d,e]-Lindelöf. Hence, the result follows. In broad terms, consider the scenario where for each z in Z, there exists a neighborhood U_z such that $W \times \mathrm{cl}_z(U_z)$ is [d,e]-Lindelöf. Let $\mathcal{U}$ be an open cover of $W \times Z$ with a cardinality of at most e. Then, based on the assumption, for each z in Z, $W \times \mathrm{cl}_z(U_z)$ is included in the union of fewer than d members of $\mathcal{U}$. Consequently, j_z is a weak [d,e] function, and according to Theorem 4.5.3, it is also a strong [d,e]-function. □

Using a similar method of proof, the following lemma is obtained:

Lemma 4.2. *Consider W as an [d,e]-Lindelöf space and Z as an $[d, \infty]$-Lindelöf space. The projection map $j_z : W \times W \to Z$ is a weak [d,e]-function if and only if, for each z in Z, there exists a neighborhood U_z of z such that $W \times cl_Z(U_z)$ is [d,e]-Lindelöf.*

Using lemmas 6.4.1 and 6.4.2 the researcher can proof the following theorems:

Theorem 4.2. *Consider countable infinite cardinals d and e, such that $\sum_{c<d} e^c = e$. Let W and Z be [d,e]-Lindelöf spaces. Then, $W \times Z$ is a [d,e]-Lindelöf space if, for each z in Z, there exists a neighborhood U_z such that $W \times cl_z(U_z)$ is a [d,e]-Lindelöf space.*

Theorem 4.3. *Assume W is a [d,e]-Lindelöf space, and let Z be an $[d, \infty]$-Lindelöf space. The space $W \times Z$ is [d,e]-Lindelöf if and only if, for each z in Z, there exists a neighborhood U_z such that $W \times cl_Z(U_z)$ is [d,e]-Lindelöf.*

Theorem 4.4. *If $g : W \to Z$ is a weak [ω_1,e]-Lindelöf perfect function, then W is pseudo-Lindelöf if g is open, and Z is pseudo-Lindelöf, provided that both W and Z are completely regular.*

Proof. Let $\mathcal{U}$ be an open cover of W with $|W| \leq \omega_1$, then there is an open cover $\mathcal{V}$ of Z with $|\mathcal{V}| \leq \omega_1$, such that for v in $\mathcal{V}$, $g^{-1}(V) \subset \bigcup_{i=1}^{\infty} U_i$. Since Z is completely regular and pseudo-Lindelöf $\mathcal{V}$ has a countable subfamily v_1, v_2 such that $W = \bigcup_{j=1}^{\infty} \mathrm{cl}(V_i)$ so Z is pseudo-Lindelöf. By the pseudo theorem we can proof the following Corollaries. □

Corollary 4.1. *W is Lindelöf if and only if Z is [d,e]-Lindelöf.*

Corollary 4.2. *$W \times Z$ is pseudo-Lindelöf if and only if Z is pseudo-Lindelöf, given that both W and Z are completely regular.*

Corollary 4.3. *$W \times Z$ is [ω_1,e]-Lindelöf if and only if Z is [ω_1,e]-Lindelöf.*

Corollary 4.4. *$W \times Z$ is [d,e]-Lindelöf, if W is [d,e]-Lindelöf and Z is $[d, \infty]$-Lindelöf.*

Corollary 4.5. *$W \times Z$ is [d,e]-Lindelöf if and only if W, Z are [d,e]-Lindelöf where e, d countable $\sum_{c<d} e^c = e$.*

5. Conclusion

The research as broadened the understanding of topological generalized structures by introducing and emphasizing the significance of [d,e]-compactness spaces. Among the various generalizations explored, [d,e]-Lindelöfness topology has been highlighted as a critical extension of [d,e]-compactness topology. This study has focused on investigating separation axioms and limit points within [d,e]-Lindelöfness spaces by utilizing [d,e]-open covers, aiming to contribute valuable insights in this domain. The research has provided detailed descriptions of these concepts and examined their behavior in relation to perfect functions and infinite products. The introduced definitions have been shown to align with traditional concepts in topological spaces, and the thesis has thoroughly explored the sufficient conditions and fundamental characteristics of these newly defined structures.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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