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Regular Fuzzy Graphs with Chromatic Numbers

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Abstract. Let $G = (\sigma, \mu)$ be a fuzzy graph on $G^* = (V, E)$. If each vertex in G has the same degree k, then G is said to be a vertex regular fuzzy graph or a k-vertex regular fuzzy graph. The minimum number of colors required to color all the vertices in such a way that no two adjacent vertices receive the same color is called the chromatic number and is denoted by $\chi(G)$. In this paper, we present results based on cubic graphs and their chromatic number with regular fuzzy graphs, which are briefly denoted by $fr(G)$.

1. Introduction

The graph $G = (V, E)$ is a finite, undirected, connected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to Chartrand and Lesniak [2].

Chellali et al. [3] first studied the concept of $[1, 2]$-sets. Yang and Wu [11] extended the study of this parameter. In [6], Renuka et al. introduced the concept of $[1, 2]$-complementary connected domination number of graphs and studied its character. In [5], Renuka et al. studied the concept of twelve cubic graphs with respect to the parameter $[1, 2]$-complementary connected domination in graphs and exhibited the results. In [4], Kolandasamy discussed the concept of $[1, 2]$-complementary connected domination number for total graphs.

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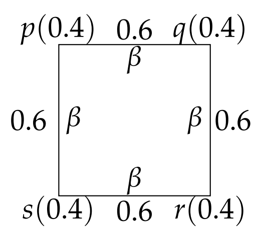
A fuzzy graph is the generalization of the crisp graph. Therefore, it is natural that many properties are similar to crisp graphs, and they deviate in many places. A fuzzy graph $G = (\sigma, \mu)$ is a nonempty set V together with a pair of functions $\mu : V \to [0, 1]$ and $\sigma : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\sigma(x, y) = \mu(x) \wedge \mu(y)$. We call μ the fuzzy vertex set of G and μ the fuzzy edge set of G, respectively. In a crisp graph, a bijection $f : V \times E \to N$ that assigns a unique natural number to each vertex and/or edge of $G = (V, E)$ is called a labeling. Additionally, this paper assumes a membership value of 0.1 for both the vertex and edge sets throughout the results to facilitate generalization in the fuzzy graph $G = (\sigma, \mu)$. In [7], Nagoorgani and Latha studied the regular fuzzy graph. In [8], Oghlan et al. studied the chromatic number of a regular fuzzy graph. In [1], Amine and Naanaa studied generating random graphs based on fuzzy proximity relationships between vertices. In [9], Radha and Kumaravel studied edge-regular fuzzy graphs. In [10], Rosenfield published an article based on fuzzy graphs. In [12], Zimmermann studied set theory and its applications.

Motivated by the above concepts, we exhibited the results in this paper based on regular fuzzy graphs with chromatic numbers.

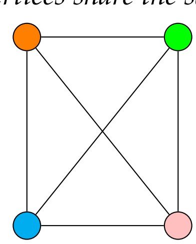
2. Main Result

In this section, we explore the chromatic properties of k-regular fuzzy graphs, particularly focusing on cubic fuzzy graphs. We introduce a key combinatorial identity relating the fuzzy regularity and chromatic number, and classify all graphs that satisfy this relationship. Our findings provide structural insights into how regularity constraints influence coloring in fuzzy graph settings.

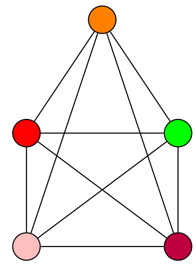
Definition 2.1. *If each vertex has the same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph. This is analogous to the definition of regular graphs in crisp graph theory.*



Definition 2.2. *The chromatic number of a graph is the smallest number of colors needed to color the vertices of a graph so that no two adjacent vertices share the same color.*



Chromatic Number $= 4$



Chromatic Number $= 5$

Throughout this section, the term fr-set refers to a specific configuration of vertices used in verifying whether the relation:

$$fr(G) + \chi(G) = 2n - 6$$

holds for a given fuzzy graph G with n vertices. The composition of an fr-set reflects the fuzzy regularity and chromatic properties being analyzed in each case.

Theorem 2.1. *For any connected graph G, $fr(G) + \chi(G) = 2n - 6$ for any $n > 3$ if and only if $G \cong K_3(5P_2)$, $C_4(P_2, P_3, 0, 0)$, P_7, $P_5(P_3)$, $C_4(P_4)$, $C_3(P_2, P_4, 0)$, $C_3(u(P_4, P_3))$, $C_3(P_4, P_3, 0)$, $K_3(u(2P_3, P_2))$, $K_3(u(2P_3))$, $K_3(u(P_3, P_2))$, $K_3(P_3, P_3, P_2)$, $K_3(u(2P_2, P_4))$, $K_3(2P_2, P_4, 0)$, $K_3(u(P_2, P_4), P_2, 0)$,*
$K_3(P_4, P_2, P_2)$, $K_4 - \{e\}(P_4)$, $K_4(4P_2)$, $K_4(3P_2, P_2, 0, 0)$, $K_4(2P_2, 2P_2, 0, 0)$, $K_4(2P_2, P_2, P_2, 0)$, $K_4P_2, P_2, P_2, P_2)$, $K_4(P_5)$, $K_3(u(P_4, P_2))$, $K_3(P_4, P_2, 0)$, $K_4(u(C_3, P_3))$, $K_4(C_3, P_3, 0, 0)$, $K_4(u(2P_2, P_3))$, $K_4(u(C_3, 2P_2))$,
$K_4(u(P_2, P_3), P_2, 0, 0)$, $K_4(u(K_3, P_2), P_2, 0, 0)$, $K_4(P_3, 2P_2, 0, 0)$, $K_4(K_3, 2P_2, 0, 0)$, $K_4(P_3, P_2, P_2, 0)$, $K_5(3P_2)$, $K_5(2P_2, P_2, 0, 0, 0)$, $K_5(P_2, P_2, P_2, 0, 0)$, $K_5(P_4)$, $K_5(u(P_3, P_2))$, $K_5(P_3, P_2, 0, 0, 0)$, $K_6(P_3)$, $K_6(2P_2)$, $K_6(P_2, P_2)$, $K_7(P_2)$, K_8.

Proof. Assume that $fr(G) + \chi(G) = 2n - 6$. This is possible only if $fr(G) = n$ and $\chi(G) = n - 6$ or $fr(G) = n - 1$ and $\chi(G) = n - 5$ or $fr(G) = n - 2$ and $\chi(G) = n - 4$ or $fr(G) = n - 3$ and $\chi(G) = n - 3$ or $fr(G) = n - 4$ and $\chi(G) = n - 2$ or $fr(G) = n - 5$ and $\chi(G) = n - 1$ or $fr(G) = n - 6$ and $\chi(G) = n$.

Case 1: $fr(G) = n$ and $\chi(G) = n - 6$. Since $fr(G) = n$, G is a star. But for a star $\chi(G) = 2$; hence, $n = 8$. Then $G \cong K_{1,7}$. If $deg(v_i) > 1$, where v_i is a pendant vertex of the star, then no graph exists.

Case 2: $fr(G) = n - 1$ and $\chi(G) = n - 5$. G contains a clique K on $n - 5$ vertices or does not contains a clique on K_{n-5} vertices. Let G contains a clique on K_{n-5} vertices. Let $S = v_1, v_2, v_3, v_4, v_5$ be the vertices other than the clique K. Then $< S >= K_5, K_5, K_5 - \{e\}, C_3(C_3), K_4K_1, K_3(P_3), C_5, C_3(P_2, P_2, 0), C_4(P_2), P_5, K_3 \cup K_2, K_{1,4}, C_4 \cup K_1, K_3(P_2) \cup K_1, P_3 \cup K_2, K_{1,3} \cup K_1, K_2 \cup K_2 \cup K_1, P_3 \cup K_2, C_3(2P_2), K_4(P_2), K_4 - \{e\} \cup K_1$.

Subcase (i): Let $< S >= K_5$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of K_5. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of $\{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, let v_i be adjacent to u_i. Then $\{u_j, u_k, v_4, v_5\}$ for some $j \neq i$ is fr-set. Hence $n = 5$, which is a contradiction; hence, no graph exists.

Subcase (ii): Let $< S >= K_5$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of K_5. Since G is connected, let all the vertices of K_5 be adjacent to vertex u_i of K_{n-5}. Then $\{u_j, u_i, v_1, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 8$ and $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let all the $K - 5$ be adjacent to u_1. Then $G \cong K_3(5P_2)$. If $d(u_j) = 3$ for $j \neq I$, $d(v_i) > 1$ for $i = 1$ to 4, then $fr = 6$ so that $n = 7$, which is a contradiction that $\chi(G) = 2$, hence no graph exists. For the remaining cases, it was observed that no graph exists.

Subcase (iii): Let $< S >= K_5 - \{e\}$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_5 - \{e\}$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of the vertices not on the cycle of $d(v_i) = 4v_1, v_2, v_3, v_4, v_5$. Without loss of generality, let v_1 be adjacent to u_i. Then $\{u_j, u_i, v_3, v_4\}$ is fr-set. Hence $n = 5$, which is a contradiction.

If u_i is adjacent to any one of the vertices not on the cycle of $d(v_i) = 3$ of $K_5 - \{e\}$, say v_5. Then $\{u_j, u_k, v_3, v_1\}$ is fr-set. Hence $n = 5$, which is a contradiction.

If u_i is adjacent to any one of the vertices on the cycle of $d(v_i) = 3$ of $K_5 - \{e\}$, say v_1. Then $\{u_j, u_k, v_4, v_5\}$ is fr-set. Hence $n = 5$, which is a contradiction.

If u_i is adjacent to any one of the vertices on the cycle of $d(v_i) = 4$ of $K_5 - \{e\}$, say v_5. Then $\{u_j, u_k, v_1, v_2\}$ is fr-set. Hence $n = 5$, which is a contradiction.

Subcase (iv): Let $< S >= C_3(C_3, 0, 0)$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $C_3(C_3, 0, 0)$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex v_i of $C_3(C_3, 0, 0)$, say v_1. Then $\{u_j, u_k, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 7$. Therefore $K = K_2$, which is a contradiction to $\chi(G) = 2$; hence, no graph exists.

Subcase (v): Let $< S >= K_4 \cup K_1$. Let v_1, v_2, v_3, v_4 be the vertices of K_4 and v_5 be an isolated vertex. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex $\{v_1, v_2, v_3, v_4\}$ and v_5. Then $\{u_j, u_k, v_2, v_3, v_5\}$ is fr-set. Hence $n = 6$. Therefore $K = K_1$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to any one of vertex $\{v_1, v_2, v_3, v_4\}$ and u_i is adjacent to v_5. Then $\{u_j, u_k, v_2, v_3, v_5\}$ is fr-set. Hence $n = 6$. Therefore $K = K_1$. This is a contradiction; hence, no graph exists.

Subcase (vi): Let $< S >= K_3(P_3)$. Let v_1, v_2, v_3 be the vertices of K_3 and v_4, v_5 be the vertices of P_3 that is adjacent to v_3. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex of K_3. Without loss of generality, let v_1 be adjacent to u_i, then $\{u_j, u_K, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 7$. Therefore $K = K_2$, which is a contradiction. Hence, no graph exists.

Subcase (vii): Let $< S >= C_5$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of C_5. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex of C_5. Without loss of generality, let v_1 be adjacent to u_i, then $\{u_j, u_k, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 7$. Therefore $K = K_2$, which is a contradiction to $\chi(G) = 2$. Hence, no graph exists.

Subcase (viii) Let $< S >= C_3(P_2, P_2, 0)$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $C_3(P_2, P_2, 0)$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex of $C_3(P_2, P_2, 0)$. Without loss of generality, let v_1 be adjacent to u_i, then $\{u_j, u_k, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 7$. Therefore $K = K_2$, which is a contradiction to $\chi(G) = 2$. Hence, no graph exists.

Subcase (ix): Let $< S >= C_4(P_2)$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $C_4(P_2)$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex of $C_3(P_2, P_2, 0)$. Without loss of generality, let v_1 be adjacent to u_i, then $\{u_j, u_k, v_2, v_3, v_4, v_5\}$ is fr-set. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong C_4(P_2, P_3, 0, 0)$. If $d(u_1) > 2$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction that no graph exists.

Subcase (x): Let $< S >= P_5$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of P_5. Since G is connected, the following are the possible cases (i) there exists u_i of K_{n-5} that is adjacent to any one of pendant vertex (ii) there exists u_i of K_{n-5} that is adjacent to other than pendant vertices. Without loss of generality, let v_1 (which is a pendant vertex) be adjacent to u_i, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong P_7$. If $d(u_1) > 2$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If v_2, which is not a pendant vertex, be adjacent to u_i, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong P_5(P_3)$. If $d(u_1) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xi) Let $< S >= K_3 \cup K_2$. Let v_1, v_2, v_3 be the vertices of K_3 and v_4, v_5 be the vertices of K_2. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of vertex of $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$. Without loss of generality, let v_1 be adjacent to u_i and v_4 be adjacent to u_j, then $fr(G) = 6$.

Hence $n = 7$. Therefore $K = K_2$, which is a contradiction to $\chi(G) = 2$. Hence, no graph exists.

If there exists u_i of K_{n-5} that is adjacent to any one of vertex of $\{v_1, v_2, v_3\}$ and there exists another vertex u_j of K_{n-5} that is adjacent to any one of vertex of $\{v_4, v_5\}$. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. This is a contradiction, as $\chi(G) = 2$. Hence, no graph exists.

Subcase (xii) Let $< S >= K_{1,4}$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_{1,4}$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to v_1, which is not a pendant vertex. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong G_4$. If $d(u_1) > 3$ or $d(u_3)$ and $d(u_i) > 2$, where $i = 2, 3$ which is a contradiction to $\chi(G) = 2$. Hence, no graph exists.

Subcase (xiii): Let $< S >= C_4 \cup K_1$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $C_4 \cup K_1$. Since G is connected, there exists a vertex u_i of K_{n-5} that is adjacent to any one of vertex of $\{v_1, v_2, v_3, v_4\}$ and $\{v_5\}$. Without loss of generality, let v_1 and v_5 be adjacent to u_i, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong G_5$. If $d(u_1) > 3$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} be adjacent to any one of vertex of $\{v_1, v_2, v_3, v_4\}$ and there exists another vertex u_j of K_{n-5} be adjacent to $\{v_5\}$, where i, j. Without loss of generality, let v_1 be adjacent to u_i and v_5 be adjacent to u_j, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong C_4(P_4)$. If $d(u_1) > 2$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xiv): Let $< S >= K_3(P_2) \cup K_1$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_3(P_2) \cup K_1$. Since G is connected, there exists a vertex u_i of K_{n-5} that is adjacent to any one vertex of $\{v_1, v_2, v_3\}$ and $\{v_5\}$. Without loss of generality, let v_1 and v_5 be adjacent to u_i, then $fr = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_3$. If $d(u_1) > 3$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} be adjacent to any one of vertex of $\{v_1, v_2, v_3, v_4\}$ and there exists another vertex u_j of K_{n-5} be adjacent to $\{v_5\}$, where i, j. Without loss of generality, let v_1 be adjacent to u_i and v_5 be adjacent to u_j, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong C - 3(P_2, P_4, 0)$. If $d(u_1) > 2$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xv): Let $< S >= P_3 \cup K_2$. Let v_1, v_2, v_3 be the vertices of P_3 and v_4, v_5 be the vertices of K_2. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$. Without loss of generality, let v_1 and v_4 be adjacent to u_i, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong C_3(u(P_4, P3))$. If $d(u_1) > 4$ or $d(u_2)$ and $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$.

This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to any one of vertex of $\{v_1, v_2, v_3\}$ and there exists another vertex u_i of K_{n-5} that is adjacent to $\{v_4, v_5\}$, where i, j. Without loss of generality, let v_1 be adjacent to u_i and v_4 be adjacent to u_j, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong C_3(P_4, P3, 0)$. If $d(u_1) > 3$ or $d(u_2) > 3$ or $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xvi): Let $< S >= K_{1,3} \cup K_1$. Let v_1, v_2, v_3 be the vertices of $K_{1,3}$ and v_4, v_5 be the vertices of K_1. Since G is connected, there exists u_i of K_{n-5} that is adjacent to $\{v_1, v_2, v_3, v_4\}$ and $\{v_5\}$. Without loss of generality, let v_1, which is not a pendant vertex, and v_4 is adjacent to u_i, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 4$ or $d(u_2)$ and $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and there exists another vertex u_j of K_{n-5} that is adjacent to v_5, where i, j. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 3$ or $d(u_3) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xvii): Let $< S >= K_3 \cup K_2$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_3 \cup K_2$. Since G is connected, there exists u_i of K_{n-5} that is adjacent to any one of $\{v_1, v_2, v_3\}$ and $\{v_4, v_5\}$. Without loss of generality, let v_1 and v_4 be adjacent to u_i, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 4$ or $d(u_2)$ and $d(u_3) > 2$, then $fr(G) + \chi(G) \cong 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to any one of vertex of $\{v_1, v_2, v_3\}$ and there exists another vertex u_j of K_{n-5} that is adjacent to $\{v_4, v_5\}$, where i, j. Without loss of generality, let v_1 be adjacent to u_i and v_4 be adjacent to u_j, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 3$ or $d(u_3) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xviii): Let $< S >= K_2 \cup K_2 \cup K_1$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_2 \cup K_2 \cup K_1$. Since G is connected, there exist U_i of K_{n-5} that is adjacent to any one of $\{v_1, v_2\}$, $\{v_3, v_4\}$ and $\{v_5\}$. Without loss of generality, let v_1, v_3 and v_5 be adjacent to u_i, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(u(2P_3, P_2))$. If $d(u_1) > 5$ or $d(u_2)$ and $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and v_3. There exists another vertex u_j of K_{n-5} that is adjacent to v_5, where i, j. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(u(2P_3), P_2, 0)$. If $d(u_1) > 4$ or $d(u_3) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex U_i of K_{n-5} that is adjacent to v_1 and v_5. There exists another vertex u_j of K_{n-5} that is adjacent to v_3, where i, j, then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(u(P_3, P_2), P3, 0)$. If $d(u_1) > 4$ or $d(u_2) > 2$ or $d(u_3) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xix): Let $< S >= P_3 \cup K_2$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $P_3 \cup K_2$. Since G is connected, there exists a vertex u_i of K_{n-5} that is adjacent to any one of $\{v_3, v_5\}$, which is the pendant vertex and v_2 and v_1. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(u(2P_2, P_4))$. If $d(u_1) > 5$ or $d(u_2)$ and $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_4 which is not a pendant vertex v_1 and v_2. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_3$. If $d(u_1) > 4$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and v_2. There exists another vertex u_j of K_{n-5} that is adjacent to v_4 which is not a pendant vertex, where i, j. Then $fr(G) = 5$. Hence $n = 6$. Therefore $K = K_1$. This is a contradiction that no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and v_2. There exists another vertex u_j of K_{n-5} that is adjacent to v_3 which is a pendant vertex, where i, j. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(2P_2, P_4, 0)$. If $d(u_1) > 4$ or $d(u_2) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction that no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and v_3 which is a pendant vertex. There exists another vertex u_j of K_{n-5} that is adjacent to v_2, where i, j. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(u(P_2, P_4), P_2, 0)$. If $d(u_1) > 4$ or $d(u_3) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and v_4 which is not a pendant vertex. There exists another vertex u_j of K_{n-5} that is adjacent to v_2, where i, j, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_3$. If $d(u_1) > 3$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_3, which is a pendant vertex. There exists another vertices u_j and u_k of K_{n-5} that is adjacent to v_1 and v_2, where i, j, k. Then $fr(G) = 7$. Hence $n = 8$. Therefore $K = K_3$. Then $G \cong K_3(P_4, P_2, P_2)$. If $d(u_1), d(u_2), d(u_3) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_4, which is not a pendant vertex. There exists another vertices u_j and u_k of K_{n-5} that is adjacent to v_1 and v_2, where i, j, k. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. This is a contradiction; hence, no graph exists.

Subcase (xx): Let $< S >= C_3(2P_2)$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $C_3(2P_2)$. Since G is connected, there exists a vertex u_i of K_{n-5} adjacent to v_1. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. This is a contradiction; hence, no graph exists.

There exists a vertex u_i of K_{n-5} adjacent to v_3. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. This is a contradiction that no graph exists.

Subcase (xxi): Let $< S >= K_3(P_2)$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_3(P_2)$. Since G is connected, there exists a vertex u_i of K_{n-5} adjacent to v_1. Then $fr(G) = 5$. Hence $n = 6$. Therefore $K = K_1$. This is a contradiction that no graph exists.

There exists a vertex u_i of K_{n-5} adjacent to v_3. Then $fr(G) = 5$. Hence $n = 6$. Therefore $K = K_1$. This is a contradiction that no graph exists.

Subcase (xxii): Let $< S >= K_4 - \{e\} \cup K_1$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $K_4 - \{e\} \cup K_1$. Since G is connected, there exists a vertex u_i of K_{n-5} adjacent to v_1 and v_5. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_3$. This is a contradiction that no graph exists.

If there exists a vertex u_i of K_{n-5} adjacent to v_4 and v_5. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong G_{12}$. If $d(u_1) > 3$ or $d(u_2) > 1$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_1 and there exists another vertex u_j of K_{n-5} that is adjacent to v_5, where i, j, then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_4 - \{e\}(P_4)$. If $d(u_1) > 2$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists a vertex u_i of K_{n-5} that is adjacent to v_4 there exists another vertex u_j of K_{n-5} that is adjacent to v_5, where i, j. Then $fr(G) = 6$. Hence $n = 7$. Therefore $K = K_2$. Then $G \cong K_4 - e(P_4)$. If $d(u_1) > 2$ or $d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Case (iii): $fr(G) = n - 2$ and $\chi(G) = n - 4$. Since $\chi(G) = n - 4$, G contains a clique K on $n - 4$ vertices or does not contains a clique on K_{n-4} vertices. Let G contains a clique on K_{n-4} vertices. Let G contain a clique K on $n - 4$ vertices. Let $S = \{v_1, v_2, v_3, v_4\}$ be the vertices other than the clique. Then the induced subgraph S is isomorphic to one of the following graphs $K_4, K_4, P_4, K_3 \cup K_1, K_{1,3}, P_3 \cup K_1, K_2 \cup K_2, K_2 \cup K_2, K_4 - \{e\}, C_3(1, 0, 0)$.

Subcase (i): Let $< S >= K_4$. Let v_1, v_2, v_3, v_4 be the vertices of K_4. Since G is connected, there exists a vertex u_i of K_{n-4} that is adjacent to v_1. Then $fr(G) = 4$. Hence $n = 6$. Therefore $K = K_2$. This is a contradiction; hence, no graph exists.

Subcase (ii): Let $< S >= K_4$. Let v_1, v_2, v_3, v_4 be the vertices of K_4. Since G is connected, let all the vertices of K_4 be adjacent to vertex u_i of K_{n-4}. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(4P_2)$. If $d(u1) > 7$, $d(u_2), d(u_3), d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Let $\{v_1, v_2, v_3\}$ vertices of K_4 be adjacent to vertex u_i of K_{n-4} and v_4 vertex of K_4 is adjacent to u_j, where i, j. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(3P_2, P_2, 0, 0)$. If $d(u_1) > 6$, $d(u_2) > 4$, $d(u_3), d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction that no graph exists.

Let $\{v_1, v_2\}$ vertices of K_4 be adjacent to vertex u_i of K_{n-4} and $\{v_3, v_4\}$ vertex of K_4 is adjacent to u_j, where i, j. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(2P_2, 2P_2, 0, 0)$. If $d(u_1) > 5$, $d(u_2) > 5$, $d(u_3), d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Let $\{v_1, v_2\}$ vertices of K_4 be adjacent to vertex u_i of K_{n-4} and $\{v_3\}$ is adjacent to vertex u_j of K_{n-4} and $\{v_4\}$ vertex of K_4 is adjacent to u_k, where i, j, k. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(2P_2, 2P_2, 0, 0)$. If $d(u_1) > 5$, $d(u_2) > 4$, $d(u_3) > 4$, $d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Let v_1 vertices of K_{n-4} be adjacent to vertex u_i of K_{n-4} and v_2 is adjacent to vertex u_j of K_{n-4} and v_3 vertex of K_{n-4} is adjacent to u_k and v_4 is adjacent to vertex u_l of K_{n-4}, where i, j, k, l. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(P_2, P_2, P_2, P_2)$. If $d(u_1), d(u_2), d(u_3), d(u_4) > 4$, then

$fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (iii): Let $< S >= P_4$. Let v_1, v_2, v_3, v_4 be the vertices of P_4. Since G is connected, the following are the possible cases (i) there exists u_i of K_{n-4} that is adjacent to any one of pendant vertex (ii) there exists u_i of K_{n-4} that is adjacent to other than pendant vertices. Without loss of generality, let v_1, which is the pendant vertex, be adjacent to u_i, then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(P_5)$. If $d(u_1) > 4$ or $d(u_2), d(u_3), d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to a vertex other than a pendant vertex. Without loss of generality, let v_2, which is not pendant vertex, be adjacent to u_i, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong G_{13}$. If $d(u_1) = 4$ or $d(u_2) = d(u_3) = d(v_1) = d(v_3) = d(v_2) = 2$, $d(v_1) = 1$, then $G \cong G_{14}$. If $d(u_1) > 4$, $d(u_2) = d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (iv): Let $< S >= P_3 \cup K_1$. Let v_1, v_2, v_3, v_4 be the vertices of $P_3 \cup K_1$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_3\}$, which is pendant vertices and v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3(u(P_4, P_2))$. If $d(u_1) > 4$, $d(u_2) = d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction that no graph exists.

If there exists u_i of K_{n-4} that is adjacent to v_2, which is not pendant vertices and v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. $G \cong G_{15}$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of v_1 and there exists another vertex u_j of K_{n-4} that is adjacent to v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3(P_4, P_2, 0)$. If $d(u_1), d(u_2) > 3$, $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of v_1, which is pendant vertices and there exists another vertex u_j of K_{n-4} that is adjacent to v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong G_{16}$. If $d(u_1)$, $d(u_2) > 3$, $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (v): Let $< S >= K_3 \cup K_1$. Let v_1, v_2, v_3, v_4 be the vertices of $K_3 \cup K_1$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2, v_3\}$ and v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 4$, $d(u_2) = d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2, v_3\}$ and there exists another vertex u_j of K_{n-4} that is adjacent to v_4, then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong G_{18}$. If $d(u_1) > 3$, $d(u_2) > 3$, $d(u_3) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (vi): Let $< S >= K_{1,3}$. Let v_1, v_2, v_3, v_4 be the vertices of$K_{1,3}$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to v_1 which is not a pendant vertex, say v_1. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_3$. If $d(u_1) > 4$ or $d(u_3), d(u_4), d(u_2) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (vii): Let $< S >= K_2 \cup K_2$. Let v_1, v_2, v_3, v_4 be the vertices of $K_2 \cup K_2$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2\}$ and $\{v_3, v_4\}$. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(2P_3)$. If $d(u_1) = 6, d(u_2) = d(u_3) = d(u_4) = 3$, then $G \cong K_4(u(C_3, P_3))$. If $d(u_1) > 6$ or $d(u_3), d(u_4)$ and $d(u_2) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction that no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of v_1 or v_2 and there exists another vertex u_j, adjacent v_3 or v_4, then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(P_3, P_3, 0, 0)$. If $d(u_1) = 5, d(u_2) = 4, d(u_3) = d(u_4) = 3$, then $G \cong K_4(C_3, P_3, 0, 0)$. If $d(u_1) > 5$ or $d(u_3), d(u_4) > 3$ and $d(u_2) > 4$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (viii): Let $< S >= K_2 \cup K_2$. Let v_1, v_2, v_3, v_4 be the vertices of $K_2 \cup K_2$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2\}$, v_3 and v_4. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(u(2P_2, P_3))$. If $d(u_1) = 7, d(u_2) = d(u_3) = d(u_4) = 3$, then $G \cong K_4(u(C_3, 2P_2))$. If $d(u_1) > 7$ or $d(u_3), d(u_4)$ and $d(u_2) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2\}$ and v_3. There exists another vertex u_j that is adjacent to v_4, where i, j. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(u(P_2, P_3), P_2, 0, 0)$. If $d(u_1) = 6$, $d(u_2) = 4$, $d(u_3) = d(u_4) = 3$, then $G \cong K_4(u(K_3, P_2), P_2, 0, 0)$. If $d(u_1) > 6$ or $d(u_3), d(u_4) > 3$ and $d(u_2) > 4$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2\}$ and there exists another vertex u_j that is adjacent to v_3 and v_4, where i, j, k. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $G \cong K_4(P_3, 2P_2, 0, 0)$. If $d(u_1) = d(u_2) = 5$, $d(u_3) = d(u_4) = 4$, then $G \cong K_4(K_3, 2P_2, 0, 0)$. If

$d(u_1) > 5$ or $d(u_3), d(u_4) > 3$ and $d(u_2) > 5$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2\}$ and there exists another vertices u_j and u_k that is adjacent to v_3 and v_4, where i, j, k. Then $fr(G) = 6$. Hence $n = 8$. Therefore $K = K_4$. Then $GK_4(P_3, P_2, P_2, 0)$. If $d(u_1), d(u_3), d(u_2) > 4$ and $d(u_4) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (ix): Let $< S >= C_4$. Let v_1, v_2, v_3, v_4 be the vertices of C_4. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$. Then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong G_{20}$. If $d(u_1) = d(u_2) = 3$ and $d(u_3) = 2$, then $G \cong K_3$. If $d(u_1) > 3$ or $d(u_3) > 2, d(u_2) > 3$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (x) Let $< S >= C_3(1, 0, 0)$. Let v_1, v_2, v_3, v_4 be the vertices of $C_3(1, 0, 0)$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of $\{v_1, v_2, v_3, v_4\}$. Then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong G_{22}$. If $d(u_1) > 3$ or $d(u_3), d(u_2) > 2$, then $fr(G) + \chi(G) \neq 2n - 6$. This is a contradiction; hence, no graph exists.

Subcase (xi): Let $< S >= K_4 - \{e\}$. Let v_1, v_2, v_3, v_4 be the vertices of $_4 - \{e\}$. Since G is connected, there exists u_i of K_{n-4} that is adjacent to any one of the vertices not on the cycle of $d(v_i) = 2$. Without loss of generality, let v_1 be adjacent to u_i. Then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) = 4$, $d(u_3), d(u_2) = 2$, then $G \cong K_3$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of the vertices not on the cycle of $d(v_i) = 3$. Without loss of generality, let v_2 be adjacent to u_i. Then $fr(G) = 4$. Hence $n = 5$. Therefore $K = K_2$. Then $G \cong K_3$. If $d(u_1) > 2$, $d(u_2) > 1$, we get a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of the vertices on the cycle of $d(v_i) = 2$. Without loss of generality, let v_1 be adjacent to u_i. Then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) > 3$, $d(u_3), d(u_2) > 2$, we get a contradiction; hence, no graph exists.

If there exists u_i of K_{n-4} that is adjacent to any one of the vertices on the cycle of $d(v_i) = 3$. Without loss of generality, let v_2 be adjacent to u_i. Then $fr(G) = 5$. Hence $n = 7$. Therefore $K = K_3$. Then $G \cong K_3$. If $d(u_1) = 4$, $d(u_2) = d(u_3) = 2$, then $G \cong K_4$. If $d(u_1) > 2$, $d(u_2) > 1$, we get a contradiction; hence, no graph exists.

Case (iv): $fr(G) = n-3$ and $\chi(G) = n-3$. Since $\chi(G) = n-3$, G contains a clique K on $n-3$ vertices or does not contains a clique on K_{n-3} vertices. Let G contains a clique on K_{n-3} vertices. Let S= $\{v_1, v_2, v_3\}$ be the vertices other than the clique. Then the induced subgraph S is isomorphic to one of the following graphs K_3, P_3, $K_2 \cup K_1$.

Subcase (i): Let $< S >= K_3$. Let v_1, v_2, v_3 be the vertices of K_3. Since G is connected, there exists u_i of K_{n-3} that is adjacent to any one of the vertices $\{v_1, v_2, v_3\}$. Without loss of generality, let v_1 be adjacent to u_i. Then $fr(G) = 4$. Hence $n = 7$. Therefore $K = K_4$. Then $G \cong K_3$. If $d(u_1) = d(u_2) = 4$, $d(u_3), d(u_2) = 3$, then $G \cong K_3$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

Subcase (ii): Let $< S >= K_3$. Let v_1, v_2, v_3 be the vertices of K_3. Since G is connected, there exists u_i of K_{n-3} that is adjacent to all vertices $\{v_1, v_2, v_3\}$. Then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(3P_2)$. If $d(u_1) = 7$, $d(u_5) = 5$, $d(u_3), d(u_2), d(u_4) = 4$, then $G \cong K_3$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exists u_i of K_{n-3} that is adjacent to all vertices $\{v_1, v_2\}$ and there exists another vertex u_j, where i, j that is adjacent to v_3, then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(2P_2, P_2, 0, 0, 0)$. If $d(u_1) = 6d(u_5)$, d$(u_2) = 5, d(u_3), d(u_4) = 4$, then $G \cong K_3$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exist u_i, u_j, u_k of K_{n-3} that is adjacent to v_1, v_2, v_3, where i, j, k, then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(P_2, P_2, P_2, 0, 0)$. If $d(u_1), d(u5), d(u_2), d(u_3) = 5$, $d(u_4) = 4$, then $G \cong G_{33}$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

Subcase (iii): Let $< S >= P_3$. Let v_1, v_2, v_3 be the vertices of P_3. Since G is connected, there exists u_i of K_{n-3} that is adjacent to any one of the vertices $\{v_1, v_2, v_3\}$. Without loss of generality, let u_i be adjacent to the pendant vertex v_1. Then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(P_4)$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exists u_i of K_{n-3} that is adjacent to v_2, then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_3$. If $d(u_1) > 5$, $d(u_2), d(u_3), d(u_2) > 4$, we get a contradiction; hence, no graph exists.

Subcase (iv): Let $< S >= K_2 \cup K_1$. Let v_1, v_2, v_3 be the vertices of $K_2 \cup K_1$. Since G is connected, there exists u_i of K_{n-3} that is adjacent v_1, v_3. Then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(u(P_3, P_2))$. If $d(u_1) > 6d(u_2), d(u_3), d(u_2) > 4$, we get a contradiction; hence, no graph exists.

If there exists u_i of K_{n-3} that is adjacent v_1, v_3, then $fr(G) = 5$. Hence $n = 8$. Therefore $K = K_5$. Then $G \cong K_5(P_3, P_2, 0, 0, 0)$. If $d(u_1), d(u_2) > 5$, $d(u_3), d(u_2) > 4$, we get a contradiction; hence, no graph exists.

Case (v): $fr(G) = n - 4$ and $\chi(G) = n - 2$. Since $\chi(G) = n - 2$, G contains a clique K on $n - 2$ vertices or does not contains a clique on K_{n-2} vertices. Let G contains a clique on K_{n-2} vertices. Let $S = \{v_1, v_2\}$ be the vertices other than the clique. Then the induced subgraph S is isomorphic to one of the following graphs K_2.

Subcase (i): Let $< S >= K_2$. Since G is connected, there exists u_i of K_{n-2} that is adjacent v_1. Then $fr(G) = 4$. Hence $n = 8$. Therefore $K = K_6$. Then $G \cong K_6(P3)$. This is a contradiction; hence, no graph exists.

Subcase (ii) Let $< S >= K_2$. Since G is connected, there exists u_i of K_{n-2} that is adjacent v_1 and v_2. Then $fr(G) = 4$. Hence $n = 8$. Therefore $K = K_6$. Then $G \cong K_6(2P_2)$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

If there exists u_i and u_j of K_{n-2} that is adjacent v_1 and v_2. Then $fr(G) = 4$. Hence $n = 8$. Therefore $K = K_6$. Then $G \cong K_6(P_2, P_2)$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

Case (vi): $fr(G) = n - 5$ and $\chi(G) = n - 1$. Since $\chi(G) = n - 1$, G contains a clique K on $n - 1$ vertices or does not contains a clique on K_{n-1} vertices. Let G contains a clique on K_{n-1} vertices. Let $S = \{v_1\}$ be the vertices other than the clique. There exists a vertex u_i that is adjacent to v_1. Then $fr(G) = 4$. Hence $n = 8$. Therefore $K = K_6$. Then $G \cong K_7(P_2)$. As the degree increases, we obtain a contradiction, indicating that no graph exists.

Case (vii): $fr(G) = n - 6$ and $\chi(G) = n$. Since $\chi(G) = n$, then $G \cong K_n$. But for a complete graph $fr(G) = 2$. hence $n = 8$ and $K = K_8$. Then $G \cong K8$. □

3. Conclusion

In this paper, we analyzed the chromatic number of cubic fuzzy graphs and explored their properties within the framework of vertex-regular fuzzy graphs. Our results provide insights into the relationship between regular fuzzy graphs and their chromatic properties. The findings contribute to a deeper understanding of fuzzy graph coloring and its implications in graph theory. Future research may extend these results to higher-order regular fuzzy graphs and explore applications in network analysis and optimization problems.

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