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Spherical Picture Fuzzy Sets: Enhancing Decision-Making with Geometric Bonferroni Methods

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Abstract. The representation of a spherical picture fuzzy set (SPFS) employs a spherical framework to depict uncertainty across positive, negative, and neutral membership functions, effectively capturing the vagueness inherent in these degrees. Through this structure, SPFS enable nuanced decision-making, supported by novel ranking mechanisms, parametric distance measures and Euclidean distance evaluations. Additionally, an extended version of the spherical picture fuzzy Bonferroni method is introduced, tailored to MCDM scenarios. Applied to diverse stakeholder contexts, this approach overcomes the limitations of traditional averaging methods, offering a comprehensive representation of collective opinions within a spherical framework. In contrast to conventional picture fuzz set theories, our research introduces the SPFS, revolutionizing decision-making paradigms with its geometric model.

1. Introduction

Multi-Criteria Decision Making (MCDM) stands as a pivotal discipline addressing the intricacies inherent in decision-making processes, particularly when confronted with multiple and often conflicting criteria. Rather than relying on singular metrics, decision-makers must weigh and balance numerous factors, each contributing to the overall outcome. MCDM offers a systematic framework and an array of methodologies to navigate such complex landscapes, empowering decision-makers to make well-informed and effective choices. Its significance spans across diverse domains, including engineering, economics, environmental management, healthcare, and public policy. For instance, in engineering, MCDM aids in selecting optimal designs among competing

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alternatives, while in healthcare, it assists in treatment selection by integrating clinical effectiveness, patient preferences, and cost-effectiveness. Within the domain of MCDM, PFSs play a pivotal role, finding applications across a myriad of fields. Several existing MCDM methods employing PFSs include PF WASPAS [36], PF Analytic Hierarchy Process (AHP) [14], PF TOPSIS [19], PF Electre [24], PF Promethee [39], PF Simple Additive Weighting (SAW) [15], PF Additive Ratio Assessment (ARAS) [20], and PF Measurement of Alternatives and Ranking according to Compromise Solution (MARCOS) [1]. In machine learning, algorithm selection and hyperparameter tuning profoundly impact model performance and predictive accuracy. Conventional approaches often fall short in capturing nuanced relationships between algorithms and parameters, leading to sub optimal outcomes. Here, a MCDM approach emerges as a promising solution, offering a systematic framework for algorithm selection and hyperparameter tuning. By considering multiple criteria simultaneously such as predictive accuracy, computational efficiency, and model interpretability this approach aims to identify the most suitable algorithm and parameter configuration for a given task.

Introducing the concept of SPFS, our research proposes a novel geometric model transcending conventional PF set theories. Leveraging SPFS, we present a sophisticated framework for algorithm selection and hyperparameter tuning, enhancing decision-making clarity and effectiveness in machine learning tasks. By integrating SPFS based representations with specialized operators and distance measures, our approach offers a robust and intuitive framework for navigating the complexities of algorithm selection and hyperparameter optimization. Building upon existing research, such as "A Multi-Criteria Decision Making Approach for Algorithm Selection and Hyperparameter Tuning in Machine Learning," our study aims to extend traditional decision-making methods by incorporating SPFS based representations and methodologies. Through empirical validation and practical applications, we seek to demonstrate the efficacy and superiority of our proposed approach in enhancing model performance and facilitating informed decision-making in machine learning tasks.

2. Literature review

The notion of FSs (FSs) introduced by Zadeh [44] . Any real integer between 0 and 1 in FSs represents the membership degree of each element in the discourse universe. A number of applications of FSs have been demonstrated [3], [7], [30], [33], and [37]. Extending this notion, Atanassov [4] used it to Intuitionistic FSs (IFSs), where each element's membership is indicated by both a membership and non-membership grade. Applications for IFSs have been studied in several studies [2], [8], [18], [27], [35]. The PF Sets (PFSs) introduced by Cuong [9] [10] are additional expansions of this framework wherein each element's membership is defined by its positive, neutral, and negative membership. The BM (BM) [6], introduced by Bonferroni in 1950, is a widely used aggregation operator in decision-making and data fusion processes. It calculates the weighted average of input values, where the weights are determined by the relative importance of each value in the aggregation process. The BM is particularly useful when dealing with uncertain or imprecise information, as it allows for the incorporation of multiple sources of data while considering their respective significance. In [40], different variations of BM based PF pattern tree models are introduced and utilized for the classification of epileptic EEG signals. Using PF Aczel-Alsina power BM operator, weighted PF Aczel-Alsina power BM operator, PF Aczel-Alsina power geometric BM operator, and weighted geometric PF Aczel-Alsina power BM operator, the selection of capable research scientists to handle MADM issues under the framework of PF values is done in [29]. Ateş, F. and Akay, D. utilized MCDM in [5] to apply an PFBM operator to the implementation of an enterprise resource planning (ERP) system for an organization.

The following motivates to present spherical picture fuzzy set. Li et. al. [25] presented a multi-attribute group decision-making method based on the Elimination and Choice Translating Reality (EDAS) technique within a PF environment. An emergency management center (EMC) aims to select an optimal emergency alternative from four feasible options. A committee of three experts evaluates the alternatives and selects the most suitable emergency alternative, with the average value of their assessments used for the collective decision matrix. Peng & Chen [31], proposed a comprehensive approach to group decision-making on a large scale is proposed, utilizing hesitancy degrees and accounting for non-cooperative behaviors within the framework of PF information. Thirty decision makers are involved in evaluating the criteria of each alternative and consolidating the values using a uniform PF linguistic scale. Kou et. al. [23] introduced an integrated framework of quantum PF rough sets incorporating golden cuts, aimed at evaluating investment decision policies for sustainable industries based on carbon footprint considerations. Three decision makers analyze the data and take the average of their evaluations to create a collective decision matrix. Kamber et al. [21] introduced a methodology for prioritizing drip-irrigation pump alternatives in agricultural settings, utilizing an integrated approach combining PF Best Worst Method (BWM) and Complex Proportional Assessment (CODAS). Two decision makers analyze the data and take the average of their evaluations to create an average decision matrix. Švadlenka et. al., [38] introduced a decision-making approach utilizing PF logic for enhancing sustainable last-mile delivery processes. Ten decision makers evaluate 6 alternatives based on 20 criteria, and the average of their evaluations is taken as the evaluation matrix. In this paper, we introduce the concept of a spherical picture fuzzy set (SPFS) for representing a decision maker's choices geometrically, akin to a sphere within the framework of PFSs. The SPFS approach offers a solution to the limitation of simply averaging the decision values of the decision maker, as discussed in [32], [12], [13], [41], [21], [23], [25], [31], [28], [38], [42], [43].

The following contributions have been made to achieve the objectives of the study:

- Introduction of SPFSs: This paper introduces SPFSs as a novel geometric representation of PFSs, leveraging the spherical framework to capture uncertainties associated with positive, negative, and neutral membership functions. The SPFSs facilitates intuitive interpretation and decision-making processes by converting collections of points into a spherical structure, enhancing sensitivity and nuance in decision-making.

- Exploration of the Spherical Picture Fuzzy Bobferroni Mean (Y) Operator: Within the realm of MCDM, the paper investigates the Y operator, employing Euclidean Distance measures to depict evaluation values of alternatives concerning various criteria. By integrating this operator into decision analysis methodologies, the paper advances theoretical understandings and practical applications in decision-making processes.
- Construction of an Innovative MCDM Framework for Algorithm Selection and Hyperparameter Adjusting: The methodical strategy suggested in this paper is designed especially for machine learning algorithm selection and hyperparameter optimization. The framework provides an approach to decision-making that is transparent and allows practitioners to examine trade-offs and make well-informed judgments based on particular requirements and restrictions. These criteria include accuracy, interpretability, computational efficiency, and resilience.
- Practical Benefits of SPFS in Decision Effectiveness: The importance along with effectiveness of SPFS and the Y operator in decision-making scenarios is demonstrated through a numerical case study. The study showcases how SPFS surpasses traditional FSs by enabling intuitive data interpretation and facilitating visualization of intricate decision landscapes, thereby enhancing decision effectiveness.

2.1. **Preliminaries.**

Definition 2.1. *[6] Let $\rho, \sigma \geq 0$, and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_\kappa$ be non negative numbers. Then BM (BM) is defined as*

$$BM^{\rho,\sigma}(\alpha_1, \alpha_2, \ldots, \alpha_\kappa) = \left(\frac{1}{\kappa(\kappa-1)} \sum_{\substack{\psi,\eta=1 \\ \psi\neq\eta}}^{\kappa} \alpha_\psi^\rho \alpha_\eta^\sigma \right)^{\frac{1}{\rho+\sigma}}$$

Definition 2.2. *[9,10] A PF set A on universe of discourse $\Gamma = \{\varsigma_1, \varsigma_2, ...\varsigma_n\}$ is an object of the form: $A = \{\langle \mu_A(\varsigma_i), \eta_A(\varsigma_i), \nu_A(\varsigma_i) \rangle \ \varsigma_i \in \Gamma\}$, where $\mu_A(\varsigma_i) : \Gamma \to [0,1]$ is called the degree of membership of ς_i in A, $\eta_A(\varsigma_i) : \Gamma \to [0,1]$ is called the degree of neutral membership of ς_i in A, and $\nu_A(\varsigma_i) : \Gamma \to [0,1]$ is called the degree of non-membership of ς_i in A, $\mu_A(\varsigma_i), \eta_A(\varsigma_i), \nu_A(\varsigma_i)$ satisfy $0 \leq \mu_A(\varsigma_i) + \eta_A(\varsigma_i) + \nu_A(\varsigma_i) \leq 1 \forall \varsigma_i \in X$, $\pi_A(\varsigma_i) = 1 - (\mu_A(\varsigma_i) + \eta_A(\varsigma_i) + \nu_A(\varsigma_i))$ is called the degree of refusal membership of ς_i in A. The set of all PF subsets on universe of discourse $\Gamma = \{\varsigma_1, \varsigma_2, ...\varsigma_n\}$ is denoted by $PFSs(\Gamma)$.*

3. Spherical Picture Fuzzy Sets

Definition 3.1. *Let Γ be the universal set containing elements known as Spherical Picture Fuzzy sets (SPFSs). Each $\varsigma_i \in \Gamma$ is defined as $\Omega_{\varsigma_i} = \{\langle \varsigma_i, \mathcal{T}(\varsigma_i), \mathcal{I}(\varsigma_i), \mathcal{N}(\varsigma_i); \xi_i \rangle : \varsigma_i \in \Gamma\}$, where*

$\mathcal{T}(\varsigma_i), \mathcal{I}(\varsigma_i), \mathcal{N}(\varsigma_i) : \Gamma \to [0,1]$, represent the degrees of membership, non-membership, and indeterminacy of ς_i. This degrees satisfy $0 \leq \mathcal{T}(\varsigma_i) + \mathcal{I}(\varsigma_i) + \mathcal{N}(\varsigma_i) \leq 1$ for all $\varsigma_i \in \Gamma$ and $i = 1, 2, ...k$. We construct the center of the sphere by

$$\langle \mathcal{T}(\varsigma_i), \mathcal{I}(\varsigma_i), \mathcal{N}(\varsigma_i) \rangle = \left\langle \frac{\sum_{j=1}^{k} \mathcal{T}_{i,j}}{k}, \frac{\sum_{j=1}^{k} \mathcal{I}_{i,j}}{k}, \frac{\sum_{j=1}^{k} \mathcal{N}_{i,j}}{k} \right\rangle \tag{3.1}$$

and the radius is

$$\xi_i = min\left\{ \overset{max}{1 \leq j \leq k} \sqrt{(\mathcal{T}(\varsigma_i) - \mathcal{T}_{i,j})^2 + (\mathcal{I}(\varsigma_i) - \mathcal{I}_{i,j})^2 + (\mathcal{N}(\varsigma_i) - \mathcal{N}_{i,j})^2}, 1 \right\} \tag{3.2}$$

Example 3.1. *Let $A = \{\langle \varsigma_1, 0.4, 0.3, 0.3 \rangle, \langle \varsigma_1, 0.2, 0.4, 0.2 \rangle, \langle \varsigma_1, 0.5, 0.2, 0.3 \rangle, \langle \varsigma_1, 0.2, 0.5, 0.2 \rangle, \langle \varsigma_1, 0.1, 0.3, 0.4 \rangle, \langle \varsigma_1, 0.4, 0.2, 0.1 \rangle\}$ and $B = \{\langle \varsigma_2, 0.4, 0.3, 0.3 \rangle, \langle \varsigma_2, 0.2, 0.4, 0.2 \rangle, \langle \varsigma_2, 0.4, 0.2, 0.4 \rangle, \langle \varsigma_2, 0.2, 0.5, 0.3 \rangle, \langle \varsigma_2, 0.2, 0.3, 0.5 \rangle, \langle \varsigma_2, 0.4, 0.2, 0.2 \rangle\}$ be the collection of PFSs.*
Then $\mathcal{T}(\varsigma_1) = \frac{0.4+0.2+0.5+0.2+0.1+0,4}{6} = 0.3$, $\mathcal{I}(\varsigma_1) = \frac{0.3+0.4+0.2+0.5+0.3+0.2}{6} = 0.32$,
$\mathcal{N}(\varsigma_1) = \frac{0.3+0.2+0.3+0.2+0.4+0.1}{6} = 0.25$, $\mathcal{T}(\varsigma_2) = \frac{0.4+0.2+0.4+0.2+0.2+0.4}{6} = 0.3$,
$\mathcal{I}(\varsigma_2) = \frac{0.3+0.4+0.2+0.5+0.3+0.2}{6} = 0.32$ and $\mathcal{N}(\varsigma_2) = \frac{0.3+0.2+0.4+0.3+0.5+0.2}{6} = 0.35$.
The radii are

$$\xi_i = min\left\{ \overset{max}{1 \leq j \leq k} \sqrt{(\mathcal{T}(\varsigma_i) - \mathcal{T}_{i,j})^2 + (\mathcal{I}(\varsigma_i) - \mathcal{I}_{i,j})^2 + (\mathcal{N}(\varsigma_i) - \mathcal{N}_{i,j})^2}, 1 \right\}$$

$$\begin{aligned}
\xi_1 = min\{\max\{ &\sqrt{(0.3-0.4)^2 + (0.32-0.3)^2 + (0.25-0.3)^2},\\
&\sqrt{(0.3-0.2)^2 + (0.32-0.4)^2 + (0.25-0.2)^2},\\
&\sqrt{(0.3-0.5)^2 + (0.32-0.2)^2 + (0.25-0.3)^2},\\
&\sqrt{(0.3-0.2)^2 + (0.32-0.5)^2 + (0.25-0.2)^2},\\
&\sqrt{(0.3-0.1)^2 + (0.32-0.3)^2 + (0.25-0.4)^2},\\
&\sqrt{(0.3-0.4)^2 + (0.32-0.2)^2 + (0.25-0.1)^2}\}, 1\}
\end{aligned}$$

$$= min\{max\{0.11, 0.14, 0.24, 0.22, 0.25, 0.22\}, 1\} = min\{0.25, 1\} = 0.25$$

$$\begin{aligned}
\xi_2 = min\{\max\{ &\sqrt{(0.3-0.4)^2 + (0.32-0.3)^2 + (0.35-0.3)^2},\\
&\sqrt{(0.3-0.2)^2 + (0.32-0.4)^2 + (0.35-0.2)^2},\\
&\sqrt{(0.3-0.4)^2 + (0.32-0.2)^2 + (0.35-0.4)^2},\\
&\sqrt{(0.3-0.2)^2 + (0.32-0.5)^2 + (0.35-0.3)^2},\\
&\sqrt{(0.3-0.2)^2 + (0.32-0.3)^2 + (0.35-0.5)^2},\\
&\sqrt{(0.3-0.4)^2 + (0.32-0.2)^2 + (0.35-0.2)^2}\}, 1\}
\end{aligned}$$

$$= min\{max\{0.11, 0.14, 0.16, 0.22, 0.18, 0.22\}, 1\} = min\{0.22, 1\} = 0.22.$$

Then the SPFSs are $\kappa_A = \langle 0.3, 0.32, 0.25; 0.22 \rangle$ and $\kappa_B = \langle 0.3, 0.32, 0.35; 0.22 \rangle$.

Definition 3.2. *Let $\kappa_A = \langle \mathcal{T}(\varsigma_1), \mathcal{I}(\varsigma_1), \mathcal{N}(\varsigma_1); \xi_1 \rangle$, $\kappa_B = \langle \mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_2); \xi_2 \rangle$ are two SPFSs over the universal Γ, and $\triangledown$, $\vartriangle$ are denotes minimum and maximum respectively. Then the following operations are defined as follows:*

1. $\kappa_A \cup \kappa_B = < \vartriangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \vartriangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangledown(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \vartriangle(\xi_1; \xi_2) >$.
2. $\kappa_A \cup \kappa_B = < \vartriangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \vartriangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangledown(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangledown(\xi_1; \xi_2) >$.
3. $\kappa_A \cup \kappa_B = < \vartriangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \vartriangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \vartriangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \vartriangle(\xi_1; \xi_2) >$.
4. $\kappa_A \cup \kappa_B = < \vartriangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \vartriangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \vartriangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangledown(\xi_1; \xi_2) >$.

5. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), (\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
6. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), (\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
7. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), 1-(\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
8. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), 1-(\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
9. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \left|\mathcal{I}(\varsigma_1) - \mathcal{I}(\varsigma_2)\right|, \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
10. $\kappa_A \cup \kappa_B =< \triangle(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \left|\mathcal{I}(\varsigma_1) - \mathcal{I}(\varsigma_2)\right|, \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
11. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \nabla(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
12. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \nabla(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
13. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \triangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
14. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \triangle(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
15. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \nabla(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
16. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \nabla(\mathcal{I}(\varsigma_1), \mathcal{I}(\varsigma_2)), \nabla(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
17. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), (\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
18. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), (\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
19. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), 1-(\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
20. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), 1-(\frac{\mathcal{I}(\varsigma_1)+\mathcal{I}(\varsigma_2)}{2}), \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.
21. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \left|\mathcal{I}(\varsigma_1) - \mathcal{I}(\varsigma_2)\right|, \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \triangle(\xi_1; \xi_2) >$.
22. $\kappa_A \cap \kappa_B =< \nabla(\mathcal{T}(\varsigma_1), \mathcal{T}(\varsigma_2)), \left|\mathcal{I}(\varsigma_1) - \mathcal{I}(\varsigma_2)\right|, \triangle(\mathcal{N}(\varsigma_1), \mathcal{N}(\varsigma_2)); \nabla(\xi_1; \xi_2) >$.

Definition 3.3. *Let* $\kappa_A = \langle \mathcal{T}(\varsigma_1), \mathcal{I}(\varsigma_1), \mathcal{N}(\varsigma_1); \xi_1 \rangle$, $\kappa_B = \langle \mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_2); \xi_2 \rangle$ *are two SPFSs over the universal* Γ. *Then the following operations are defined as follows:*
1. $\kappa_A = \kappa_B$ *if and only if* $\xi_1 = \xi_2$ *and* $\mathcal{T}(\varsigma_1) = \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) = \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) = \mathcal{N}(\varsigma_2)$.
2. $\kappa_A = \kappa_B$ *if and only if* $\xi_1 = \xi_2$ *and* $\mathcal{T}(\varsigma_1) \leq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \leq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \leq \mathcal{N}(\varsigma_2)$.
3. $\kappa_A = \kappa_B$ *if and only if* $\xi_1 = \xi_2$ *and* $\mathcal{T}(\varsigma_1) \geq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \geq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \geq \mathcal{N}(\varsigma_2)$.
4. $\kappa_A \subset \kappa_B$ *if and only if* $\xi_1 < \xi_2$ *and* $\mathcal{T}(\varsigma_1) < \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) < \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) > \mathcal{N}(\varsigma_2)$.
5. $\kappa_A \subseteq \kappa_B$ *if and only if* $\xi_1 \leq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \leq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \leq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \geq \mathcal{N}(\varsigma_2)$.
6. $\kappa_A \subseteq \kappa_B$ *if and only if* $\xi_1 \leq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \leq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \geq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \geq \mathcal{N}(\varsigma_2)$.
7. $\kappa_A \subseteq \kappa_B$ *if and only if* $\xi_1 \leq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \leq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) < \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) > \mathcal{N}(\varsigma_2)$.
8. $\kappa_A \subseteq \kappa_B$ *if and only if* $\xi_1 < \xi_2$ *and* $\mathcal{T}(\varsigma_1) \leq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \geq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \geq \mathcal{N}(\varsigma_2)$.
9. $\kappa_A \supset \kappa_B$ *if and only if* $\xi_1 > \xi_2$ *and* $\mathcal{T}(\varsigma_1) > \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) > \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) < \mathcal{N}(\varsigma_2)$.
10. $\kappa_A \supset \kappa_B$ *if and only if* $\xi_1 > \xi_2$ *and* $\mathcal{T}(\varsigma_1) > \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) < \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) < \mathcal{N}(\varsigma_2)$.
11. $\kappa_A \supseteq \kappa_B$ *if and only if* $\xi_1 \geq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \geq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \geq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \leq \mathcal{N}(\varsigma_2)$.
12. $\kappa_A \supseteq \kappa_B$ *if and only if* $\xi_1 \geq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \geq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \leq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \leq \mathcal{N}(\varsigma_2)$.
13. $\kappa_A \supseteq \kappa_B$ *if and only if* $\xi_1 \geq \xi_2$ *and* $\mathcal{T}(\varsigma_1) \geq \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) < \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) < \mathcal{N}(\varsigma_2)$.
14. $\kappa_A \supseteq \kappa_B$ *if and only if* $\xi_1 \geq \xi_2$ *and* $\mathcal{T}(\varsigma_1) > \mathcal{T}(\varsigma_2)$, $\mathcal{I}(\varsigma_1) \leq \mathcal{I}(\varsigma_2)$, $\mathcal{N}(\varsigma_1) \leq \mathcal{N}(\varsigma_2)$.

Example 3.2. *Let* $A = \{\langle \varsigma_1, 0.4, 0.3, 0.3 \rangle, \langle \varsigma_1, 0.2, 0.4, 0.2 \rangle, \langle \varsigma_1, 0.5, 0.2, 0.3 \rangle, \langle \varsigma_1, 0.2, 0.5, 0.2 \rangle, \langle \varsigma_1, 0.1, 0.3, 0.4 \rangle, \langle \varsigma_1, 0.4, 0.2, 0.1 \rangle\}$, $B = \{\langle \varsigma_2, 0.4, 0.3, 0.3 \rangle, \langle \varsigma_2, 0.2, 0.4, 0.2 \rangle, \langle \varsigma_2, 0.4, 0.2, 0.4 \rangle, \langle \varsigma_2, 0.2, 0.5, 0.3 \rangle, \langle \varsigma_2, 0.2, 0.3, 0.5 \rangle, \langle \varsigma_2, 0.4, 0.2, 0.2 \rangle\}$, $C = \{\langle \varsigma_3, 0.3, 0.4, 0.3 \rangle, \langle \varsigma_3, 0.2, 0.6, 0.2 \rangle,$

$\langle \varsigma_3, 0.3, 0.3, 0.4\rangle, \langle \varsigma_3, 0.4, 0.4, 0.2\rangle, \langle \varsigma_3, 0.4, 0.2, 0.4\rangle . \langle \varsigma_3, 0.6, 0.2, 0.2\rangle\}$, $D = \{\langle \varsigma_4, 0.6, 0.3, 0.1\rangle,$ $\langle \varsigma_4, 0.4, 0.4, 0.2\rangle, \langle \varsigma_4, 0.4, 0.2, 0.4\rangle, \langle \varsigma_4, 0.2, 0.5, 0.3\rangle, \langle \varsigma_4, 0.2, 0.4, 0.4\rangle . \langle \varsigma_4, 0.3, 0.4, 0.3\rangle\}$ *be the collection of PFSs and SPFSs are* $\kappa_A = \langle 0.3, 0.32, 0.25; 0.25\rangle$, $\kappa_B = \langle 0.3, 0.32, 0.35; 0.21\rangle$, $\kappa_C = \langle 0.37, 0.35, 0.28; 0.28\rangle$, $\kappa_D = \langle 0.35, 0.37, 0.28; 0.21\rangle$. *Then*

By Definition 3.2, the union of two SPFSs is

3.2 1.	$\kappa_A \cup \kappa_B = \langle 0.3, 0.32, 0.35; 0.25\rangle$	*3.2 2.*	$\kappa_C \cup \kappa_D = \langle 0.37, 0.37, 0.28; 0.21\rangle$
3.2 3.	$\kappa_C \cup \kappa_D = \langle 0.37, 0.36, 0.28; 0.28\rangle$	*3.2 4.*	$\kappa_C \cup \kappa_D = \langle 0.37, 0.36, 0.28; 0.28\rangle$
3.2 5.	$\kappa_A \cup \kappa_B = \langle 0.3, 0.68, 0.25; 0.21\rangle$	*3.2 6.*	$\kappa_A \cup \kappa_B = \langle 0.3, 0.68, 0.25; 0.25\rangle$
3.2 7.	$\kappa_A \cup \kappa_B = \langle 0.3, 0.00, 0.35; 0.25\rangle$	*3.2 8.*	$\kappa_A \cup \kappa_B = \langle 0.3, 0, 0.35; 0.21\rangle$
3.2 9.	$\kappa_C \cup \kappa_D = \langle 0.35, 0.35, 0.28; 0.28\rangle$	*3.2 10.*	$\kappa_A \cup \kappa_B = \langle 0.3, 0.32, 0.35; 0.21\rangle$

By Definition 3.2, the intersection of two SPFSs is

3.2 11.	$\kappa_C \cap \kappa_D = \langle 0.35, 0.37, 0.28; 0.28\rangle$	*3.2 12.*	$\kappa_C \cap \kappa_D = \langle 0.35, 0.37, 0.28; 0.21\rangle$
3.2 13.	$\kappa_A \cap \kappa_B = \langle 0.3, 0.32, 0.25; 0.25\rangle$	*3.2 14.*	$\kappa_C \cap \kappa_D = \langle 0.35, 0.35, 0.28; 0.21\rangle$
3.2 15.	$\kappa_C \cap \kappa_D = \langle 0.35, 0.36, 0.28; 0.28\rangle$	*3.2 16.*	$\kappa_C \cap \kappa_D = \langle 0.35, 0.36, 0.28; 0.21\rangle$
3.2 17.	$\kappa_A \cap \kappa_B = \langle 0.3, 0.68, 0.35; 0.25\rangle$	*3.2 18.*	$\kappa_A \cap \kappa_B = \langle 0.3, 0.68, 0.35; 0.21\rangle$
3.2 19.	$\kappa_C \cap \kappa_D = \langle 0.35, 0.02, 0.28; 0.28\rangle$	*3.2 20.*	$\kappa_C \cap \kappa_D = \langle 0.35, 0.02, 0.28; 0.21\rangle$

Definition 3.4. *Let* $\kappa = \langle \mathcal{T}(\varsigma), \mathcal{I}(\varsigma), \mathcal{N}(\varsigma); \xi\rangle$, $\kappa_A = \langle \mathcal{T}(\varsigma_1), \mathcal{I}(\varsigma_1), \mathcal{N}(\varsigma_1); \xi_1\rangle$, $\kappa_B = \langle \mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_2); \xi_2\rangle$ *are three SPFSs over the universal* Γ *and* $0 \leq \alpha \leq 1$. *Then the following operations are defined as follows:*

(1) $\kappa_A \oplus \kappa_B =< \mathcal{T}(\varsigma_1) + \mathcal{T}(\varsigma_2) - \mathcal{T}(\varsigma_1)\mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_1)\mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_1)\mathcal{N}(\varsigma_2); \xi_1 + \xi_2 - \xi_1\xi_2 >$
(2) $\kappa_A \otimes \kappa_B =< \mathcal{T}(\varsigma_1)\mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_1) + \mathcal{I}(\varsigma_2) - \mathcal{I}(\varsigma_1)\mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_1) + \mathcal{N}(\varsigma_2) - \mathcal{N}(\varsigma_1)\mathcal{N}(\varsigma_2); \xi_1\xi_2 >$
(3) $\alpha\kappa =< 1 - (1 - \mathcal{T}(\varsigma))^\alpha, (\mathcal{I}(\varsigma))^\alpha, (\mathcal{N}(\varsigma))^\alpha; 1 - (1 - \xi)^\alpha >$
(4) $\kappa^\alpha =< \mathcal{T}(\varsigma)^\alpha, 1 - (1 - \mathcal{I}(\varsigma))^\alpha, 1 - (1 - \mathcal{N}(\varsigma))^\alpha; \xi^\alpha >$
(5) $\neg\kappa =< \mathcal{N}(\varsigma), \mathcal{I}(\varsigma), \mathcal{T}(\varsigma); \xi >$
(6) $\neg\kappa =< \mathcal{N}(\varsigma), 1 - \mathcal{I}(\varsigma), \mathcal{T}(\varsigma); \xi >$
(7) $\neg\kappa =< 1 - \mathcal{T}(\varsigma), 1 - \mathcal{I}(\varsigma), 1 - \mathcal{N}(\varsigma); \xi >$
(8) $\neg\kappa =< 1 - \mathcal{T}(\varsigma), \mathcal{I}(\varsigma), 1 - \mathcal{N}(\varsigma); \xi >$
(9) $\kappa^c =< \mathcal{N}(\varsigma), \mathcal{I}(\varsigma), \mathcal{T}(\varsigma); 1 - \xi >$

Example 3.3. *Let* $A = \{\langle \varsigma_1, 0.4, 0.3, 0.3\rangle, \langle \varsigma_1, 0.2, 0.4, 0.2\rangle, \langle \varsigma_1, 0.5, 0.2, 0.3\rangle, \langle \varsigma_1, 0.2, 0.5, 0.2\rangle,$ $\langle \varsigma_1, 0.1, 0.3, 0.4\rangle, \langle \varsigma_1, 0.4, 0.2, 0.1\rangle\}$, $B = \{\langle \varsigma_2, 0.4, 0.3, 0.3\rangle, \langle \varsigma_2, 0.2, 0.4, 0.2\rangle, \langle \varsigma_2, 0.4, 0.2, 0.4\rangle,$ $\langle \varsigma_2, 0.2, 0.5, 0.3\rangle, \langle \varsigma_2, 0.2, 0.3, 0.5\rangle, \langle \varsigma_2, 0.4, 0.2, 0.2\rangle\}$ *be the collection of PFSs and the SPFSs are* $\kappa_A = \langle 0.3, 0.32, 0.25; 0.25\rangle$, $\kappa_B = \langle 0.3, 0.32, 0.35; 0.21\rangle$. *Then*

Proposition 3.1. *For any three SPFSs* $\kappa_A, \kappa_B, \kappa_C$, *the following results are valid.*

(1) $\kappa_A \cap \kappa_B = \kappa_B \cap \kappa_A$
(2) $\kappa_A \cup \kappa_B = \kappa_B \cup \kappa_A$

3.4 1. $\kappa_A \oplus \kappa_B = \langle 0.51, 0.12, 0.09; 0.41\rangle$ 3.4 2. $\kappa_A \otimes \kappa_B = \langle 0.09, 0.54, 0.51; 0.05\rangle$
3.4 3. $\alpha\kappa_A = \langle 0.16, 0.57, 0.5; 0.13\rangle$ 3.4 4. $\kappa_A^{\alpha} = \langle 0.55, 0.18, 0.13; 0.5\rangle$
3.4 5. $\neg\kappa_A = \langle 0.25, 0.32, 0.3; 0.25\rangle$ 3.4 6. $\neg\kappa_A = \langle 0.25, 0.68, 0.3; 0.25\rangle$
3.4 7. $\neg\kappa_A = \langle 0.7, 0.68, 0.75; 0.75\rangle$ 3.4 8. $\neg\kappa_A = \langle 0.7, 0.32, 0.75; 0.25\rangle$
3.4 9. $\kappa_A^c = \langle 0.25, 0.32, 0.3; 0.75\rangle$.

(3) $\kappa_A \oplus \kappa_B = \kappa_B \oplus \kappa_A$
(4) $\kappa_A \otimes \kappa_B = \kappa_B \otimes \kappa_A$
(5) $(\kappa_A \cap \kappa_B) \cap \kappa_C = \kappa_A \cap (\kappa_B \cap \kappa_C)$
(6) $(\kappa_A \cup \kappa_B) \cup \kappa_C = \kappa_A \cup (\kappa_B \cup \kappa_C)$
(7) $(\kappa_A \oplus \kappa_B) \oplus \kappa_C = \kappa_A \oplus (\kappa_B \oplus \kappa_C)$
(8) $(\kappa_A \otimes \kappa_B) \otimes \kappa_C = \kappa_A \otimes (\kappa_B \otimes \kappa_C)$
(9) $(\kappa_A \cup \kappa_B) \cap \kappa_C = (\kappa_A \cap \kappa_C) \cup (\kappa_B \cap \kappa_C)$
(10) $(\kappa_A \cap \kappa_B) \cup \kappa_C = (\kappa_A \cup \kappa_C) \cap (\kappa_B \cup \kappa_C)$
(11) $(\kappa_A \cap \kappa_B) \oplus \kappa_C = (\kappa_A \oplus \kappa_C) \cap (\kappa_B \oplus \kappa_C)$
(12) $(\kappa_A \cap \kappa_B) \otimes \kappa_C = (\kappa_A \otimes \kappa_C) \cap (\kappa_B \otimes \kappa_C)$
(13) $(\kappa_A \cup \kappa_B) \oplus \kappa_C = (\kappa_A \oplus \kappa_C) \cup (\kappa_B \oplus \kappa_C)$
(14) $(\kappa_A \cup \kappa_B) \otimes \kappa_C = (\kappa_A \otimes \kappa_C) \cup (\kappa_B \otimes \kappa_C)$
(15) $\kappa_A \cap \kappa_A = \kappa_A$
(16) $\kappa_A \cup \kappa_A = \kappa_A$
(17) $\neg(\neg\kappa_A \cap \neg\kappa_B) = \kappa_A \cup \kappa_B$
(18) $\neg(\neg\kappa_A \cup \neg\kappa_B) = \kappa_A \cap \kappa_B$
(19) $\neg(\neg\kappa_A \oplus \neg\kappa_B) = \kappa_A \otimes \kappa_B$
(20) $\neg(\neg\kappa_A \otimes \neg\kappa_B) = \kappa_A \oplus \kappa_B$

Definition 3.5. *Let κ_A and κ_B be a SPFSs and r any real number such that for each $0 \le r \le 1$ the convex combination of κ_A and κ_B is defined as follows:*
$C_r(\kappa_A, \kappa_B) = (\mathcal{T}_{c_r}(\varsigma), \mathcal{I}_{c_r}(\varsigma), \mathcal{N}_{c_r}(\varsigma); \xi_{c_r}(\varsigma))$, *where*
$\mathcal{T}_{c_r}(\varsigma) = r.\mathcal{T}_{\kappa_A}(\varsigma) + (1-r)\mathcal{T}_{\kappa_B}(\varsigma)$
$\mathcal{I}_{c_r}(\varsigma) = r.\mathcal{I}_{\kappa_A}(\varsigma) + (1-r)\mathcal{I}_{\kappa_B}(\varsigma)$
$\mathcal{N}_{c_r}(\varsigma) = r.\mathcal{N}_{\kappa_A}(\varsigma) + (1-r)\mathcal{N}_{\kappa_B}(\varsigma)$
$\xi_{c_r}(\varsigma) = r.\xi_{\kappa_A}(\varsigma) + (1-r)\xi_{\kappa_B}(\varsigma)$.

Example 3.4. *Let $A = \{\langle\varsigma_1, 0.4, 0.3, 0.3\rangle, \langle\varsigma_1, 0.2, 0.4, 0.2\rangle, \langle\varsigma_1, 0.5, 0.2, 0.3\rangle, \langle\varsigma_1, 0.2, 0.5, 0.2\rangle, \langle\varsigma_1, 0.1, 0.3, 0.4\rangle, \langle\varsigma_1, 0.4, 0.2, 0.1\rangle\}$, $B = \{\langle\varsigma_2, 0.4, 0.3, 0.3\rangle, \langle\varsigma_2, 0.2, 0.4, 0.2\rangle, \langle\varsigma_2, 0.4, 0.2, 0.4\rangle, \langle\varsigma_2, 0.2, 0.5, 0.3\rangle, \langle\varsigma_2, 0.2, 0.3, 0.5\rangle, \langle\varsigma_2, 0.4, 0.2, 0.2\rangle\}$, be the collection of PFSs, $r = 0.2$, and SPFSs are $\kappa_A = \langle 0.3, 0.32, 0.25; 0.25\rangle$, $\kappa_B = \langle 0.3, 0.32, 0.35; 0.21\rangle$. Then $\mathcal{T}_{c_r}(\varsigma) = 0.3$, $\mathcal{I}_{c_r}(\varsigma) = 0.3$, $\mathcal{N}_{c_r}(\varsigma) = 0.33$, $\xi_{c_r}(\varsigma) = 0.22$ and $C_r(\kappa_A, \kappa_B) = (0.3, 0.3, 0.33, 0.22)$.*

Proposition 3.2. *Let κ_A and κ_B be two SPFSs. Let r be a real number such that $0 \le r \le 1$. Then*

(1) *If $r = 1$, then $C_r(\kappa_A, \kappa_B) = \kappa_A$ and if $r = 0$, then $C_r(\kappa_A, \kappa_B) = \kappa_B$*
(2) *If $\kappa_A \subseteq \kappa_B$, then $\forall r$, $\kappa_A \subseteq C_r(\kappa_A, \kappa_B) \subseteq \kappa_B$*
(3) *If $\kappa_A \supseteq \kappa_B$ and $r_1 \ge r_2$, then $C_{r_1}(\kappa_A, \kappa_B) \ge C_{r_2}(\kappa_A, \kappa_B)$.*

Definition 3.6. *For any two SPFSs κ_A and κ_B defined as $\kappa_A = \langle \mathcal{T}(\varsigma_1), \mathcal{I}(\varsigma_1)), \mathcal{N}(\varsigma_1)); \xi_1\rangle$, $\kappa_B = \langle \mathcal{T}(\varsigma_2), \mathcal{I}(\varsigma_2), \mathcal{N}(\varsigma_2); \xi_2\rangle$ the distance between two SPFSs defined by:*
$D = \sqrt{(\mathcal{T}(\varsigma_1) - \mathcal{T}(\varsigma_2))^2 + (\mathcal{I}(\varsigma_1) - \mathcal{I}(\varsigma_2))^2 + (\mathcal{N}(\varsigma_1) - \mathcal{N}(\varsigma_2))^2}$-$(\xi_1 + \xi_2)$

For example, consider the two picture fuzzy sets (0.6, 0.3, 0.1) and (0.6, 0.3, 0.1), then center of spherical picture fuzzy set is (0.6, 0.3, 0.1) and the radius of spherical picture fuzzy set is 0.1. the distance between the of spherical picture fuzzy set is $d = 0$ since they are the same. This indicating Two spherical picture fuzzy sets overlap completely.

Separation of two spherical picture fuzzy sets

Two SPFSs are separate. Two PFSs (0.7, 0.2, 0.1), (0.8, 0.1, 0.1) then center of SPFS is (0.75, 0.15, 0.1) and the radius 0.1

Two PFSs (0.4, 0.4, 0.2) (0.5, 0.3, 0.2) then center of SPFS is (0.45, 0.35, 0.2) and the radius 0.1
Distance Calculation: d = $\sqrt{(0.75 - 0.45)^2 + (0.15 - 0.35)^2 + (0.1 - 0.2)^2}$
$= \sqrt{(0.3)^2 + (-0.2)^2 + (-0.1)^2}$
$= \sqrt{0.09 + 0.04 + 0.01} = \sqrt{0.14} \approx 0.374$
$D = d - (\xi_1 + \xi_2) = 0.374 - (0.1 + 0.1) = 0.374 - 0.2 = 0.174$. indicating separation of two spherical picture fuzzy sets.

Intersection of two spherical picture fuzzy sets

Two SPFSs intersect : Degrees: (0.5, 0.4, 0.1) (0.6, 0.3, 0.1) (0.55, 0.35, 0.1) Radius: $\xi_1 = 0.1$ $(0.6, 0.2, 0.2)(0.5, 0.3, 0.2)(0.55, 0.25, 0.2)$ Radius: $\xi_2 = 0.1$

Distance Calculation D = $\sqrt{(0.55 - 0.55)^2 + (0.35 - 0.25)^2 + (0.1 - 0.2)^2}$ - (0.1 + 0.1)
$= \sqrt{(0)^2 + (0.1)^2 + (-0.1)^2}$ -0.2
$D = -0.0586$ indicating intersection of two spherical picture fuzzy sets.

4. Spherical Picture Fuzzy Bonferroni Mean

Definition 4.1. *Let $pm_i = \langle \mathcal{T}_{n_i}, \mathcal{I}_{n_i}, \mathcal{N}_{n_i}; \xi_{n_i}\rangle$ where $i = 1, 2, \ldots, \vartheta$ is a collection of $\mathcal{N}$. For any $\psi, \eta > 0$, the spherical picture fuzzy Bonferroni mean (Υ) is defined as*

$$\Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) = \left(\frac{1}{\vartheta(\vartheta - 1)} \bigoplus_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (pm_\alpha^\psi \otimes pm_\beta^\eta) \right)^{\frac{1}{\psi+\eta}} \tag{4.1}$$

Theorem 4.1. *For $\psi, \eta > 0$ and a collection of SPFSs $pm_\alpha = \langle \mathcal{T}_{n_\alpha}, \mathcal{I}_{n_\alpha}, \mathcal{N}_{n_\alpha}; \xi_{n_\alpha} \rangle$, $\alpha = 1, 2, \ldots, \vartheta$, the aggregation Υ is also a SPFS and it is of the form*

$$\Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) = \begin{cases} \left(\left(1 - \left(\prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} \left(1 - \mathcal{T}_{pm_\alpha}^{\psi} \mathcal{T}_{pm_\beta}^{\eta}\right)^{\frac{1}{\vartheta(\vartheta-1)}}\right)\right)^{\frac{1}{\psi+\eta}}\right), \\ 1 - \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} \left(1 - (1 - \mathcal{I}_{pm_\alpha})^{\psi} (1 - \mathcal{I}_{pm_\beta})^{\eta}\right)^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}, \\ 1 - \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} \left(1 - (1 - \mathcal{N}_{pm_\alpha})^{\psi} (1 - \mathcal{N}_{pm_\beta})^{\eta}\right)^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}, \\ \left(\left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} \left(1 - \xi_{pm_\alpha}^{\psi} \xi_{pm_\beta}^{\eta}\right)^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}\right. \end{cases} \tag{4.2}$$

Proof:

From the basic operations, we get

$$pm_\alpha^{\psi} = \left(\mathcal{T}_{pm_\alpha}^{\psi}, 1 - (1 - \mathcal{I}_{pm_\alpha})^{\psi}, 1 - (1 - \mathcal{N}_{pm_\alpha})^{\psi}; \xi_{pm_\alpha}^{\psi}\right)$$

and

$$pm_\beta^{\eta} = \left(\mathcal{T}_{pm_\beta}^{\eta}, 1 - (1 - \mathcal{I}_{pm_\beta})^{\eta}, 1 - (1 - \mathcal{N}_{pm_\beta})^{\eta}; \xi_{pm_\beta}^{\eta}\right)$$

Then,

$$pm_\alpha^{\psi} \otimes pm_\beta^{\eta} = \begin{cases} \mathcal{T}_{pm_\alpha}^{\psi} \mathcal{T}_{pm_\beta}^{\eta}, \\ 1 - (1 - \mathcal{I}_{pm_\alpha})^{\psi} (1 - \mathcal{I}_{pm_\beta})^{\eta}, \\ 1 - (1 - \mathcal{N}_{pm_\alpha})^{\psi} (1 - \mathcal{N}_{pm_\beta})^{\eta}, \\ \xi_{pm_\alpha}^{\psi} \xi_{pm_\beta}^{\eta} \end{cases} \tag{4.3}$$

First let us prove

$$\bigoplus_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (pm_\alpha^{\psi} \otimes pm_\beta^{\eta}) = \begin{cases} 1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (1 - \mathcal{T}_{pm_\alpha}^{\psi} \mathcal{T}_{pm_\beta}^{\eta}), \\ \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (1 - (1 - \mathcal{I}_{pm_\alpha})^{\psi} (1 - \mathcal{I}_{pm_\beta})^{\eta}), \\ \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (1 - (1 - \mathcal{N}_{pm_\alpha})^{\psi} (1 - \mathcal{N}_{pm_\beta})^{\eta}), \\ 1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{\vartheta} (1 - \xi_{pm_\alpha}^{\psi} \xi_{pm_\beta}^{\eta}) \end{cases} \tag{4.4}$$

by mathematical induction principle on ϑ.

For $\vartheta = 2$, we get

$$\bigoplus_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{2} (pm_\alpha^{\psi} \otimes pm_\beta^{\eta}) = (pm_1^{\psi} \otimes pm_2^{\eta}) \oplus (pm_2^{\psi} \otimes pm_1^{\eta})$$

$$\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(pm_\alpha^\psi \otimes pm_\beta^\eta) = \begin{cases} 1-(1-\mathcal{T}_{pm_1}^{\psi}\mathcal{T}_{pm_2}^{\eta})(1-\mathcal{T}_{pm_2}^{\psi}\mathcal{T}_{pm_1}^{\eta}), \\ (1-(1-\mathcal{I}_{pm_1})^{\psi}(1-\mathcal{I}_{pm_2})^{\eta})(1-(1-\mathcal{I}_{pm_2})^{\psi}(1-\mathcal{I}_{pm_1})^{\eta}), \\ (1-(1-\mathcal{N}_{pm_1})^{\psi}(1-\mathcal{N}_{pm_2})^{\eta})(1-(1-\mathcal{N}_{pm_2})^{\psi}(1-\mathcal{N}_{pm_1})^{\eta}), \\ 1-(1-\xi_{pm_1}^{\psi}\xi_{pm_2}^{\eta})(1-\xi_{pm_2}^{\psi}\xi_{pm_1}^{\eta}) \end{cases}$$

$$\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(pm_\alpha^\psi \otimes pm_\beta^\eta) = \begin{cases} 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(1-\mathcal{T}_{pm_\alpha}^{\psi}\mathcal{T}_{pm_\beta}^{\eta}), \\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(1-(1-\mathcal{I}_{pm_\alpha})^{\psi}(1-\mathcal{I}_{pm_\beta})^{\eta}), \\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(1-(1-\mathcal{N}_{pm_\alpha})^{\psi}(1-\mathcal{N}_{pm_\beta})^{\eta}), \\ 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2}(1-\xi_{pm_\alpha}^{\psi}\xi_{pm_\beta}^{\eta}) \end{cases}$$

Assume that equation (4.4) holds for $\vartheta = \omega$, that is,

$$\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(pm_\alpha^\psi \otimes pm_\beta^\eta) = \begin{cases} 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}\left(1-\mathcal{T}_{pm_\alpha}^{\psi}\mathcal{T}_{pm_\beta}^{\eta}\right), \\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}\left(1-\left(1-\mathcal{I}_{pm_\alpha}\right)^{\psi}\left(1-\mathcal{I}_{pm_\beta}\right)^{\eta}\right), \\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}\left(1-\left(1-\mathcal{N}_{pm_\alpha}\right)^{\psi}\left(1-\mathcal{N}_{pm_\beta}\right)^{\eta}\right), \\ 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}\left(1-\xi_{pm_\alpha}^{\psi}\xi_{pm_\beta}^{\eta}\right) \end{cases} \tag{4.5}$$

Now let $\vartheta = \omega + 1$, then

$$\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(pm_\alpha^\psi \otimes pm_\beta^\eta) = \left(\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(pm_\alpha^\psi \otimes pm_\beta^\eta)\right) \oplus \left(\bigoplus_{\alpha=1}^{\omega}(pm_\alpha^\psi \otimes pm_{\omega+1}^\eta)\right) \tag{4.6}$$

$$\oplus \left(\bigoplus_{\beta=1}^{\omega}(pm_{\omega+1}^\psi \otimes pm_\beta^\eta)\right) \otimes \left(\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(pm_\alpha^\psi \otimes pm_\beta^\eta)\right) \tag{4.7}$$

By operations 1)-3) given in Definition 3.6, we get

$$\bigoplus_{\alpha=1}^{\omega}(pm_\alpha^\psi \otimes pm_{\omega+1}^\eta) = \begin{cases} 1-\prod_{\alpha=1}^{\omega}(1-\mathcal{T}_{pm_\alpha}^{\psi}\mathcal{T}_{pm_{\omega+1}}^{\eta}), \\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{I}_{pm_\alpha})^{\psi}(1-\mathcal{I}_{pm_{\omega+1}})^{\eta}), \\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{N}_{pm_\alpha})^{\psi}(1-\mathcal{N}_{pm_{\omega+1}})^{\eta}); \\ 1-\prod_{\alpha=1}^{\omega}(1-\xi_{pm_\alpha}^{\psi}\xi_{pm_{\omega+1}}^{\eta}) \end{cases} \tag{4.8}$$

and

$$\bigoplus_{\beta=1}^{\omega}(pm_{\omega+1}^{\psi}\otimes pm_{\beta}^{\eta})=\begin{cases}1-\prod_{\alpha=1}^{\omega}(1-\mathcal{T}_{pm_{\omega+1}}^{\psi}\mathcal{T}_{pm_{\beta}}^{\eta}),\\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{I}_{pm_{\omega+1}})^{\psi}(1-\mathcal{I}_{pm_{\beta}})^{\eta}),\\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{N}_{pm_{\omega+1}})^{\psi}(1-\mathcal{N}_{pm_{\beta}})^{\eta});\\ 1-\prod_{\alpha=1}^{\omega}(1-\xi_{pm_{\omega+1}}^{\psi}\xi_{pm_{\beta}}^{\eta})\end{cases} \tag{4.9}$$

From equations (4.5-4.9) we get

$$\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(pm_{\alpha}^{\psi}\otimes pm_{\beta}^{\eta})=\begin{cases}1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(1-\mathcal{T}_{pm_{\alpha}}^{\psi}\mathcal{T}_{pm_{\beta}}^{\eta}),\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(1-(1-\mathcal{I}_{pm_{\alpha}})^{\psi}(1-\mathcal{I}_{pm_{\beta}})^{\eta}),\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(1-(1-\mathcal{N}_{pm_{\alpha}})^{\psi}(1-\mathcal{N}_{pm_{\beta}})^{\eta});\\ 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega}(1-\xi_{pm_{\alpha}}^{\psi}\xi_{pm_{\beta}}^{\eta})\end{cases}$$

$$\oplus\begin{cases}1-\prod_{\alpha=1}^{\omega}(1-\mathcal{T}_{pm_{\alpha}}^{\psi}\mathcal{T}_{pm_{\omega+1}}^{\eta}),\\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{I}_{pm_{\alpha}})^{\psi}(1-\mathcal{I}_{pm_{\omega+1}})^{\eta}),\\ \prod_{\alpha=1}^{\omega}(1-(1-\mathcal{N}_{pm_{\alpha}})^{\psi}(1-\mathcal{N}_{pm_{\omega+1}})^{\eta});\\ 1-\prod_{\alpha=1}^{\omega}(1-\xi_{pm_{\alpha}}^{\psi}\xi_{pm_{\omega+1}}^{\eta})\end{cases}\oplus\begin{cases}1-\prod_{\beta=1}^{\omega}(1-\mathcal{T}_{pm_{\omega+1}}^{\psi}\mathcal{T}_{pm_{\beta}}^{\eta}),\\ \prod_{\beta=1}^{\omega}(1-(1-\mathcal{I}_{pm_{\omega+1}})^{\psi}(1-\mathcal{I}_{pm_{\beta}})^{\eta}),\\ \prod_{\beta=1}^{\omega}(1-(1-\mathcal{N}_{pm_{\omega+1}})^{\psi}(1-\mathcal{N}_{pm_{\beta}})^{\eta});\\ 1-\prod_{\beta=1}^{\omega}(1-\xi_{pm_{\omega+1}}^{\psi}\xi_{pm_{\beta}}^{\eta})\end{cases}$$

$$=\begin{cases}1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(1-\mathcal{T}_{pm_{\alpha}}^{\psi}\mathcal{T}_{pm_{\beta}}^{\eta}),\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(1-(1-\mathcal{I}_{pm_{\alpha}})^{\psi}(1-\mathcal{I}_{pm_{\beta}})^{\eta}),\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(1-(1-\mathcal{N}_{pm_{\alpha}})^{\psi}(1-\mathcal{N}_{pm_{\beta}})^{\eta});\\ 1-\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\omega+1}(1-\xi_{pm_{\alpha}}^{\psi}\xi_{pm_{\beta}}^{\eta})\end{cases}$$

Therefore the result is true for $\vartheta=\omega+1$. Hence by mathematical induction the equation (4.1) holds for all ϑ. Now,

$$\frac{1}{\vartheta(\vartheta-1)}\bigoplus_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\vartheta}(pm_{\alpha}^{\psi}\otimes pm_{\beta}^{\eta})=\begin{cases}1-\left(\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\vartheta}(1-\mathcal{T}_{pm_{\alpha}}^{\psi}\mathcal{T}_{pm_{\beta}}^{\eta})\right)^{\frac{1}{\vartheta(\vartheta-1)}},\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\vartheta}(1-(1-\mathcal{I}_{pm_{\alpha}})^{\psi}(1-\mathcal{I}_{pm_{\beta}})^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}},\\ \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\vartheta}(1-(1-\mathcal{N}_{pm_{\alpha}})^{\psi}(1-\mathcal{N}_{pm_{\beta}})^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}};\\ 1-\left(\prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{\vartheta}(1-\xi_{pm_{\alpha}}^{\psi}\xi_{pm_{\beta}}^{\eta})\right)^{\frac{1}{\vartheta(\vartheta-1)}}\end{cases}$$

$$\text{Therefore } \Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) = \begin{cases} \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{\vartheta} (1 - \mathcal{T}_{pm_\alpha}^{\psi} \mathcal{T}_{pm_\beta}^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}, \\ 1 - \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{\vartheta} (1 - (1 - \mathcal{I}_{pm_\alpha})^{\psi} (1 - \mathcal{I}_{pm_\beta})^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}, \\ 1 - \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{\vartheta} (1 - (1 - \mathcal{N}_{pm_\alpha})^{\psi} (1 - \mathcal{N}_{pm_\beta})^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}}; \\ \left(1 - \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{\vartheta} (1 - \xi_{pm_\alpha}^{\psi} \xi_{pm_\beta}^{\eta})^{\frac{1}{\vartheta(\vartheta-1)}}\right)^{\frac{1}{\psi+\eta}} \end{cases}$$

Also Υ has the following propositionerties.

Proposition 4.1 (Idempotency). *If the SPFSs, $pm_\alpha = fn = \left(\mathcal{T}_{pm_\alpha}, \mathcal{I}_{pm_\alpha}, \mathcal{N}_{pm_\alpha}; \xi_{pm_\alpha}\right)$ for all $\alpha = 1, 2, \ldots, \vartheta$, then $\Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) = fn$.*

Proposition 4.2 (Monotonicity). *Consider two SPFSs $pm_\alpha = (\mathcal{T}_{pm_\alpha}, \mathcal{I}_{pm_\alpha}, \mathcal{N}_{pm_\alpha}; \xi_{pm_\alpha})$ and $fm_\alpha = (\mathcal{T}_{fm_\alpha}, \mathcal{I}_{fm_\alpha}, \mathcal{N}_{fm_\alpha}; \xi_{fm_\alpha})$ for α=1, 2, 3, . . . , ϑ. If $\mathcal{T}_{pm_\alpha} \leq \mathcal{T}_{fm_\alpha}, \mathcal{I}_{pm_\alpha} \geq \mathcal{I}_{fm_\alpha}, \mathcal{N}_{pm_\alpha} \geq \mathcal{N}_{fm_\alpha}; \xi_{pm_\alpha} \leq \xi_{fm_\alpha}$ for each α, then $\Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) \leq \Upsilon^{\psi,\eta}(fm_1, fm_2, \ldots, fm_\vartheta)$.*

Proposition 4.3 (Commutativity). *Consider pm_α= $(\mathcal{T}_{pm_\alpha}, \mathcal{I}_{pm_\alpha}, \mathcal{N}_{pm_\alpha}; \xi_{pm_\alpha})$ as a set of SPFSs. Then $\Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta)$=$\Upsilon^{\psi,\eta}(p\dot{m}_1, p\dot{m}_2, \ldots, p\dot{m}_\vartheta)$ where $(p\dot{m}_1, p\dot{m}_2, \ldots, p\dot{m}_\vartheta)$ is any one arrangement of $(pm_1, pm_2, \ldots, pm_\vartheta)$.*

Proposition 4.4 (Boundedness). *Consider pm_α = $(\mathcal{T}_{pm_\alpha}, \mathcal{I}_{pm_\alpha}, \mathcal{N}_{pm_\alpha}; \xi_{pm_\alpha})$, $\alpha = 1, 2, 3, ..., \vartheta$ as a set of SPFSs, and let*

$$fn^- = (\min_\alpha\{\mathcal{T}_{pm_\alpha}\}, \max_\alpha\{\mathcal{I}_{pm_\alpha}\}, \max_\alpha\{\mathcal{N}_{pm_\alpha}\}; \min_\alpha\{\xi_{pm_\alpha}\})$$
$$fn^+ = (\max_\alpha\{\mathcal{T}_{pm_\alpha}\}, \min_\alpha\{\mathcal{I}_{pm_\alpha}\}, \min_\alpha\{\mathcal{N}_{pm_\alpha}\}; \max_\alpha\{\xi_{pm_\alpha}\}).$$

Then $fn^- \leq \Upsilon^{\psi,\eta}(pm_1, pm_2, \ldots, pm_\vartheta) \leq fn^+$.

5. Multi-Criteria Decision Making (MCDM) Approach Using Spherical Picture Fuzzy Information

In this section, we propose a Multi-Criteria Decision Making (MCDM) approach using the spherical picture fuzzy Bonferroni mean operator.

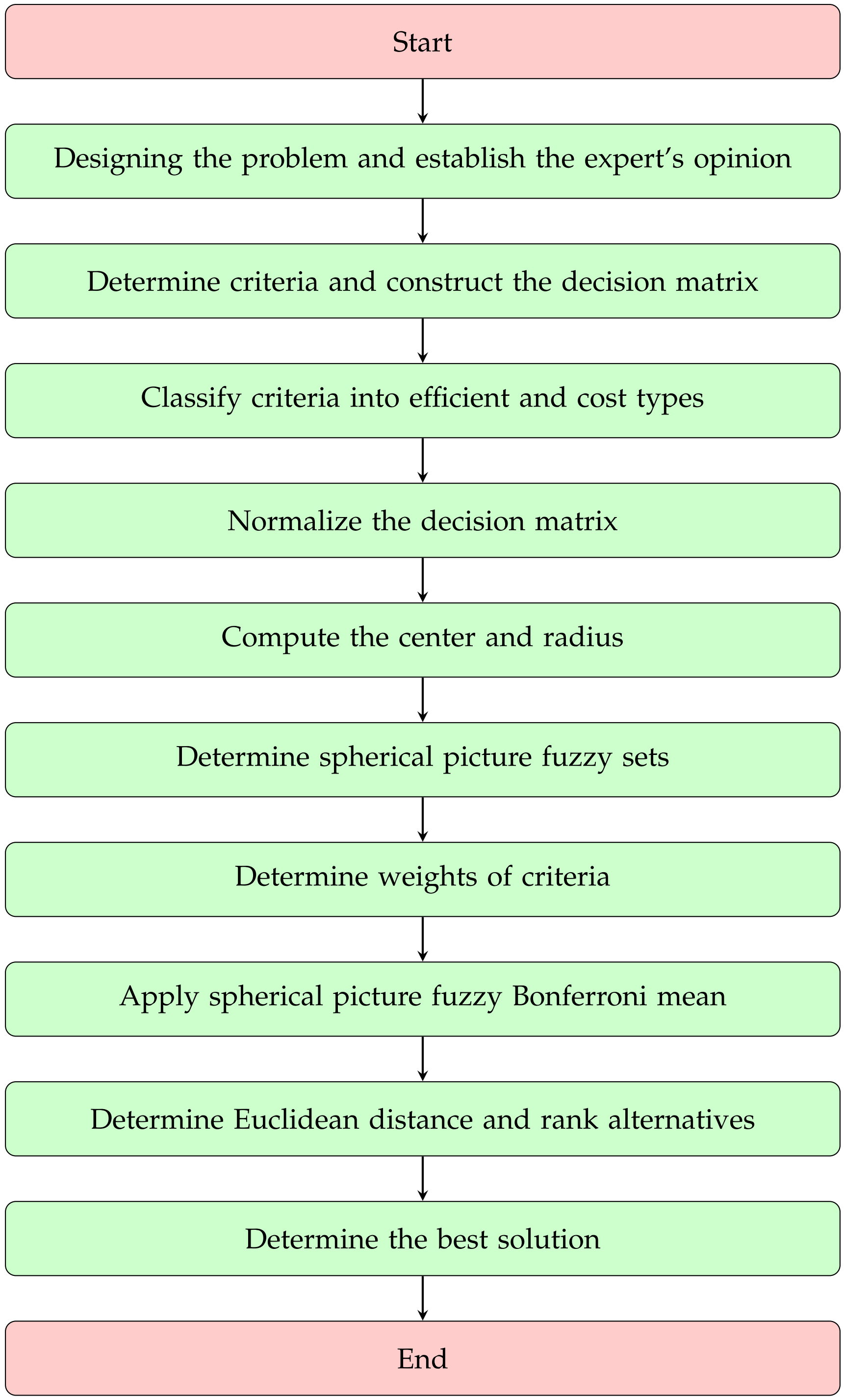
Let $\Theta = \{\Theta_1, \Theta_2 \ldots \Theta_\lambda\}$ be a set of alternatives and $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2 \ldots \mathbb{C}_\lambda\}$ be a set of criteria. Suppose $(\delta_{\alpha\varepsilon})_{m\times n} = \langle \mathcal{T}_{\delta_\alpha}(\nu_\varepsilon), \mathcal{I}_{\delta_\alpha}(\nu_\varepsilon), \mathcal{N}_{\delta_\alpha}(\nu_\varepsilon)\rangle_{m\times n}$ is a picture fuzzy decision matrix, where $\mathcal{T}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree of membership of alternatives Θ_ϵ, $\mathcal{I}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree of neutral membership of alternatives Θ_ϵ, and $\mathcal{N}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree non-membership of alternatives Θ_ϵ, each alternatives Θ_ϵ satisfy $0 \leq \mathcal{T}_{\delta_\alpha}(\nu_\varepsilon) + \mathcal{I}_{\delta_\alpha}(\nu_\varepsilon) + \mathcal{N}_{\delta_\alpha}(\nu_\varepsilon) \leq 1$.

We propose the following algorithm to solve MCDM problem with spherical picture fuzzy information using spherical picture fuzzy Bonferroni mean operator.

Algorithm 5.1 Multi-Criteria Decision Making (MCDM) Process

1. Start.
2. Input: To select the best alternative.
3. We employ the decision information given in matrix $(\delta_{\alpha\varepsilon})_{m\times n}$.
4. For each alternatives Θ_ϵ, $(\epsilon = 1, 2 \ldots, \lambda)$ construct the spherical picture fuzzy set $\Omega_\epsilon = \langle \mathcal{T}(\varsigma_\epsilon), \mathcal{I}(\varsigma_\epsilon), \mathcal{N}(\varsigma_\epsilon); \xi_\epsilon\rangle$ where $\langle \mathcal{T}(\varsigma_\epsilon), \mathcal{I}(\varsigma_\epsilon), \mathcal{N}(\varsigma_\epsilon)\rangle$ is the center of Ω_ϵ and ξ_ϵ is the radius of the spherical picture fuzzy set Ω_ϵ for all $\epsilon = 1, 2, \ldots \lambda$ from the decision matrix $(\delta_{\alpha\varepsilon})_{m\times n}$.
5. Operate spherical picture fuzzy Bonferroni mean operator $\mathbb{PM}_\epsilon = \Upsilon^{\psi,\eta}(\Omega_1, \Omega_2, \ldots, \Omega_\lambda)$ to obtain the overall preference values $\mathbb{PM}_\epsilon$ $(\epsilon = 1, 2, \ldots \lambda)$ of the alternative $\Theta_\epsilon(\epsilon = 1, 2, \ldots \lambda)$.
6. Calculate the Euclidean distance $D(\mathbb{PM}_\epsilon, \Omega_{\mathbb{I}})$ $(\epsilon = 1, 2, \ldots \lambda)$, where $\Omega_{\mathbb{I}} = (1, 0, 0; 1)$ is the positive ideal sphere.
7. The smallest distance value of $D(\mathbb{PM}_\epsilon, \Omega_{\mathbb{I}})$ $(\epsilon = 1, 2, \ldots \lambda)$, is the better alternative Θ_ϵ, because it is close to the positive ideal alternative $\Omega_{\mathbb{I}}$.
8. Rank the alternatives Θ_ϵ, $(\epsilon = 1, 2, \ldots \lambda)$ based on the spherical picture fuzzy Bonferroni mean operator $\mathbb{PM}_\epsilon$, $(\epsilon = 1, 2, \ldots \lambda)$ evaluations and Euclidean distance $D(\mathbb{PM}_\epsilon, \Omega_{\mathbb{I}})$ $(\epsilon = 1, 2, \ldots \lambda)$.
9. Output : Best alternative.
10. End.

The presented flowchart outlines a systematic decision-making process using spherical picture fuzzy information.


Start
Designing the problem and establish the expert's opinion
Determine criteria and construct the decision matrix
Classify criteria into efficient and cost types
Normalize the decision matrix
Compute the center and radius
Determine spherical picture fuzzy sets
Determine weights of criteria
Apply spherical picture fuzzy Bonferroni mean
Determine Euclidean distance and rank alternatives
Determine the best solution
End

6. Numerical Example : A MCDM Approach for Algorithm Selection and Hyperparameter Tuning in Machine Learning

The rapid evolution of machine learning techniques has led to an abundance of algorithms and models, each with distinct advantages and limitations. In the pursuit of building effective machine learning systems, selecting the most suitable algorithm and fine-tuning its hyperparameters are pivotal steps. However, this decision is often intricate, involving a delicate balance of multiple competing objectives. Traditional approaches to algorithm selection and hyperparameter tuning typically prioritize a single objective, such as maximizing accuracy, potentially neglecting other critical factors like model interpretability or computational efficiency. To surmount this challenge, we propose a novel MCDM approach that concurrently considers multiple evaluation criteria, enabling more informed decisions in algorithm selection and hyperparameter tuning.

Criteria Considered in the MCDM Framework.

- **Accuracy:** Accuracy reflects a machine learning model's capability to correctly classify or predict outcomes. It is assessed through metrics such as classification accuracy, precision, recall, F1-score, or mean squared error for regression tasks. While prioritizing accuracy is vital, an exclusive focus on this metric may lead to overfitting or favoring complex, difficult-to-interpret models.
- **Interpretability:** Interpretability measures how easily a model's predictions can be understood and explained by humans. Factors such as model simplicity, transparency in decision-making, and the ability to provide meaningful insights contribute to interpretability. Particularly crucial in real-world applications requiring transparency and trust, interpretable models are easier to validate, debug, and integrate, potentially at the expense of predictive performance or complexity.
- **Computational Efficiency:** Computational efficiency quantifies the time and resources required for training and deploying a machine learning model. It encompasses aspects like training time, inference time, memory usage, and scalability to large datasets or distributed environments. Given practical constraints, efficient resource utilization is crucial for real-time or resource-constrained deployments, even if it entails opting for simpler models at the t of accuracy or complexity.
- **Robustness:** Robustness characterizes a model's ability to withstand variations or perturbations in input data and generalize effectively to unseen data. Attributes like model stability, generalization error, and performance across different datasets or domains contribute to robustness. Essential for ensuring the reliability and effectiveness of machine learning models in dynamic real-world scenarios, enhancing robustness often involves incorporating regularization techniques or data augmentation, potentially influencing other criteria such as accuracy or interpretability.

Selecting Decision-Makers for the Problem. Selecting decision-makers for a MCDM problem in algorithm selection and hyperparameter tuning in machine learning involves identifying individuals or stakeholders who possess relevant expertise, perspectives, and authority to contribute meaningfully to the decision-making process. Here's a systematic approach to selecting decision-makers: Identify Relevant Stakeholders, Assess Expertise and Knowledge, Consider Perspectives and Interests, Evaluate Decision-Making Authority, Balance Participation and Efficiency, Facilitate Collaboration and Communication, Document Decision-Making Criteria. By following these steps, we can select decision-makers who are well-equipped to navigate the complexities of algorithm selection and hyperparameter tuning in machine learning projects, fostering collaboration, and achieving consensus towards optimal decision outcomes. Based on the outlined criteria, here are three decision-makers selected for the MCDM problem in algorithm selection and hyperparameter tuning in machine learning: Data Scientist (DS), Domain Expert (DE), Project Manager (PM). These decision-makers collectively represent a diverse range of expertise, perspectives, and decision-making authority required to address the complexities of algorithm selection and hyperparameter tuning in machine learning projects.

Linguistic Terms	PFNs $(\mathcal{T}, \mathcal{I}, \mathcal{N})\ \times 10^{-2}$
Extremely good (ζ_1)	(100, 0, 0)
Very good (ζ_2)	(75, 15, 10)
Good (ζ_3)	(65, 20, 15)
Medium (ζ_4)	(55, 30, 15)
Bad (ζ_5)	(35, 45, 20)
Very bad (ζ_6)	(25, 55, 20)
Extremely bad (ζ_7)	(0, 85, 15)

Table 1. Linguistic Terms and Corresponding PF Numbers

Step 1: Three Decision Makers DS, DE and PM evaluates six machine learning algorithm Θ_1, Θ_2, Θ_3, Θ_4, Θ_5, and Θ_6 with four criteria $\mathbb{C}_1$ = Accuracy, $\mathbb{C}_2$=Interpretability, $\mathbb{C}_3$ = Computational Efficiency and $\mathbb{C}_4$ = Robustness to select the best algorithm to solve a given problem. Each evaluators decisions in linguistic phrase are given in Table 2. The picture fuzzy numbers that match to the linguistic phrases in Table 1 will be substituted in Table 3.

Step 2: For each alternatives Θ_ϵ, $\epsilon = 1, 2, \ldots 6$, we construct spherical picture fuzzy numbers Ω_ϵ $=(\mathcal{T}_\epsilon, \mathcal{I}_\epsilon, \mathcal{N}_\epsilon; \xi_\epsilon)$ using Equation (2) & (3) and shown in Table 4.

Decision Makers	DS				DE				PM			
Algorithm / Criteria	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
Θ_1	ζ_1	ζ_6	ζ_3	ζ_4	ζ_3	ζ_7	ζ_4	ζ_5	ζ_2	ζ_5	ζ_7	ζ_7
Θ_2	ζ_3	ζ_4	ζ_3	ζ_6	ζ_4	ζ_5	ζ_1	ζ_7	ζ_7	ζ_7	ζ_2	ζ_5
Θ_3	ζ_6	ζ_3	ζ_4	ζ_4	ζ_7	ζ_4	ζ_5	ζ_5	ζ_5	ζ_7	ζ_7	ζ_7
Θ_4	ζ_4	ζ_3	ζ_6	ζ_3	ζ_5	ζ_1	ζ_7	ζ_5	ζ_7	ζ_2	ζ_5	ζ_4
Θ_5	ζ_2	ζ_1	ζ_4	ζ_5	ζ_4	ζ_3	ζ_5	ζ_7	ζ_4	ζ_2	ζ_3	ζ_6
Θ_6	ζ_7	ζ_3	ζ_6	ζ_3	ζ_5	ζ_4	ζ_7	ζ_4	ζ_6	ζ_3	ζ_5	ζ_3

Table 2. DM's evaluation of each criteria in Linguistic terms

DM's	Algorithm / Criteria	$\mathbb{C}_1$ $(\mathcal{T},\mathcal{I},\mathcal{N})\times 10^{-2}$	$\mathbb{C}_2$ $(\mathcal{T},\mathcal{I},\mathcal{N})\times 10^{-2}$	$\mathbb{C}_3$ $(\mathcal{T},\mathcal{I},\mathcal{N})\times 10^{-2}$	$\mathbb{C}_4$ $(\mathcal{T},\mathcal{I},\mathcal{N})\times 10^{-2}$
DS	Θ_1	(100, 0, 0)	(25, 55, 20)	(65, 20, 15)	(55, 30, 15)
	Θ_2	(65, 20, 15)	(55, 30, 15)	(65, 20, 15)	(25, 55, 20)
	Θ_3	(25, 55, 20)	(65, 20, 15)	(55, 30, 15)	(55, 30, 15)
	Θ_4	(55, 30, 15)	(65, 20, 15)	(25, 55, 20)	(65, 20, 15)
	Θ_5	(75, 15, 10)	(100, 0, 0)	(55, 30, 15)	(35, 45, 20)
	Θ_6	(0, 85, 15)	(65, 20, 15)	(25, 55, 20)	(65, 20, 15)
DE	Θ_1	(65, 20, 15)	(55, 30, 15)	(55, 30, 15)	(35, 45, 20)
	Θ_2	(55, 30, 15)	(35, 45, 20)	(100, 0, 0)	(0, 85, 15)
	Θ_3	(65, 20, 15)	(55, 30, 15)	(35, 45, 20)	(35, 45, 20)
	Θ_4	(35, 45, 20)	(100, 0, 0)	(75, 15, 10)	(35, 45, 20)
	Θ_5	(55, 30, 15)	(65, 20, 15)	(35, 45, 20)	(0, 85, 15)
	Θ_6	(35, 45, 20)	(55, 30, 15)	(55, 30, 15)	(55, 30, 15)
PM	Θ_1	(75, 15, 10)	(35, 45, 20)	(65, 20, 15)	(65, 20, 15)
	Θ_2	(65, 20, 15)	(65, 20, 15)	(75, 15, 10)	(35, 45, 20)
	Θ_3	(35, 45, 20)	(65, 20, 15)	(65, 20, 15)	(65, 20, 15)
	Θ_4	(65, 20, 15)	(75, 15, 10)	(35, 45, 20)	(55, 30, 15)
	Θ_5	(55, 30, 15)	(75, 15, 10)	(65, 20, 15)	(25, 55, 20)
	Θ_6	(25, 55, 20)	(65, 20, 15)	(35, 45, 20)	(65, 20, 15)

Table 3. DM's decision about each algorithm Θ_ϵ in picture fuzzy $(\mathcal{T}_\epsilon, \mathcal{I}_\epsilon, \mathcal{N}_\epsilon)$ values.

Algorithm / Criteria	$\mathbb{C}_1$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-2}$	$\mathbb{C}_2$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-2}$	$\mathbb{C}_3$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-2}$	$\mathbb{C}_4$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-2}$
Θ_1	(80, 12, 8; 25)	(38, 43, 18; 22)	(62, 23, 15; 9)	(52, 32, 17; 22)
Θ_2	(62, 23, 15; 9)	(52, 32, 17; 22)	(80, 12, 8; 25)	(20, 62, 18; 31)
Θ_3	(42, 40, 18; 31)	(62, 23, 15; 9)	(52, 32, 17; 22)	(52, 32, 17; 22)
Θ_4	(52, 32, 17; 22)	(80, 12, 8; 25)	(45, 38, 17; 39)	(52, 32, 17; 22)
Θ_5	(62, 25, 13; 17)	(80, 12, 8; 25)	(52, 32, 17; 22)	(20, 62, 18; 31)
Θ_6	(20, 62, 18; 31)	(62, 23, 15; 9)	(38, 43, 18; 22)	(62, 23, 15; 9)

TABLE 4. Spherical picture fuzzy representation $\Omega_\epsilon = (\mathcal{T}_\epsilon, \mathcal{I}_\epsilon, \mathcal{N}_\epsilon; \xi_\epsilon)$ of DM's evaluation of each alternatives Θ_ϵ.

Step 3: we operate spherical picture fuzzy Bonferroni mean operator $\mathbb{PM}_\epsilon = \Upsilon^{\psi,\eta}(\Omega_1, \Omega_2, \ldots, \Omega_\lambda)$ and obtained the overall preference values $\mathbb{PM}_\epsilon$ $(\epsilon = 1, 2, \ldots 6)$ of the alternative $\Theta_\epsilon(\epsilon = 1, 2, \ldots 6)$ and shown in Table 5.

Algorithm / $\Upsilon^{\psi,\eta}$	$\Upsilon^{\psi=1,\eta=1}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$	$\Upsilon^{\psi=1,\eta=2}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$	$\Upsilon^{\psi=2,\eta=2}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$
Θ_1	(567, 212, 69 ; 72)	(236, 532, 277 ; 9)	(123, 705, 458 ; 2)
Θ_2	(482, 280, 68 ; 93)	(222, 583, 260 ; 18)	(96, 762, 456 ; 3)
Θ_3	(470, 275, 88 ; 85)	(187, 586, 306 ; 12)	(74, 775, 510 ; 2)
Θ_4	(545, 231, 69 ; 137)	(229, 549, 271 ; 27)	(109, 729, 460 ; 5)
Θ_5	(469, 296, 66 ; 111)	(179, 636, 272 ; 21)	(90, 778, 451 ; 3)
Θ_6	(381, 355, 88 ; 58)	(154, 638, 303 ; 6)	(54, 827, 509 ; 1)
Algorithm / $\Upsilon^{\psi,\eta}$	$\Upsilon^{\psi=2,\eta=1}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$	$\Upsilon^{\psi=2,\eta=3}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$	$\Upsilon^{\psi=3,\eta=2}$ $(\mathcal{T},\mathcal{I},\mathcal{N};\xi)\times 10^{-3}$
Θ_1	(299, 475, 248 ; 11)	(67, 794, 547 ; 0)	(308, 338, 75 ; 1)
Θ_2	(212, 566, 261 ; 12)	(67, 821, 533 ; 1)	(205, 464, 81 ; 1)
Θ_3	(185, 591, 309 ; 15)	(39, 844, 590 ; 0)	(149, 509, 122 ; 1)
Θ_4	(266, 510, 256 ; 25)	(60, 811, 543 ; 2)	(256, 393, 79 ; 3)
Θ_5	(234, 553, 242 ; 16)	(52, 851, 539 ; 1)	(221, 463, 71 ; 1)
Θ_6	(130, 684, 310 ; 10)	(31, 876, 587 ; 0)	(108, 622, 123 ; 1)

TABLE 5. spherical picture fuzzy Bonferroni mean $\Upsilon^{\psi,\eta}$ values for the alternatives Θ_ϵ.

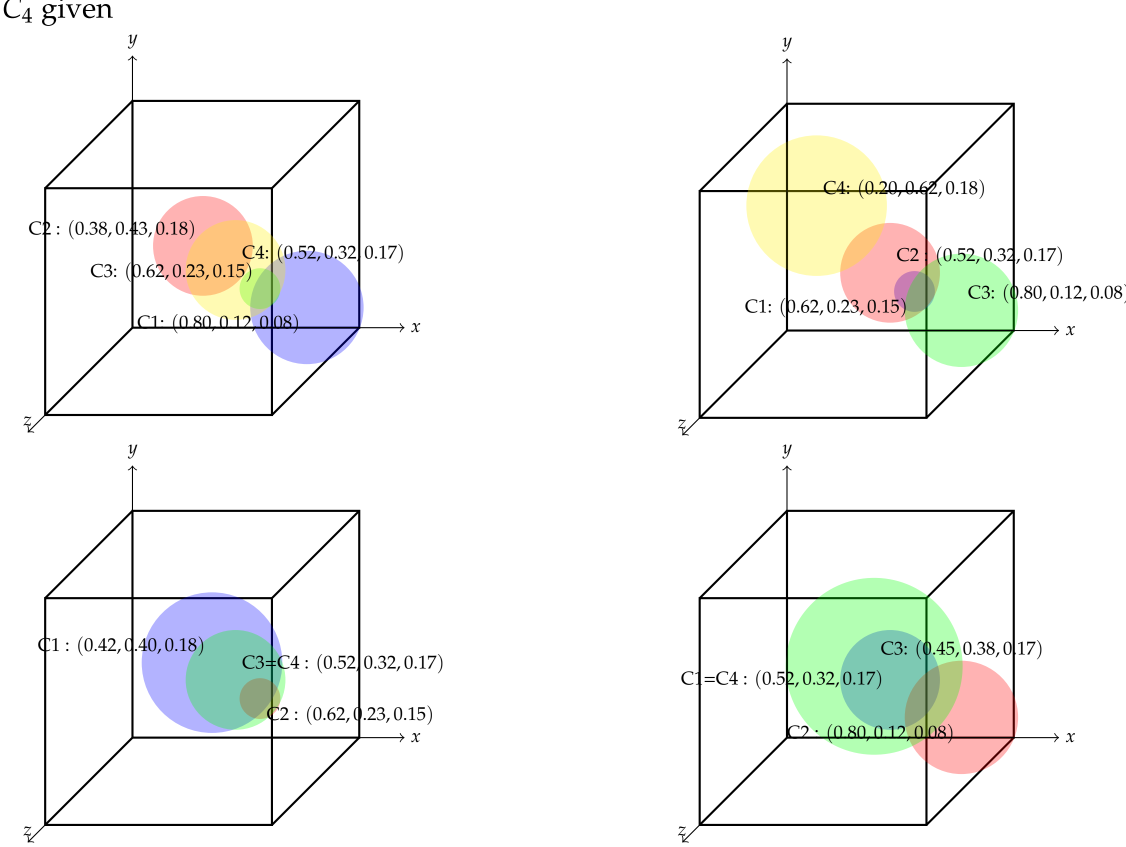
Step 4: Table 6 will apply the Euclidean distance formula $D(\mathbb{PM}_\epsilon, \Omega_{\mathbb{I}})$ to find the similarity between each alternative Θ_ϵ, $\epsilon = 1, 2, \ldots 6$, and the ideal sphere $\Omega_{\mathbb{I}} = (1, 0, 0; 1)$.

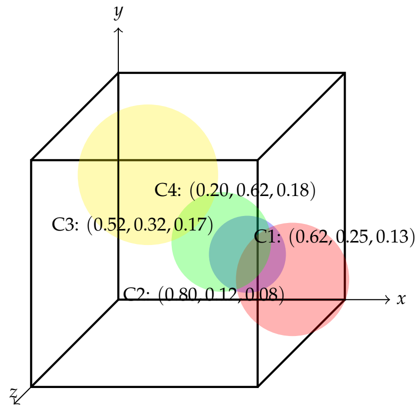
Step 5: The ranking results with proposed method and existing methods are shown in Table 7.

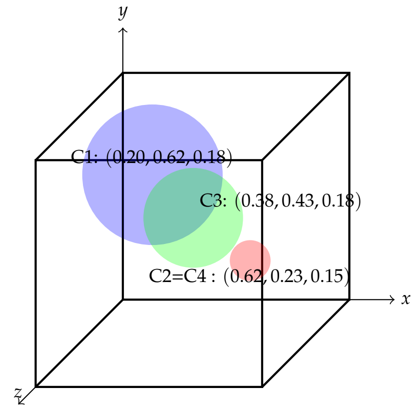
$Y^{\psi,\eta}$ / D(Θ_ϵ, $\Omega_{\mathbb{I}}$)	D(Θ_1, $\Omega_{\mathbb{I}}$)	D(Θ_2, $\Omega_{\mathbb{I}}$)	D(Θ_3, $\Omega_{\mathbb{I}}$)	(Θ_4, $\Omega_{\mathbb{I}}$)	D(Θ_5, $\Omega_{\mathbb{I}}$)	D(Θ_6, $\Omega_{\mathbb{I}}$)
$Y^{\psi=1,\eta=1}$	-0.585	-0.501	-0.482	-0.622	-0.499	-0.339
$Y^{\psi=1,\eta=2}$	-0.038	-0.012	0.036	-0.043	0.053	0.096
$Y^{\psi=2,\eta=2}$	0.213	0.264	0.308	0.234	0.276	0.354
$Y^{\psi=2,\eta=1}$	-0.128	-0.008	0.038	-0.095	-0.041	0.139
$Y^{\psi=2,\eta=3}$	0.341	0.351	0.408	0.353	0.382	0.432
$Y^{\psi=3,\eta=2}$	-0.228	-0.077	-0.002	-0.157	-0.093	0.094

TABLE 6. Euclidean Distance between alternatives Θ_ϵ and positive ideal sphere $\Omega_{\mathbb{I}}$=(1,0,0;1)

6.1. **Visualization.** In this section we visualize the spherical picture fuzzy sets for each criteria C_1 - C_4 given



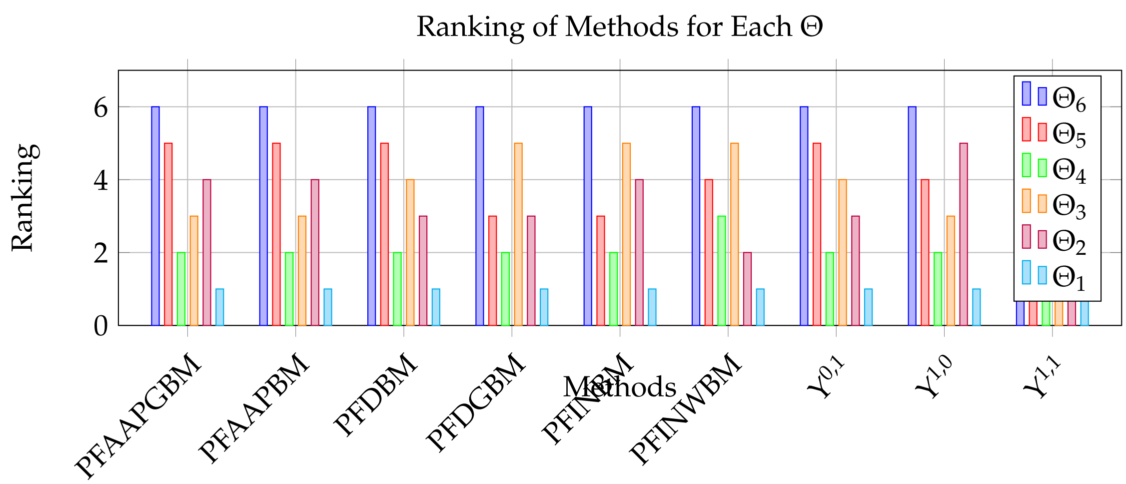




6.2. **Comparative Analysis and Limitations.** We compare our results with the existing methods of PFAAPGBM [29], PFAAPBM [29], PFDBM [16] PFDGBM [16], PFINBM [26] and PFINWBM [26] their visualization given. Table 7 displays the ranking orders. It is clear that the suggested method and the ranking results from the current methods are nearly identical. This confirms even more how applicable the suggested techniques are. The suggested approach produces more accurate findings than the current method by getting over the restriction of averaging the values of the decision makers.

Methods	Ranking	Best Algorithm
PFAAPGBM [29]	$\Theta_6 > \Theta_5 > \Theta_2 > \Theta_3 > \Theta_4 > \Theta_1$	Θ_1
PFAAPBM [29]	$\Theta_6 > \Theta_5 > \Theta_2 > \Theta_3 > \Theta_4 > \Theta_1$	Θ_1
PFDBM [16]	$\Theta_6 > \Theta_5 > \Theta_3 > \Theta_2 > \Theta_4 > \Theta_1$	Θ_1
PFDGBM [16]	$\Theta_6 > \Theta_3 > \Theta_5 > \Theta_2 > \Theta_4 > \Theta_1$	Θ_1
PFINBM [26]	$\Theta_6 > \Theta_3 > \Theta_2 > \Theta_5 > \Theta_4 > \Theta_1$	Θ_1
PFINWBM [26]	$\Theta_6 > \Theta_3 > \Theta_5 > \Theta_4 > \Theta_2 > \Theta_1$	Θ_1
Proposed $Y^{0,1}$	$\Theta_6 > \Theta_5 > \Theta_3 > \Theta_2 > \Theta_4 > \Theta_1$	Θ_1
Proposed $Y^{1,0}$	$\Theta_6 > \Theta_2 > \Theta_5 > \Theta_3 > \Theta_4 > \Theta_1$	Θ_1
Proposed $Y^{1,1}$	$\Theta_6 > \Theta_5 > \Theta_2 > \Theta_3 > \Theta_4 > \Theta_1$	Θ_1

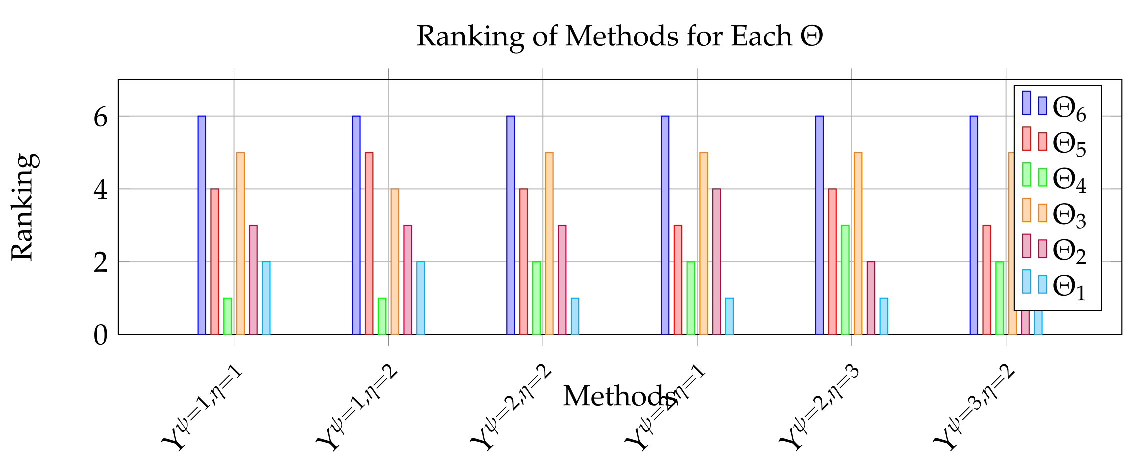
Table 7. The ranking results with different methods



We utilize several values for the parameters of the $\Upsilon^{\psi,\eta}$ operator to show how the parameters ψ and η influence the case. Table 8 displays the outcomes of the ranking. Table 8 illustrates how the ranking of the hyperparameter optimization and machine learning algorithm selection using various values of parameters ψ and η in the aggregation process differs slightly, but Θ_6 is the best algorithm for all combinations of parameters.

Methods	Ranking	Best Algorithm
$\Upsilon^{\psi=1,\eta=1}$	$\Theta_6 \succ \Theta_3 \succ \Theta_5 \succ \Theta_2 \succ \Theta_1 \succ \Theta_4$	Θ_4
$\Upsilon^{\psi=1,\eta=2}$	$\Theta_6 \succ \Theta_5 \succ \Theta_3 \succ \Theta_2 \succ \Theta_1 \succ \Theta_4$	Θ_4
$\Upsilon^{\psi=2,\eta=2}$	$\Theta_6 \succ \Theta_3 \succ \Theta_5 \succ \Theta_2 \succ \Theta_4 \succ \Theta_1$	Θ_1
$\Upsilon^{\psi=2,\eta=1}$	$\Theta_6 \succ \Theta_3 \succ \Theta_2 \succ \Theta_5 \succ \Theta_4 \succ \Theta_1$	Θ_1
$\Upsilon^{\psi=2,\eta=3}$	$\Theta_6 \succ \Theta_3 \succ \Theta_5 \succ \Theta_4 \succ \Theta_2 \succ \Theta_1$	Θ_1
$\Upsilon^{\psi=3,\eta=2}$	$\Theta_6 \succ \Theta_3 \succ \Theta_2 \succ \Theta_5 \succ \Theta_4 \succ \Theta_1$	Θ_1

Table 8. Ranking results of different parameters



In MCDM, the decision maker's involvement plays a crucial role in determining the weights and preferences associated with different criteria. It has been suggested that the decision maker's influence should be greater than 1 to enable the creation of a sphere representation in SPFSs. This requirement reflects the need for a significant level of involvement to ensure the meaningful representation of preferences and uncertainties. Understanding the limitations of SPFSs is essential for their effective utilization in MCDM. By addressing constraints such as the decision maker's involvement, researchers and practitioners can enhance the applicability and reliability of SPFSs in real-world decision-making contexts.

7. Conclusion

This paper uses Multi-Criteria Decision Making (MCDM) approaches to give a comprehensive framework for machine learning algorithm selection and hyperparameter optimization. We have aided informed decision-making for maximizing model performance through a series of methodical processes, such as weighing criteria, choosing decision makers, evaluating language, converting to fuzzy numbers for spherical pictures, using the BM operator, measuring Euclidean distance and ranking algorithms. Table 6 displays the total rating of algorithms, which represents the culmination of our efforts. With a clear hierarchy of algorithm performance, the table identifies the optimal algorithm based on assessments and similarity computations. It is noteworthy that, for various Y operator parameter settings, Θ_1 constantly shows up as the best-performing algorithm.

The significance of using structured decision-making frameworks to manage the complexity of algorithm selection and hyperparameter tuning is highlighted by this study. We have shown a reliable strategy for increasing model efficacy in machine learning tasks by combining MCDM approaches with cutting-edge methods like the BM operator and spherical picture fuzzy numbers. Our methodology will be a useful resource going ahead for scholars and practitioners looking to optimize the processes of hyperparameter tweaking and algorithm selection. Stakeholders can enhance model performance and decision-making results in machine learning applications by utilizing the insights obtained from this study to make better-informed judgments.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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