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Computational Analysis of Certified Reinforcement Numbers Across Specialized Graph Classes

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Abstract. A certified dominating set D of a graph G is a dominating set in which every vertex in D must have either no neighbors or at least two neighbors in $V\backslash D$, where V denotes the set of all vertices in G. A certified domination number of G represented by $\gamma_{cer}(G)$ is defined as the smallest size of such a certified dominating set of G. The reinforcement number $r(G)$ is defined to be the cardinality of minimum number of edges $F \subset E(\bar{G})$ such that $\gamma(G+F) < \gamma(G)$, broadened this parameter to encompass certified domination and we define certified reinforcement number of a graph G, $r_{cer}(G)$ to be the cardinality of the minimum number of edges $F \subset E(\bar{G})$ such that $\gamma_{cer}(G+F) < \gamma_{cer}(G)$ that is minimum number of edges to be added to decrease the certified domination number of G at least by one. In this paper, we characterize the graph G for which $r_{cer}(G) = 1$ and determine the values of certified reinforcement number for various classes of graphs.

1. Introduction

A dominating set in a graph G is a set. $R \subseteq V$ with the property that for each vertex $u \in V\backslash R$, there exists at least a vertex $x \in R$ adjacent to u. The least cardinality among all dominating sets of G is the domination number $\gamma(G)$, and R is called a γ-set of G if R is minimum. Haynes [9] presented a key concept in graph theory, widely recognized and researched as domination in

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graphs. Numerous advanced studies are being conducted across various aspects of domination terminology: [1, 5, 8, 13]. One of the most recent concepts is certified domination, introduced by Magda Dettlaff et al. [3].

A certified dominating set is defined as $D \subseteq V$, where D is a dominating set of a graph G, and each vertex in D has either no neighbors or at least two neighbors in $V \backslash D$. The certified domination number $\gamma_{cer}(G)$ is the least cardinality of a certified dominating set of G, and D is the $\gamma_{cer}(G)$-set of G if D is minimal. Further results on this parameter can be found in [4, 15–17].

Recently, numerous studies have explored how the domination parameter of a graph changes with the addition or removal of edges or vertices. Fink et al. [6] introduced the concept of the bondage number for a graph G. The bondage number b(G) is defined to be the cardinality of the minimum number of edges. $F \subset E(G)$ such that $\gamma(G-F) > \gamma(G)$. Additionally, sharp bounds for $b(G)$ were established, and exact values for various classes of graphs were determined in [7]. In [19, 20], the authors introduced the certified bondage number, which is defined as the certified bondage number of a graph G, $b^{+}_{cer}(G)$ $[b^{-}_{cer}(G)]$ to be the cardinality of the minimum number of edges $F \subset E(G)$ such that $\gamma_{cer}(G-F) > \gamma_{cer}(G)$ $[\gamma_{cer}(G-F) < \gamma_{cer}(G)]$. To increase or decrease the certified domination number of G, one must remove the minimum number of edges. Mynhardt and Kok introduced the concept of the reinforcement number to study the impact of edge addition on the domination number of a graph [19]. The reinforcement number $r(G)$ is defined to be the cardinality of the minimum number of edges. $F \subset E(\bar{G})$ such that $\gamma(G+F) < \gamma(G)$ [11, 14, 18].

In this paper we introduce the certified reinforcement number of a graph G, which we denote by $r_{cer}(G)$ and define to be the cardinality of the minimum number of edges $F \subset E(\bar{G})$ such that $\gamma_{cer}(G+F) < \gamma_{cer}(G)$, which is the minimum number of edges to be added to decrease the certified domination number of G at least by one. Also, we characterize the graph G for which $r_{cer}(G) = 1$ and determine the values of the certified reinforcement number for various classes of graphs.

1.1. **Motivation.** The goal of studying the certified reinforcement number of graphs is to improve and optimize network structures with little human input, making sure that they are both efficient and strong. Certified domination, in which vertices in the dominating set must meet certain neighbor constraints, is useful for real-life situations like making sure that communication and resource allocation systems are stable or redundant. The certified reinforcement number quantifies the least number of edges required to reduce the certified domination number, providing insights into improving network performance while minimizing cost. By exploring this parameter across various graph classes, this research bridges theoretical advancements with practical applications in designing resilient and efficient networks.

1.2. **Novelty.** The novelty of this work lies in extending the concept of reinforcement in graph theory to the domain of certified domination, introducing the certified reinforcement number $r_{cer}(G)$, which measures the least number of edges required to decrease the certified domination number of a graph. Traditional reinforcement focuses on lowering the standard domination number. The certified variant, on the other hand, adds extra structural constraints that are more useful in real life, like making sure networks are robust and have backups. This study not only expands on previous ideas by finding $r_{cer}(G)$ for different types of graphs, but it also gives us new theoretical insights and tools for improving network performance when domination requirements are stricter.

2. Notation

Let $G = (V, E)$ denote a connected, simple graph of order $|V| = n$. We use Harary [10] for graph theoretic notation. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$; the closed neighborhood of S is $N[S] = N(S) \cup S$; $N_X(v)$ denotes the neighbors of v in X, where X is a subset of V; and the *private neighborhood* $pn(v, S)$ of a vertex $v \in S$ is defined by $pn(v, S) = \{u \in V - S : N(u) \cap S = \{v\}\}$.

A path is a walk with no repeated vertices, and a non-trivial closed path is called a cycle. A connected graph without any cycles is referred to as a tree. In a tree, a vertex with degree one is termed a *leaf*, while a vertex directly connected to one or more leaves is called a *support*. A support connected to only one leaf is classified as a *weak support*, whereas a support connected to two or more leaves is termed a *strong support*. A *complete binary tree* is a special type of tree where all leaves are at the same depth, and every internal vertex has a degree of three. In a complete binary tree rooted at vertex v, the set of all vertices situated at a depth of k is referred to as the vertices at level k.

3. Main Results

Theorem 3.1. *Let G be a graph of order n and D be a γ_{cer}-set of G and G' be a subgraph induced by D.Choose a vertex $x \in D$ such that $|pn(x, D)| \leq |pn(y, D)|$ for all $y \in D$.Then*

$$r_{cer}(G) = \begin{cases} (|pn(x, D)| + 1 & \text{if } x \text{ is an isolated vertex in } G' \\ (|pn(x, D)| & \text{otherwise.} \end{cases}$$

Proof. Let G be a graph and D be a γ_{cer}-set of G, and let G' be a subgraph induced by D.

Choose a vertex $x \in D$, so that $|pn(x, D)| \leq |pn(y, D)|$ for all $y \in D$.

Let $x_1, x_2, x_3, \cdots, x_m \in pn(x, D)$. Let G_1 be a graph obtained by joining the vertices of $pn(x, D)$ to a vertex, say $y \in D$, resulting in $pn(x, D) = \emptyset$.

Note that m number of edges are added in G. If x is not an isolated vertex in G', then x is adjacent to some vertex of $D_1 = D - \{x\}$, where D_1 is a γ_{cer}-set of G_1.

Hence $r_{cer}(G) = m = |pn(x, D)|$. Otherwise, x is an isolated vertex in G', so x is not dominated by $D_1 = D - \{x\}$.

Hence, adding another edge xy results in D_1 being a γ_{cer}-set of $G_1 + xy$. Therefore, $r_{cer}(G) = m + 1 = |pn(x, D)| + 1$. □

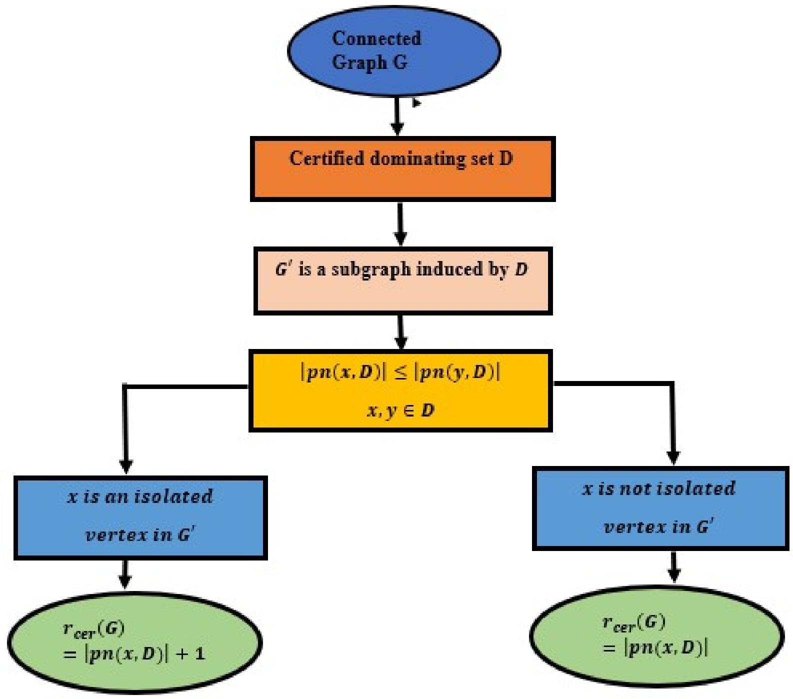

Figure 1. Logical frame work for the proof of Theorem 3.1

Theorem 3.2. *Let G be a graph of order n and D be a γ_{cer}- set of G, then $r_{cer}(G) = 1$ if and only if one of the following holds.*
(i) $|pn(v,D)| = 0$ for at least one vertex $v \in D$.
(ii) $|pn(v,D)| = 1$ for at least one vertex $v \in D$ with $N(v) \cap D = \emptyset$.
(iii) $\triangle(G) = n$-2.

Proof. Let G be a graph of order n and D be a γ_{cer}-set of G. Let $r_{cer}(G) = 1$.

Assuming that there is a vertex $v \in D$, so that $|pn(v,D)| = 0$. Then either $N_{V-D}(v) = \emptyset$ or $N_{V-D}(v) > 1$.

Now, joining the vertex v to the vertex $u \in D \backslash N(v)$ by an edge $e = uv$ results in a graph G', and the γ_{cer}-set of G' is $D' \leq D - \{v\}$.

Therefore $|D'| < |D|$. Otherwise $|pn(v,D)| > 0$ for all $v \in D$, here either $|pn(v,D)| = 1$ or $|pn(v,D)| > 1$.

In the former case, choose $u \in pn(v,D)$. By adding an edge $e = uw$ for some $w \in D$, we ensure that $D - \{v\}$ becomes a γ_{cer}-set of $G + e$. This condition is possible only if $N(v) \cap D \neq \emptyset$. Hence (ii) is proved.

In the lateral case, $|pn(v,D)| > 1$ for all $v \in D$. In this case, suppose $\triangle(G) = n-3$ and $deg(v) = n-3$. Then $v \in \triangle(G)$, and except for $v_1, v_2 \in V(G)$, all other vertices are adjacent to v.

Suppose $v \in D$, then there exists one more vertex in D, say $w \in D$, such that $|pn(w,D)| > 1$. If not, by (i) and (ii) we are through, hence $|pn(w,D)| > 1$.

Now, by Theorem 3.1, we have $r_{cer}(G) = |pn(w, D)| + 1 > 2$, a contradiction. Hence $\triangle(G)$ = n-2.

Conversely, suppose $|pn(v, D)| = 0$ for at least one vertex $v \in D$. Then either $N(v) \cap D = \emptyset$ or $N(v) \cap D \neq \emptyset$.

In both cases, joining the vertices u and v for some $u \in D \backslash N(v)$ results in a graph G' with a γ_{cer}-set $D' = D - \{v\}$. Hence $|D| > |D'|$. Therefore, $r_{cer}(G) = 1$. □

Theorem 3.3. *[4] For cycle C_n, $n \geq 3$, $\gamma_{cer}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Theorem 3.4. *For cycle C_n, $n \geq 3$,* $r_{cer}(C_n) = \begin{cases} 1 & \text{if } n \equiv 1 \ (mod\ 3) \\ 2 & \text{if } n \equiv 2 \ (mod\ 3) \\ 3 & \text{if } n \equiv 0 \ (mod\ 3) \end{cases}$

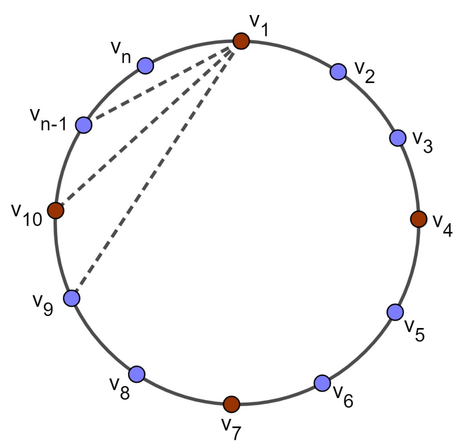

Figure 2. $r_{cer}(C_{12}) = 3$

Proof. Let $G \cong C_n$, $n \geq 3$, by Theorem 3.3, $\gamma_{cer}(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Let D be a γ_{cer}-set of G and each vertex in D dominates at the most three vertices including itself.

Hence $|pn(v, D)| \leq 2$, for all $v \in D$. Now consider the following cases.

Case(i): $n \equiv 1$ (mod 3)

In this case, $D = \{v_{3k-2} : 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil\}$ is a γ_{cer}-set of G.

Note that $|pn(v_n, D)| = 1$ and also v_n is adjacent to v_1 where $v_1 \in D$.

Now by the Theorem 3.1, adding the edge v_1v_{n-1} results a graph G_1 with γ_{cer}-set $D_1 = D - \{v_n\}$.

Therefore, $\gamma_{cer}(G_1) = \left\lceil \frac{n}{3} \right\rceil - 1$. Hence $r_{cer}(G) = 1$.

Case(ii): $n \equiv 2$ (mod 3)

Here $D = \{v_{3k-2} : 1 \leq k \leq \left\lceil \frac{n+1}{3} \right\rceil\}$ is a γ_{cer}-set of G.

Also, $|pn(v_1, D)| = 1$ and $|pn(v_{n-1}, D)| = 1$. Now by the Theorem 3.1, adding the edges $v_{n-1}v_1$ and $v_{n-2}v_1$ results a graph G_1 with γ_{cer}-set $D_1 = D - \{v_{n-1}\}$.

Therefore, $\gamma_{cer}(G_1) = \left\lceil \frac{n}{3} \right\rceil - 1$. Hence $r_{cer}(G) = 2$.

Case(iii): $n \equiv 0 \pmod 3$

Here $D = \{v_{3k-2} : 1 \leq k \leq \lceil \frac{n}{3} \rceil\}$ is a γ_{cer}-set of G.

Also $|pn(v, D)| = 2$, for every $v \in D$. Now by the Theorem 3.1, joining the vertex v_{n-2} and the private neighbors of v_{n-2} [i.e. v_{n-1}, v_{n-3}] to the vertex $v_1 \in D$ results a graph G_1 with γ_{cer}-set $D_1 = D - \{v_{n-2}\}$.

Therefore, $\gamma_{cer}(G_1) = \lceil \frac{n}{3} \rceil - 1$. Hence $r_{cer}(G) = 3$. □

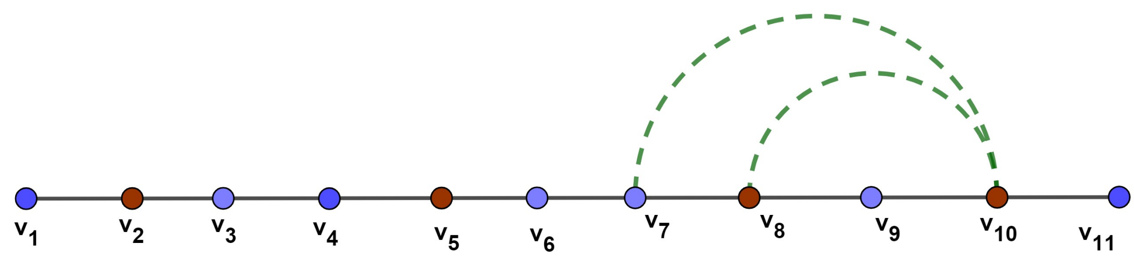

FIGURE 3. $r_{cer}(P_{11}) = 2$

Theorem 3.5. *[4] For path P_n,* $\gamma_{cer}(P_n) = \begin{cases} 1 & \text{if } n=1 \text{ or } n=3; \\ 2 & \text{if } n = 2; \\ 4 & \text{if } n = 4; \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$

Theorem 3.6. *For path P_n, $n > 4$,* $r_{cer}(P_n) = \begin{cases} 1 & \text{if } n \equiv 1 \ (mod\ 3); \\ 2 & \text{if } n \equiv 2 \ (mod\ 3); \\ 3 & \text{if } n \equiv 0 \ (mod\ 3). \end{cases}$

Proof. Let $G \cong P_n$, $n > 4$, by Theorem 3.5, $\gamma_{cer}(P_n) = \lceil \frac{n}{3} \rceil$. Let D be a γ_{cer}-set of G.

Case(i): $n \equiv 1 \pmod 3$.

In this case, $D = \{v_{3k-1} : 1 \leq k \leq \frac{n-4}{3}\} \cup \{v_{n-1}, v_{n-3}\}$ is a γ_{cer}-set of G.

Also, $|pn(v_{n-1}, D)| = 1$ and $|pn(v_{n-3}, D)| = 0$. So by Theorem 3.1, adding the edge $e = v_{n-1}v_{n-3}$ results a graph G_1 with certified dominating set $D' = D - \{v_{n-3}\}$.

This implies $\gamma_{cer}(G_1) = \lceil \frac{n}{3} \rceil - 1$. Hence $r_{cer}(G) = 1$.

Case(ii): $n \equiv 2 \pmod 3$.

Here $D = \{v_{3k-1} : 1 \leq k \leq \frac{n-2}{3}\} \cup \{v_{n-1}\}$ is a γ_{cer}-set of G. Also $|pn(v_{n-1}, D)| = |pn(v_{n-3}, D)| = 1$.

Now by the Theorem 3.1, joining the vertex v_{n-3} and the vertex $v_{n-4} \in pn(v_{n-3}, D)$ to any of the vertex of D say v_{n-1} results a graph G_1 with γ_{cer}-set $D_1 = D - \{v_{n-3}\}$.

Therefore, $\gamma_{cer}(G_1) = \lceil \frac{n}{3} \rceil - 1$. Hence $r_{cer}(G) = 2$.

Case(iii): $n \equiv 0 \pmod 3$.

Here $D = \{v_{3k-1} : 1 \leq k \leq \lceil \frac{n}{3} \rceil\}$ is a γ_{cer}-set of G and each vertex in D has exactly two private neighbours.

Hence by the Theorem 3.1, joining the vertex v_{n-4} and the vertices $v_{n-3}, v_{n-5} \in pn(v_{n-4}, D)$ to any of the vertex of D say v_{n-1} results a graph G_1 with γ_{cer}-set $D_1 = D - \{v_{n-4}\}$.

Therefore, $\gamma_{cer}(G_1) = \lceil \frac{n}{3} \rceil - 1$. Hence $r_{cer}(G) = 3$. □

We derive the follow lemma in order to find certified reinforcement number for binary trees T.

Lemma 3.1. *For binary tree T of level k, $0 \leq k \leq n$,*

$$\gamma_{cer}(T) = \begin{cases} \frac{(2^{k+2}+3)}{7} & \text{if } k \equiv 0, 2 \ (mod\ 3); \\ \frac{(2^{k+2}-2)}{7} & \text{if } k \equiv 1 \ (mod\ 3). \end{cases}$$

Proof. Let S_i be the set of vertices in level i. Then $|S_i| = 2^i$. We define

$$D = \begin{cases} S_0 \cup S_2 \cup S_5 \cup \cdots \cup S_{k-4} \cup S_{k-1} \ for\ k \equiv 0, 2 \pmod 3; \\ S_1 \cup S_4 \cup S_7 \cup \cdots \cup S_{k-4} \cup S_{k-1} \ for\ k \equiv 1 \pmod 3. \end{cases}$$

Clearly D is a γ_{cer}-set of T. Hence

$$\gamma_{cer}(T) = \begin{cases} 2^0 + 2^2 + 2^5 + \cdots + 2^{k-4} + 2^{k-1} for\ k \equiv 0, 2 \ (\text{mod}3); \\ 2^1 + 2^4 + 2^7 + \cdots + 2^{k-4} + 2^{k-1} \ for\ k \equiv 1 \pmod 3. \end{cases}$$

Case (i): $k \equiv 0, 2$ (mod3). Now,

$$\begin{aligned} \gamma_{cer}(T) &= 2^0 + 2^2 + 2^5 + \cdots + 2^{k-4} + 2^{k-1} \\ &= 1 + 4(\frac{2^k - 1}{7}) \\ &= \frac{2^{k+2} + 3}{7}. \end{aligned}$$

Case(ii): $k \equiv 1 \pmod 3$. Now,

$$\begin{aligned} \gamma_{cer}(T) &= 2^1 + 2^4 + 2^7 + \cdots + 2^{k-4} + 2^{k-1} \\ &= \frac{2(2^{k+1} - 1)}{7} \\ &= \frac{2^{k+2} - 2}{7}. \end{aligned}$$

□

Theorem 3.7. *For a binary tree T of level k, $0 \leq k \leq n$,*

$$r_{cer}(T) = \begin{cases} 1 & \text{if } k \equiv 0, 2 \ (mod\ 3); \\ 3 & \text{if } k \equiv 1 \ (mod\ 3). \end{cases}$$

Proof. Let T be a binary tree of level k and S_i be the set of vertices in level i. Let $\{v_0\}, \{v_{11}, v_{12}\}$, $\{v_{21}, v_{22}, v_{23}, v_{24}\}$, $\{v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}, v_{37}, v_{38}\} \cdots$ be the vertex set in level 0,1,2,3,... respectively and D be a γ_{cer}-set of T.

Every vertex $v \in D$ has at most three private neighbors that is $|pn(v, D)| \leq 3$ for all $v \in D$.

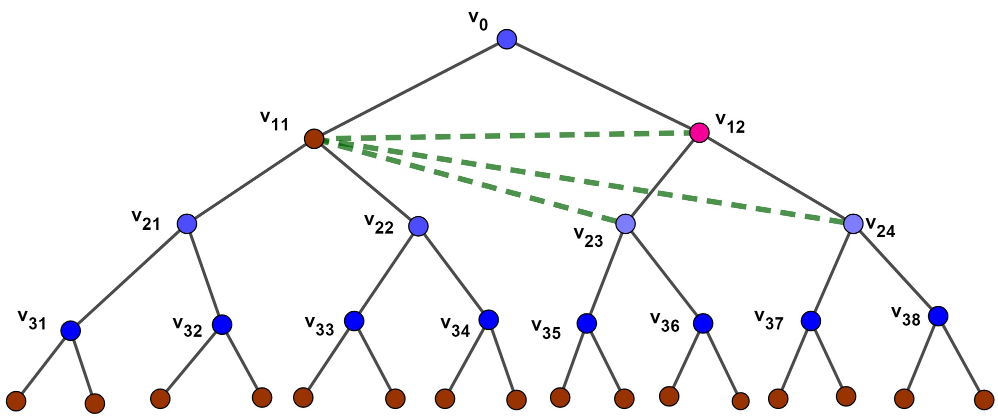

FIGURE 4. $r_{cer}(P_{11}) = 2$

Case(i): $k \equiv 0, 2 \pmod 3$.

By the above lemma, $\gamma_{cer}(T) = \frac{2^{k+2}+3}{7}$ for $k \equiv 0, 2 \pmod 3$ and $v_0 \in D$ with $|pn(v_0, D)| = 0$ as level 2 vertices are in D.

Now joining v_0 to one of the vertices of D results a graph T' with γ_{cer}-set $D' = D - \{v_0\}$.

Hence $r_{cer}(T) = 1$ for $k \equiv 0, 2 \pmod 3$.

Case(ii): $k \equiv 1 \pmod 3$.

By the above lemma, $\gamma_{cer}(T) = \frac{2^{k+2}-2}{7}$ for $k \equiv 1 \pmod 3$ and $v_{11} \in D$ with $|pn(v_{11}, D))| = 2$.

Here $v_{21}, v_{22} \in pn(v_{11}, D)$ and each vertex in G' [where G' is a subgraph induced by D] is isolated , by Theorem 3.1, $r_{cer}(T) = 3$. □

4. CONCLUSION

In this paper, we explored the concept of the certified reinforcement number $r_{cer}(G)$, which represents the minimum number of edges that must be added to a graph G to reduce its certified domination number $\gamma_{cer}(G)$. We characterized the cases where $r_{cer}(G) = 1$ and established the values of $r_{cer}(G)$ in terms of private neighbors of vertices in a minimum certified dominating set. Furthermore, we determined the certified reinforcement numbers for various classes of graphs, providing insights into how edge additions influence certified domination. These findings contribute to a deeper understanding of domination-based parameters in graph theory and their structural implications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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