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A Novel Approach to Fractal Generation through Strong Coupled Fixed Points in Intuitionistic Fuzzy Metric Spaces

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ABSTRACT. In this manuscript, we explore the concept of strong-coupled fixed points in the context of intuitionistic fuzzy metric spaces (IFMS). Our approach is grounded in the idea of intuitionistic fuzzy contractive couplings (IFCCs), which provide a framework for understanding fixed points in fuzzy settings. We begin by introducing a novel formulation of coupling, which combines the principles of coupled fuzzy contractions with cyclic mappings. This combination leads to a more generalized and effective method of identifying strong-coupled fixed points, extending previous results in fuzzy metric spaces. A key contribution to this paper is the proof of the existence of a unique strong-coupled fixed point. We establish this result through rigorous theoretical analysis and provide a corollary that strengthens the foundation of our work. Several non-trivial examples are presented to demonstrate the applicability of the theory and the robustness of the strong-coupled fixed point in various scenarios. Additionally, we present a practical application of our findings: the construction of a strong-coupled fractal set within the framework of intuitionistic fuzzy metric spaces. This is achieved by applying an intuitionistic iterated function system (IIFS), which is based on a family of intuitionistic fuzzy contractive couplings. The fractal generation process is illustrated through several examples, demonstrating the theoretical results in action. To further solidify the applicability of our approach, we introduce an intuitionistic fuzzy version of the Hausdorff distance between compact sets, a crucial tool in measuring the "closeness" of sets within the intuitionistic fuzzy context. Several examples are provided to clarify the fractal generation process, showing how the intuitionistic fuzzy metrics and couplings contribute to the creation of self-similar fractals. This work not only enhances the understanding of fixed points in intuitionistic fuzzy spaces but also provides new insights into their application in fractal geometry, offering both theoretical advancements and practical tools for future research in this area.

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1. Introduction

There are several uses for fractals in computer graphics, quantum physics, biology, and other scientific fields. Ancient mathematicians began using the theory of discrete dynamical systems, often known as the notion of iterated function systems, extensively to create self-similar and fractal sets. In addition to being a fractal set's advantage, self-similarity may also be utilized to create fractal and self-similar sets. Hutchinson [1] used the Banach contraction principle to construct a theory known as Hutchinson-Barnsley (HB) theory, which is based in the construction of an unvarying compact subset of a metric space (MS) generated by the iterated function system (IFS) of contractive mappings.

Fractal image compression, also known as fractal image encoding, is a popular application of fractal theory, particularly the use of self-similarity property. One of the primary goals of this perception is to consider the fractal transform operator, which is directly obtained from the perception by examining the fractal transform maps, and the undertaken image function will be approximated by the attractor of the associated contractive operator. Rajkumar and Uthayakumar [2] developed a fuzzy point distance function and used it to create a comprehensive MS of fuzzy-valued image functions. They added a fractal transform operator to the newly generated complete MS.

The theory of fuzzy sets (FSs) and other concepts related to fuzzy metric space (FMS) were pioneered by Zadeh [3], and their properties have been analyzed by many succeeding mathematicians [4, 5]. Park [6] developed the concept of intuitionistic fuzzy metric space (IFMS), which was a common idea of fuzzy metric spaces (FMS) introduced by George and Veeramani [4]. There are several research papers available on the concept of generalized fuzzy topological spaces, which are like FMS.

Coker [7] introduced intuitionistic fuzzy topological space (IFTS). In this connection, Saadati and Park [8] launched the notions of intuitionistic fuzzy normed spaces (IFNS), and some of the results relating to the convergence of sequences in these spaces were encountered by several mathematicians [9 -12].

Then, the idea of fuzzy Banach spaces and their quotients was introduced by Saadati and Vaezpour [13]. Moreover, a study of intuitionistic fuzzy metric spaces was carried out by Pandit et al. [14] and general structures like L-topological vector space were studied (e.g., [15]). Researchers provided the necessary and sufficient criteria for L-topological vector space to be L-fuzzy normable, along with a definition of L-fuzzy normed linear space [16].

Schweizer and Sklar [17] introduced the statistical MSs, with particular emphasis on triangle inequality. George, and Veeramani, [18] defined the Hausdorff topology in FMSs and proved Baire's theorem in this space. A technique for creating a Hausdorff fuzzy metric on the collection of nonempty compact subsets of a given FMS was presented by Rodríguez-López and

Romaguera [19]. Bhaskar and Lakshmikantham [20] generalized the FP results by using partial order. Kirk et al. [21] generalized the FP results by using cyclical contractive conditions. An SCFP theorem for a generalized coupling between two subsets of an MS was proven by Choudhury and Chakraborty [22]. Some new definitions of compatible mappings in IFMS were demonstrated by Alaca et al. [23] (cfr. [24-26] for more related results).

In this work, we study SCFP results for IFMS by introducing an intuitionistic fuzzy contractive coupling (IFCC). Next, using such couplings, an intuitionistic fuzzy iterated coupling system (IFICS) is shown. Furthermore, we apply to the Hutchinson-Barnsley operator to SCFs using the fixed-point result. In the subsequent work, we demonstrate that an existing fuzzy coupled fixed-point (FCFP) result is successfully generalized by our SCFP result. Examples are provided for both the FP theorem and the fractal-generating process.

2. Preliminaries

We utilize the following notations throughout the paper: The sets of all non-empty subsets and compact subsets for any topological space (Ξ, τ), are denoted by the symbols $\mathcal{P}(\Xi)$ and $\kappa(\Xi)$, respectively. $\mathbb{N}_{\mathfrak{n}}$ stands for the set of the first $\mathfrak{n}$ natural numbers.

Definition 2.1 [17]. A mapping $*: [0,1]^2 \longrightarrow [0,1]$ is said to be a continuous t-norm (CTN), if it verifies the following axioms:

i. $*$ is associative and commutative,

ii. $\vartheta * 1 = \vartheta$ for all $\vartheta \in [0,1]$,

iii. $\vartheta * b \leq c * d$ whenever $\vartheta \leq c$ and $b \leq d$ for each $\vartheta, b, c, d \in [0,1]$,

iv. $*$ is continuous.

Definition 2.2 [17]. A mapping Δ: $[0,1]^2 \longrightarrow [0,1]$ is said to be a continuous t-conorm (CTCN), if it satisfies the following assertions:

i. Δ is associative and commutative,

ii. $\vartheta \Delta 0 = \vartheta$ for all $\vartheta \in [0,1]$,

iii. $\vartheta \Delta b \leq c \Delta d$ whenever $\vartheta \leq c$ and $b \leq d$ for each $\vartheta, b, c, d \in [0,1]$,

iv. Δ is continuous.

Throughout the study, CTN $\vartheta * b = \vartheta \cdot b$ and CTCN $\vartheta \Delta b = \max\{\vartheta, b\}$ are denoted by $*_p$ and Δ_p, respectively. We have that $\Delta_p(a,b) \leq \Delta(a,b) for\ any\ a, b \in [0,1]\ and$ any CTCN Δ. We provide several definitions provided by George et al. [18].

Definition 2.3 [18]. A 3-tuple $(\Xi, \Phi, *)$ is said to be an FMS if Ξ set on $\Xi \times \Xi \times (0, \infty)$, fulfill the below conditions for all $\pi, \mathfrak{p}, \mathcal{Z} \in \Xi$ and $\mathfrak{t}, \mathfrak{g} > 0$;

FMS1. $\lim_{\mathfrak{t} \to 0} \Phi(\pi, \mathfrak{p}, \mathfrak{t}) = 0$

FMS2. $\Phi(\pi, \mathfrak{p}, \mathfrak{t}) = 1$ iff $\pi = \mathfrak{p}$,

FMS3. $\Phi(\pi, \mathfrak{p}, \mathfrak{t}) = \Phi(\mathfrak{p}, \pi, \mathfrak{t})$, FMS4. $\Phi(\pi,\mathfrak{p},\mathrm{t})*\Phi(\mathfrak{p},\mathcal{Z},\mathfrak{g})\leq\Phi(\pi,\mathcal{Z},\mathrm{t}+\mathfrak{g})$,

FMS5. $\Phi(\pi,\mathfrak{p},.):(0,\infty)\to(0,1]$ is continuous,

FMS6. for all $\pi,\mathfrak{p}\in\Xi,\lim_{\mathrm{t}\to\infty}\Phi(\pi,\mathfrak{p},\mathrm{t})=1$.

Definition 2.4 [18]. Let $(\Xi,\Phi,*)$ be an FMS. For $\mathrm{t}>0$ and $0<r<1$, $\Sigma(\pi,r,\mathrm{t})$ is an open ball with center $\pi\in\Xi$, shown as

$$\Sigma(\pi,r,\mathrm{t})=\{\mathfrak{p}\in\Xi:\Phi(\pi,\mathfrak{p},\mathrm{t})>1-r\}.$$

The family $\{\Sigma(\pi,r,\mathrm{t}):\pi\in\Xi,0<r<1,\mathrm{t}>0\}$ as a basis for a Hausdorff topology on Ξ has been proved in [18].

Definition 2.5 [18]. Let $(\Xi,\Phi,*)$ be an FMS.

i. If there exists some $\pi\in\Xi$, such that $\lim_{\mathfrak{n}\to\infty}\Phi(\pi_{\mathfrak{n}},\pi,\mathrm{t})=1$, for all $\mathrm{t}>0$, then

sequence $\{\pi_{\mathfrak{n}}\}\in\Xi$ is said to be a convergent sequence to π.

ii. If $\lim_{m,\mathfrak{n}\to\infty}\Phi(\pi_{\mathfrak{n}},\pi_m,\mathrm{t})=1$, for all $\mathrm{t}>0$, then $\{\pi_{\mathfrak{n}}\}\in\Xi$, called a Cauchy sequence. If every Cauchy sequence is convergent, then $(\Xi,\Phi,*)$ is said complete.

Definition 2.6 [20]. Let $\zeta:\Xi\times\Xi\to\Xi$ be a mapping. An element $(\pi,\mathfrak{p})\in\Xi\times\Xi$ is called coupled fixed point (CFP) of ζ if $\zeta(\pi,\mathfrak{p})=\pi$, and $\zeta(\mathfrak{p},\pi)=\mathfrak{p}$.

If $\pi=\mathfrak{p}$, then the CFP is said to be an SCFP, in which case we have $\zeta(\pi,\pi)=\pi$. The point $(\pi,\pi)\in\Xi\times\Xi$ (or simply $\pi\in\Xi$) is called a SCFP.

Definition 2.7 [22]. Let Ξ has two non-empty subsets Ω and Σ. Let be a mapping $\zeta:\Xi\times\Xi\to\Xi$. We denote with $\hat{\zeta}:\kappa(\Xi)\times\kappa(\Xi)\to\kappa(\Xi)$ the mapping defined as $\hat{\zeta}(C,D)=\{\zeta(\vartheta,b):\vartheta\in C,b\in D\}=\zeta(C\times D) for\ any\ C,D\in\kappa(\Xi)$. The mapping ζ is called "coupling concerning Ω and Σ" if $\zeta(\pi,\mathfrak{p})\in\Sigma$ and $\zeta(\mathfrak{p},\pi)\in\Omega$ whenever $\pi\in\Omega$ and $\mathfrak{p}\in\Sigma$.

Definition 2.8 [6]. A 5-tuple $(\Xi,\Phi,\Psi,*,\Delta)$ is called an IFMS if Ξ a nonempty set, $*$ is a CTN, Δ is CTCN, Φ,Ψ are FSs in $\Xi\times\Xi\times(0,\infty)$ satisfying the following conditions for all $\pi,\mathfrak{p},\mathcal{Z}\in\Xi$ and $\mathrm{t},\mathfrak{g}>0$:

FM1. $\Phi(\pi,\mathfrak{p},\mathrm{t})+\Psi(\pi,\mathfrak{p},\mathrm{t})\leq 1$,

FM2. for all $\pi,\mathfrak{p}\in\Xi,\lim_{\mathrm{t}\to 0}\Phi(\pi,\mathfrak{p},\mathrm{t})=0$,

FM3. $\Phi(\pi,\mathfrak{p},\mathrm{t})=1$ iff $\pi=\mathfrak{p}$,

FM4. $\Phi(\pi,\mathfrak{p},\mathrm{t})=\Phi(\mathfrak{p},\pi,\mathrm{t})$,

FM5. $\Phi(\pi,\mathfrak{p},\mathrm{t})*\Phi(\mathfrak{p},\mathcal{Z},\mathfrak{g})\leq\Phi(\pi,\mathcal{Z},\mathrm{t}+\mathfrak{g})$,

FM6. $\Phi(\pi,\mathfrak{p},.):(0,\infty)\to(0,1]$ is continuous,

FM7. for all $\pi,\mathfrak{p}\in\Xi,\lim_{\mathrm{t}\to\infty}\Phi(\pi,\mathfrak{p},\mathrm{t})=1$,

FM8. $for\ all\ \pi,\mathfrak{p}\in\Xi,\ \lim_{\mathrm{t}\to 0+}\Psi(\pi,\mathfrak{p},t)=1$,

FM9. $\Psi(\pi,\mathfrak{p},\mathrm{t})=0$ iff $\pi=\mathfrak{p}$,

FM10. $\Psi(\pi,\mathfrak{p},\mathrm{t})=\Psi(\mathfrak{p},\pi,\mathrm{t})$,

FM11. $\Psi(\pi,\mathfrak{p},\mathrm{t})\,\Delta\,\Psi(\mathfrak{p},\mathcal{Z},\mathfrak{g})\geq\Psi(\pi,\mathcal{Z},\mathrm{t}+\mathfrak{g})$, FM12. $\Psi(\pi,\mathfrak{p},.):(0,\infty)\to(0,1]$ is continuous,

FM13. for all $\pi,\mathfrak{p}\in\Xi$, $\lim_{t\to\infty}\Psi(\pi,\mathfrak{p},t)=0$.

Strictly speaking, we have that $(\Xi,\Phi,,*)$ must be a FMS. If $\Psi'=1-\Phi$ and further define $\Delta'(a,b)=1-*(1-a,1-b)$ *for any* $a,b\in[0,1]$, we point out that also $(\Xi,\Psi',,\Delta')$ is a FMS. Note that FM5 implies that $\Phi(\pi,\mathfrak{p},t)$ is increasing in $t\in(0,\infty)$. Similarly, FM11 implies that $\Psi(\mathfrak{p},\pi,t)$ is decreasing in $t\in(0,\infty)$. Some of the above properties will be tacitly used in the sequel. Slightly changing Definition 2.5, we now give the following definition:

Definition 2.9. (i)Let $\{\pi_\mathfrak{n}\}$ be a sequence in Ξ. If there exists some $\pi\in\Xi$ such that $\lim_{\mathfrak{n}\to\infty}\Phi(\pi_\mathfrak{n},\pi,t)=1$ and $\lim_{\mathfrak{n}\to\infty}\Psi(\pi_\mathfrak{n},\pi,t)=0$ for all t >0, then the sequence $\{\pi_\mathfrak{n}\}$ is said to be convergent to π.

(ii) If $\lim_{m,\mathfrak{n}\to\infty}\Phi(\pi_\mathfrak{n},\pi_m,t)=1$ and $\lim_{m,\mathfrak{n}\to\infty}\Psi(\pi_\mathfrak{n},\pi_m,t)=0$ for all $t>0$, then $\{\pi_\mathfrak{n}\}$ is called a Cauchy sequence. If every Cauchy sequence is convergent, then $(\Xi,\Phi,\Psi,*,\Delta)$ is said complete.

3. Coupled Fixed-Point

In the sequel we consider the topology defined in IFMS $(\Xi,\Phi,\Psi,*,\Delta)$ (cfr.,e.g., [27]). We recall that the set B(x,r,t) = {y∈Ξ:Φ(x,y,t)>1−r, Ψ(x,y,t) < r} is called open ball with center x and radius r with respect to t. Then this topology is defined as a topology on Ξ which has as base the family of open sets of the form {B(x,r,t):x∈Ξ, r∈(0,1), t>0}. The properties of this topology shall be tacitly used in the sequel. In this section, we study the SCF mapping in a complete IFMS.

Definition 3.1. Let $(\Xi,\Phi,\Psi,*,\Delta)$ be a IFMS and $\hat{\eta}:\kappa(\Xi)\times\kappa(\Xi)\to\kappa(\Xi)$ be mapping. $\Omega\in\kappa(\Xi)$ is called a SCF of $\hat{\eta}$ if $\hat{\eta}(\Omega,\Omega)=\Omega$.

Definition 3.2. An IFICS consists of a IFMS $(\Xi,\Phi,\Psi,*,\Delta)$, with two closed subsets Ω,Σ of Ξ and of a finite collection of couplings $\zeta_i:\Xi\times\Xi\to\Xi$ concerning Ω,Σ for all $i\in\mathbb{N}_\mathfrak{n}$. We denote it by $\langle(\Xi,\Phi,\Psi,*,\Delta);\Omega,\Sigma,\zeta_i,i\in\mathbb{N}_\mathfrak{n}\rangle$.

Definition 3.3. Let $(\Xi,\Phi,\Psi,*,\Delta)$ be an IFMS and Ω and Σ be two non-empty subsets of Ξ. We call a coupling $\zeta:\Xi\times\Xi\to\Xi$ concerning Ω and Σ is an IFCC if there exists $\sigma\in(0,1)$ such that

$$\Phi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma t)\geq\left(\Phi(\pi,\mathfrak{u},t)\right)^{\frac{1}{2}}*\left(\Phi(\mathfrak{p},\mathfrak{v},t)\right)^{\frac{1}{2}},\tag{1}$$

$$\Psi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma t)\leq\left(\Psi(\pi,\mathfrak{u},t)\right)^{\frac{1}{2}}\Delta\left(\Psi(\mathfrak{p},\mathfrak{v},t)\right)^{\frac{1}{2}}.\tag{2}$$

where $\pi,\mathfrak{v}\in\Omega$ and $\mathfrak{u},\mathfrak{p}\in\Sigma$. Here the constant σ is the CF.

Theorem 3.1. *Assume that the complete IFMS* $(\Xi,\Phi,\Psi,*,\Delta_p)$ *has two subsets* Ω,Σ *closed of* $\Xi,\Omega\cap\Sigma\neq\phi$. *Let* $\zeta:\Xi\times\Xi\to\Xi$ *be an IFCC concerning* Ω *and* Σ. *Let* $*$ *be such that* $a*b\geq a*_p b$ *for any* $a,b\in[0,1]$. *For any* $\pi_0\in\Omega$ *and* $\mathfrak{p}_0\in\Sigma$, *define* $\{\pi_\mathfrak{n}\},\{\mathfrak{p}_\mathfrak{n}\}$ *two sequences as* $\pi_{\mathfrak{n}+1}=\zeta(\mathfrak{p}_\mathfrak{n},\pi_\mathfrak{n})$ *and* $\mathfrak{p}_{\mathfrak{n}+1}=\zeta(\pi_\mathfrak{n},\mathfrak{p}_\mathfrak{n})$ for all $\mathfrak{n}=0,1,2,\ldots$ *Then they converge to the unique SCFP.*

Proof. By definition 2.8, the two sequences $\{\pi_\mathfrak{n}\},\{\mathfrak{p}_\mathfrak{n}\}$ as defined in the statement fulfill that $\pi_\mathfrak{n}\in\Omega$ and $\mathfrak{p}_\mathfrak{n}\in\Sigma$ for all $\mathfrak{n}=0,1,2,\ldots$. Then

$$\Phi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = \Phi(\zeta(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1}),\zeta(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1}),\mathfrak{t}) \geq \left(\Phi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} * \left(\Phi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}}$$

$$\geq \left(\Phi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} = \Phi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)$$

$$= \Phi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2}),\zeta(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2}),\frac{\mathfrak{t}}{\sigma}\right) \geq \left(\Phi\left(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}} * \left(\Phi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}}$$

$$\geq \left(\Phi\left(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}} = \Phi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)$$

$$\geq \cdots \geq \Phi\left(\pi_0,\mathfrak{p}_0,\frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right),$$

and

$$\Psi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = \Psi(\zeta(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1}),\zeta(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1}),\mathfrak{t}) \leq \left(\Psi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}}$$

$$= \left(\Psi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \leq \Psi\left(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)$$

$$= \Psi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2}),\zeta(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2}),\frac{\mathfrak{t}}{\sigma}\right) \leq \left(\Psi\left(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}}$$

$$= \left(\Psi\left(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)\right)^{\frac{1}{2}} \leq \Psi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathfrak{t}}{\sigma^2}\right)$$

$$\leq \cdots \leq \Psi\left(\pi_0,\mathfrak{p}_0,\frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right).$$

By $\mathfrak{n} \to \infty$ and using (FM7) and (FM13), we get,

$$\lim_{\mathfrak{n}\to\infty} \Phi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = 1, \tag{3}$$

$$\lim_{\mathfrak{n}\to\infty} \Psi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = 0, \tag{4}$$

for all $\mathfrak{t} > 0$. Again for all $\mathfrak{n} \in \mathbb{N}$ and $\mathfrak{t} > 0$,

$$\Phi(\pi_{\mathfrak{n}+1},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = \Phi(\zeta(\mathfrak{p}_{\mathfrak{n}},\pi_{\mathfrak{n}}),\zeta(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1}),\mathfrak{t})$$

$$\geq \left(\Phi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} * \left(\Phi\left(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \geq \left(\Phi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} = \mathrm{A}$$

and

$$\Psi(\pi_{\mathfrak{n}+1},\mathfrak{p}_{\mathfrak{n}},\mathfrak{t}) = \Psi(\zeta(\mathfrak{p}_{\mathfrak{n}},\pi_{\mathfrak{n}}),\zeta(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1}),\mathfrak{t})$$

$$\leq \left(\Psi\left(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}-1},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} = B$$

So we have that

$$\mathrm{A} = \sqrt{\Phi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2},\pi_{\mathfrak{n}-2}),\zeta(\pi_{\mathfrak{n}-1},\mathfrak{p}_{\mathfrak{n}-1}),\frac{\mathfrak{t}}{\sigma}\right)} \quad *_p \sqrt{\Phi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1}),\zeta(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-2}),\frac{\mathfrak{t}}{\sigma}\right)}$$

$$\geq \left(\left(\Phi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \right)$$

$$*_p \left(\left(\Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \right)$$

$$= \Phi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) *_p \ \Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right)$$

$$= \Phi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2}, \pi_{\mathfrak{n}-2}), \zeta(\mathfrak{p}_{\mathfrak{n}-3}, \pi_{\mathfrak{n}-3}), \frac{\mathfrak{t}}{\sigma^2}\right)$$

$$*_p \ \Phi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-3}, \pi_{\mathfrak{n}-3}), \zeta(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-2}), \frac{\mathfrak{t}}{\sigma^2}\right)$$

$$\geq \left(\left(\Phi\left(\pi_{\mathfrak{n}-3}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-3}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} \right)$$

$$*_p \ \left(\left(\Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-3}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi_{\mathfrak{n}-3}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} \right)$$

$$= \Phi\left(\pi_{\mathfrak{n}-3}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^3}\right) *_p \Phi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-3}, \frac{\mathfrak{t}}{\sigma^3}\right)$$

$$\ldots \geq \Phi\left(\pi_0, \mathfrak{p}_1, \frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right) *_p \Phi\left(\pi_1, \mathfrak{p}_0, \frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right) \tag{5}$$

and

$$B = \Psi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2}, \pi_{\mathfrak{n}-2}), \zeta(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-1}), \frac{\mathfrak{t}}{\sigma}\right) \Delta_p \ \Psi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-1}, \pi_{\mathfrak{n}-1}), \zeta(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-2}), \frac{\mathfrak{t}}{\sigma}\right)$$

$$\leq \left(\left(\Psi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \right)$$

$$\Delta_p \left(\left(\Psi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \right)$$

$$= \left(\Psi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \right)^{\frac{1}{2}}$$

$$\leq \Psi\left(\pi_{\mathfrak{n}-1}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^2}\right) \Delta_p \ \Psi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-1}, \frac{\mathfrak{t}}{\sigma^2}\right),$$

$$= \Psi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-2}, \pi_{\mathfrak{n}-2}), \zeta(\mathfrak{p}_{\mathfrak{n}-3}, \pi_{\mathfrak{n}-3}), \frac{\mathfrak{t}}{\sigma^2}\right)$$

$$\Delta_p \ \Psi\left(\zeta(\mathfrak{p}_{\mathfrak{n}-3}, \pi_{\mathfrak{n}-3}), \zeta(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-2}), \frac{\mathfrak{t}}{\sigma^2}\right)$$

$$\leq \left(\left(\Psi\left(\pi_{\mathfrak{n}-3}, \mathfrak{p}_{\mathfrak{n}-2}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi_{\mathfrak{n}-2}, \mathfrak{p}_{\mathfrak{n}-3}, \frac{\mathfrak{t}}{\sigma^3}\right) \right)^{\frac{1}{2}} \right)$$

$$\Delta_p\left(\left(\Psi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-3},\frac{\mathrm{t}}{\sigma^3}\right)\right)^{\frac{1}{2}}\Delta_p\left(\Psi\left(\pi_{\mathfrak{n}-3},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathrm{t}}{\sigma^3}\right)\right)^{\frac{1}{2}}\right),$$
$$\leq \Psi\left(\pi_{\mathfrak{n}-3},\mathfrak{p}_{\mathfrak{n}-2},\frac{\mathrm{t}}{\sigma^3}\right)\Delta_p\Psi\left(\pi_{\mathfrak{n}-2},\mathfrak{p}_{\mathfrak{n}-3},\frac{\mathrm{t}}{\sigma^3}\right),$$
$$\leq$$
$$\dots\Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right). \tag{6}$$

Continuing this process with the sequence $\Phi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathrm{t})$, $\Psi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathrm{t})$, for al$l$ $\mathfrak{n}\in\mathbb{N}$ and $\mathrm{t}>0$, we obtain that

$$\Phi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathrm{t}) \geq \Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right)*_p\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right), \tag{7}$$
$$\Psi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathrm{t}) \leq \Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right). \tag{8}$$

By setting for al$l$ $\mathfrak{n}\in\mathbb{N}$ and $\mathrm{t}>0$,

$$\gamma_{\mathfrak{n}}(\mathrm{t}) = \Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right)*_p\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right),$$
$$\beta_{\mathfrak{n}}(\mathrm{t}) = \Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}}{\sigma^{\mathfrak{n}}}\right).$$

using (5)-(8), (FM6), and (FM13), we get $\lim_{\mathfrak{n}\to\infty}\gamma_{\mathfrak{n}}(\mathrm{t})=1$, $\lim_{\mathfrak{n}\to\infty}\beta_{\mathfrak{n}}(\mathrm{t})=0$, for all $\mathrm{t}>0$. Note that for $m>\mathfrak{n}$ and $0<\sigma<1$,

$$1 > 1-\sigma^{m-\mathfrak{n}} = (1-\sigma)(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-1}).$$

Therefore, for every $\mathrm{t}>0$,

$$\mathrm{t} > \mathrm{t}(1-\sigma)(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-1}).$$

Using definition 2.15, we prove that $\{\pi_{\mathfrak{n}}\}$ is a Cauchy sequence in Ω. For $m>\mathfrak{n}$, we can have two cases:

Case I: $m-\mathfrak{n}$ is even.

$$\Phi(\pi_{\mathfrak{n}},\pi_m,\mathrm{t}) \geq \Phi\big(\pi_{\mathfrak{n}},\pi_m,\mathrm{t}(1-\sigma)(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-1})\big)$$
$$\geq \Phi\big(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathrm{t}(1-\sigma)\big)*\Phi(\mathfrak{p}_{\mathfrak{n}+1},\pi_{\mathfrak{n}+2},\mathrm{t}(1-\sigma)\sigma)*\cdots$$
$$*\,\Phi(\pi_{m-2},\mathfrak{p}_{m-1},\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1})*\Phi(\mathfrak{p}_{m-1},\pi_m,\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1})$$
$$\geq\left(\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)*_p\ \Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\right)*\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)\sigma}{\sigma^{\mathfrak{n}+1}}\right)*_p\left(\Phi\left(,\frac{\mathrm{t}(1-\sigma)\sigma}{\sigma^{\mathfrak{n}+1}}\right)\right)\right)\right.$$
$$\left.*\dots*\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{\sigma^{m-1}}\right)*_p\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{\sigma^{m-1}}\right)\right)$$
$$=\underbrace{\gamma_{\mathfrak{n}}\big(\mathrm{t}(1-\sigma)\big)*\gamma_{\mathfrak{n}}\big(\mathrm{t}(1-\sigma)\big)*\cdots*\gamma_{\mathfrak{n}}\big(\mathrm{t}(1-\sigma)\big)}_{m-\mathfrak{n}\ \text{times}},$$

using (5) and (7) and

$$\Psi(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}) \leq \Psi\big(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}(1-\sigma)(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-1})\big)$$
$$\leq \Psi\big(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{t}(1-\sigma)\big)\Delta_p\,\Psi(\mathfrak{p}_{\mathfrak{n}+1},\pi_{\mathfrak{n}+2},\mathfrak{t}(1-\sigma)\sigma)$$
$$\Delta_p\cdots\Delta_p\,\Psi(\pi_{m-2},\mathfrak{p}_{m-1},\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1})\Delta_p\,\Psi(\mathfrak{p}_{m-1},\pi_m,\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1})$$
$$\leq\left(\Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathfrak{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\Delta_p\,\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\Delta_p\,\Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathfrak{t}(1-\sigma)\sigma}{\sigma^{\mathfrak{n}+1}}\right)\right)$$
$$\Delta_p\,\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{\sigma^{m-1}}\right)$$
$$=\underbrace{\beta_{\mathfrak{n}}\big(\mathfrak{t}(1-\sigma)\big)\Delta\,\beta_{\mathfrak{n}}\big(\mathfrak{t}(1-\sigma)\big)\Delta\cdots\Delta\,\beta_{\mathfrak{n}}\big(\mathfrak{t}(1-\sigma)\big)}_{m-\mathfrak{n}\text{ times}}$$

using (6) and (8).

Case II: $m-\mathfrak{n}$ is odd.

$$\Phi(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}) \geq \Phi\left(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}(1-\sigma)\left(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-2}+\frac{\sigma^{m-\mathfrak{n}-1}}{2}+\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)\right)$$
$$\geq \Phi\big(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{t}(1-\sigma)\big)*\Phi(\mathfrak{p}_{\mathfrak{n}+1},\pi_{\mathfrak{n}+2},\mathfrak{t}(1-\sigma)\sigma)*\ldots$$
$$*\,\Phi(\pi_{m-1},\mathfrak{p}_m,\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-2})*\Phi\left(\pi_{m-1},\mathfrak{p}_m,\mathfrak{t}(1-\sigma)\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)$$
$$*\,\Phi\left(\mathfrak{p}_m,\pi_m,\mathfrak{t}(1-\sigma)\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)$$
$$\geq\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathfrak{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)*_p\,\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\right)$$
$$*\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathfrak{t}(1-\sigma)}{\sigma^{\mathfrak{n}+!}}\right)*_p\,\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)\sigma}{\sigma^{\mathfrak{n}+!}}\right)\right)*\ldots*$$
$$*\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{2\sigma^{m-1}}\right)*_p\,\Phi\left(\pi_1,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-2}}{\sigma^{m-2}}\right)\right)$$
$$*\,\Phi\left(\pi_0,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{2\sigma^m}\right),$$

using (3), (5) and (7)

$$=\underbrace{\gamma_{\mathfrak{n}}\big(\mathfrak{t}(1-\sigma)\big)*\gamma_{\mathfrak{n}}\big(\mathfrak{t}(1-\sigma)\big)*\cdots*\gamma_{\mathfrak{n}}\left(\frac{\mathfrak{t}(1-\sigma)}{2}\right)}_{m-\mathfrak{n}\text{ times}}*\Phi\left(\pi_0,\mathfrak{p}_0,\frac{\mathfrak{t}(1-\sigma)}{2\sigma^{\mathfrak{n}+1}}\right).$$

and

$$\Psi(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}) \leq \Psi\left(\pi_{\mathfrak{n}},\pi_m,\mathfrak{t}(1-\sigma)\left(1+\sigma+\sigma^2+\cdots+\sigma^{m-\mathfrak{n}-2}+\frac{\sigma^{m-\mathfrak{n}-1}}{2}+\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)\right)$$
$$\leq \Psi\big(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}+1},\mathfrak{t}(1-\sigma)\big)\Delta\,\Psi(\mathfrak{p}_{\mathfrak{n}+1},\pi_{\mathfrak{n}+2},\mathfrak{t}(1-\sigma)\sigma)$$

$$\Delta\cdots\Delta\,\Psi(\pi_{m-1},\mathfrak{p}_m,\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-2})\Delta\,\Psi\left(\pi_{m-1},\mathfrak{p}_m,\mathrm{t}(1-\sigma)\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)$$

$$\Delta\,\Psi\left(\mathfrak{p}_m,\pi_m,\mathrm{t}(1-\sigma)\frac{\sigma^{m-\mathfrak{n}-1}}{2}\right)$$

$$\leq\left(\Phi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)}{\sigma^{\mathfrak{n}}}\right)\right)$$

$$\Delta_p\left(\Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)}{\sigma^{\mathfrak{n}+!}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)\sigma}{\sigma^{\mathfrak{n}+!}}\right)\right)$$

$$\Delta_p\cdots\Delta_p\left(\Psi\left(\pi_0,\mathfrak{p}_1,\frac{\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{2\sigma^{m-1}}\right)\Delta_p\Psi\left(\pi_1,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-2}}{\sigma^{m-2}}\right)\right)$$

$$\Delta_p\,\Psi\left(\pi_0,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)\sigma^{m-\mathfrak{n}-1}}{2\sigma^m}\right),$$

using (4), (6) and (8)

$$=\underbrace{\beta_{\mathfrak{n}}(\mathrm{t}(1-\sigma))\Delta_p\,\beta_{\mathfrak{n}}(\mathrm{t}(1-\sigma))\Delta_p\cdots\Delta_p\,\beta_{\mathfrak{n}}\left(\frac{\mathrm{t}(1-\sigma)}{2}\right)}_{m-\mathfrak{n}\text{ times}}\Delta_p\Psi\left(\pi_0,\mathfrak{p}_0,\frac{\mathrm{t}(1-\sigma)}{2\sigma^{\mathfrak{n}+1}}\right).$$

Combining the above two cases, (FM6), (FM13) and $\gamma_{\mathfrak{n}}(\mathrm{t})\to 1,\beta_{\mathfrak{n}}(\mathrm{t})\to 0$ as $\mathfrak{n}\to\infty$ for all $\mathrm{t}>0$. Then we see that $\{\pi_{\mathfrak{n}}\}\in\Omega$ is a Cauchy sequence, and similarly we can prove that $\{\mathfrak{p}_{\mathfrak{n}}\}\in\Sigma$ is a Cauchy sequence. Since, Ω,Σ are closed subsets, there exists $\pi\in\Omega$ and $\mathfrak{p}\in\Sigma$, then

$$\lim_{\mathfrak{n}\to\infty}\Phi(\pi_{\mathfrak{n}},\pi,\mathrm{t})=1,\quad\forall\mathrm{t}>0,$$
$$\lim_{\mathfrak{n}\to\infty}\Phi(\mathfrak{p}_{\mathfrak{n}},\mathfrak{p},\mathrm{t})=1,\quad\forall\mathrm{t}>0,\tag{9}$$

and

$$\lim_{\mathfrak{n}\to\infty}\Psi(\pi_{\mathfrak{n}},\pi,\mathrm{t})=0,\quad\forall\mathrm{t}>0,$$
$$\lim_{\mathfrak{n}\to\infty}\Psi(\mathfrak{p}_{\mathfrak{n}},\mathfrak{p},\mathrm{t})=0,\quad\forall\mathrm{t}>0,\tag{10}$$

Now,

$$\Phi(\pi,\mathfrak{p},\mathrm{t})\geq\Phi\left(\pi,\pi_{\mathfrak{n}},\frac{\mathrm{t}(1-\sigma)}{2}\right)*\Phi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\sigma\mathrm{t})*\Phi\left(\mathfrak{p}_{\mathfrak{n}},\mathfrak{p},\frac{\mathrm{t}(1-\sigma)}{2}\right),\tag{11}$$

$$\Psi(\pi,\mathfrak{p},\mathrm{t})\leq\Psi\left(\pi,\pi_{\mathfrak{n}},\frac{\mathrm{t}(1-\sigma)}{2}\right)\Delta_p\Psi(\pi_{\mathfrak{n}},\mathfrak{p}_{\mathfrak{n}},\sigma\mathrm{t})\Delta_p\Psi\left(\mathfrak{p}_{\mathfrak{n}},\mathfrak{p},\frac{\mathrm{t}(1-\sigma)}{2}\right),\tag{12}$$

as $\mathfrak{n}\to\infty$ in (11), (12) and using (3), (4), (9) and (10), we obtain $\pi=\mathfrak{p}$. Moreover, $\Omega\cap\Sigma\neq\phi$ and $\pi=\mathfrak{p}\in\Omega\cap\Sigma$. Also,

$$\Phi(\pi_{\mathfrak{n}},\zeta(\pi,\mathfrak{p}),\mathrm{t})\geq\Phi(\pi_{\mathfrak{n}},\zeta(\pi,\mathfrak{p}),\sigma\mathrm{t})=\Phi(\zeta(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1}),\zeta(\pi,\mathfrak{p}),\sigma\mathrm{t})$$
$$\geq\left(\Phi(\mathfrak{p}_{\mathfrak{n}-1},\pi,\mathrm{t})\right)^{\frac{1}{2}}*\left(\Phi(\pi_{\mathfrak{n}-1},\mathfrak{p},\mathrm{t})\right)^{\frac{1}{2}}=\left(\Phi(\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{p},\mathrm{t})\right)^{\frac{1}{2}}*\left(\Phi(\pi_{\mathfrak{n}-1},\pi,\mathrm{t})\right)^{\frac{1}{2}},$$

and

$$\Psi(\pi_{\mathfrak{n}},\zeta(\pi,\mathfrak{p}),\mathrm{t})\leq\Psi(\pi_{\mathfrak{n}},\zeta(\pi,\mathfrak{p}),\sigma\mathrm{t})=\Psi(\zeta(\mathfrak{p}_{\mathfrak{n}-1},\pi_{\mathfrak{n}-1}),\zeta(\pi,\mathfrak{p}),\sigma\mathrm{t})$$

$$\leq \left(\Psi(\mathfrak{p}_{\mathfrak{n}-1},\pi,\mathfrak{t})\right)^{\frac{1}{2}}\Delta_p \left(\Psi(\pi_{\mathfrak{n}-1},\mathfrak{p},\mathfrak{t})\right)^{\frac{1}{2}} = \left(\Psi(\mathfrak{p}_{\mathfrak{n}-1},\mathfrak{p},\mathfrak{t})\right)^{\frac{1}{2}}\Delta_p\left(\Psi(\pi_{\mathfrak{n}-1},\pi,\mathfrak{t})\right)^{\frac{1}{2}}.$$

Taking the limit as $\mathfrak{n} \to \infty$ in the above inequalities and using (9), and (10) we get $\pi_{\mathfrak{n}} \to \zeta(\pi,\pi)$. Since the topology of the IFMS is Hausdorff [27], we have $\zeta(\pi,\pi) = \pi$. Thus, (π,π) is a SCFP of ζ. To show the uniqueness of the SCFP, let $\mathcal{Z} \neq \pi \in \Xi$ be another SCFP of ζ and $\zeta(\mathcal{Z},\mathcal{Z}) = \mathcal{Z}$. Then

$$\Phi(\pi,\mathcal{Z},\mathfrak{t}) = \Phi(\zeta(\pi,\pi),\zeta(\mathcal{Z},\mathcal{Z}),\mathfrak{t}) \geq \left(\Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} * \left(\Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}}$$

$$\geq \left(\Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} *_p \left(\Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} = \Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right), \tag{13}$$

$$\Psi(\pi,\mathcal{Z},\mathfrak{t}) = \Psi(\zeta(\pi,\pi),\zeta(\mathcal{Z},\mathcal{Z}),\mathfrak{t}) \leq \left(\Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \Delta_p \left(\Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}} \leq \left(\Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right)\right)^{\frac{1}{2}}. \tag{14}$$

By a repeated application of (13) and (14) we have for all $\mathfrak{n}$:

$$\Phi(\pi,\mathcal{Z},\mathfrak{t}) \geq \Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right) \geq \Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma^2}\right) \geq \cdots \geq \Phi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right),$$

$$\Psi(\pi,\mathcal{Z},\mathfrak{t}) \leq \Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma}\right) \leq \Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma^2}\right) \leq \cdots \leq \Psi\left(\pi,\mathcal{Z},\frac{\mathfrak{t}}{\sigma^{\mathfrak{n}}}\right).$$

Taking limit as $\mathfrak{n} \to \infty$in the above inequality, by using (FM7) and (FM13), we get $\Phi(\pi,\mathcal{Z},\mathfrak{t}) = 1$, $\Psi(\pi,\mathcal{Z},\mathfrak{t}) = 0$. Hence $\pi = \mathcal{Z}$. Thus, ζ has a unique SCFP.

Example 3.1. Assume that $\Xi = \mathbb{R}$ and $\Omega = \left[0,\frac{1}{2}\right], \Sigma = \left[-\frac{1}{2},0\right]$, and the IFMS $(\Xi,\Phi,\Psi,*,\Delta_p)$ with CTN $\vartheta * b = \vartheta \cdot b$, and CTCN $\vartheta \, \Delta_p \, b = \max\{\vartheta,b\}$. Φ and Ψ are FSs on $\Xi^2 \times (0,\infty)$ defined by,

$$\Phi(\pi,\mathfrak{p},\mathfrak{t}) = e^{-\frac{|\pi-\mathfrak{p}|}{\mathfrak{t}}},$$

$$\Psi(\pi,\mathfrak{p},\mathfrak{t}) = 1 - e^{-\frac{|\pi-\mathfrak{p}|}{\mathfrak{t}}}.$$

Let $\zeta: \Xi \times \Xi \to \Xi$ be a mapping given by

$$\zeta(\pi,\mathfrak{p}) = \begin{cases} \dfrac{\mathfrak{p}-\pi}{6}, & \text{if } (\pi,\mathfrak{p}) \in \left[0,\frac{1}{2}\right] \times \left[-\frac{1}{2},0\right] \\ 2\pi, & \text{otherwise} \end{cases}$$

By definition, ζ is a coupling and ζ is also an IFCC concerning Ω,Σ. Suppose that $\sigma = \frac{1}{3}$.

For $\pi,\mathfrak{v} \in \Omega$ and $\mathfrak{p},\mathfrak{u} \in \Sigma$, we have

$$\left(\Phi(\pi,\mathfrak{u},\mathfrak{t})\right)^{\frac{1}{2}} = e^{-\frac{|\pi-\mathfrak{u}|}{2\mathfrak{t}}} \text{ and } \left(\Phi(\mathfrak{v},\mathfrak{p},\mathfrak{t})\right)^{\frac{1}{2}} = e^{-\frac{|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}},$$

$$\left(\Psi(\pi,\mathfrak{u},\mathfrak{t})\right)^{\frac{1}{2}} = 1 - e^{-\frac{|\pi-\mathfrak{u}|}{2\mathfrak{t}}} \text{ and } \left(\Psi(\mathfrak{v},\mathfrak{p},\mathfrak{t})\right)^{\frac{1}{2}} = 1 - e^{-\frac{|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}},$$

and

$$\Phi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma\mathfrak{t}) = e^{-\frac{|\zeta(\pi,\mathfrak{p})-\zeta(\mathfrak{u},\mathfrak{v})|}{\sigma\mathfrak{t}}} = e^{-3\frac{|(\mathfrak{p}-\pi)-(\mathfrak{v}-\mathfrak{u})|}{6t}} = e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}},$$

also,

$$|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})| \leq |\mathfrak{u}-\pi| + |\mathfrak{p}-\mathfrak{v}|,$$

or

$$\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}} \leq \frac{|\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}$$

or

$$e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}} \geq e^{-\frac{|\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}}$$

or

$$\Phi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma\mathfrak{t}) \geq \left(\Phi(\pi,\mathfrak{u},\mathfrak{t})\right)^{\frac{1}{2}} * \left(\Phi(\mathfrak{p},\mathfrak{v},\mathfrak{t})\right)^{\frac{1}{2}}. \tag{15}$$

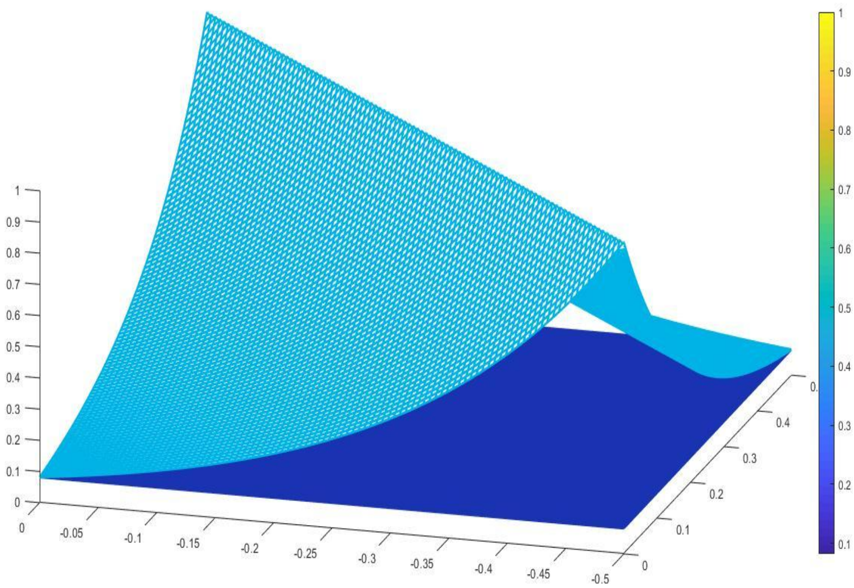

Figure 1. The left-hand side with light color blue and the right-hand side with the blue color of the last inequality (15)

and

$$\Psi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma\mathfrak{t}) = 1-e^{-\frac{|\zeta(\pi,\mathfrak{p})-\zeta(\mathfrak{u},\mathfrak{v})|}{\sigma\mathfrak{t}}} = 1-e^{-\frac{|(\mathfrak{p}-\pi)-(\mathfrak{v}-\mathfrak{u})|}{6\sigma\mathfrak{t}}} = 1-e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}},$$

also,

$$|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})| \leq |\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|,$$

or

$$\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}} \leq \frac{|\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}$$

or

$$e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}} \geq e^{-\frac{|\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}}$$

$$1-e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}} \leq 1-e^{-\frac{|\mathfrak{u}-\pi|+|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}},$$

$$1-e^{-\frac{|(\mathfrak{u}-\pi)+(\mathfrak{p}-\mathfrak{v})|}{2\mathfrak{t}}} \leq \max\left\{1-e^{-\frac{|\mathfrak{u}-\pi|}{2\mathfrak{t}}}, 1-e^{-\frac{|\mathfrak{p}-\mathfrak{v}|}{2\mathfrak{t}}}\right\},$$

or

$$\Psi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma t) \leq \left(\Psi(\pi,\mathfrak{u},t)\right)^{\frac{1}{2}}\Delta\left(\Psi(\pi,\mathfrak{v},t)\right)^{\frac{1}{2}}. \quad (16)$$

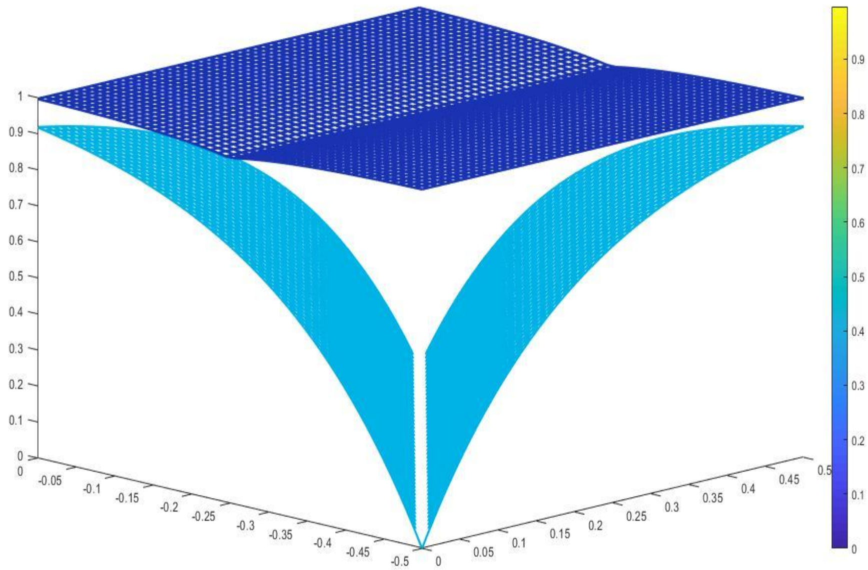

Figure 2. The left-hand side with light color blue and the right-hand side with the blue color of the last inequality (16)

We determine that ζ is an IFCC with a contractivity factor $\sigma = 1/3$. Therefore, in Theorem 3.1 the conditions are fulfilled. This theorem states that there is an SCFP of ζ, which is $(0,0)$.

Corollary 3.1. Let $(\Xi,\Phi,\Psi,*,\Delta_p)$ be a complete IFMS with $*$ as CTN and Δ_p as CTCN. *Let* $*$ *be such that* $a* b \geq a *_p b$ *for any* $a,b\in[0,1]$ Let $\zeta: \Xi\times\Xi\to\Xi$ be a mapping to fulfill the below inequalities for all $\pi,\mathfrak{p},\mathfrak{u},\mathfrak{v}\in\Xi, t>0$ and for some $\sigma\in(0,1)$:

$$\Phi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma t) \geq \left(\Phi(\pi,\mathfrak{u},t)\right)^{\frac{1}{2}} * \left(\Phi(\pi,\mathfrak{v},t)\right)^{\frac{1}{2}},$$

$$\Psi(\zeta(\pi,\mathfrak{p}),\zeta(\mathfrak{u},\mathfrak{v}),\sigma t) \leq \left(\Psi(\pi,\mathfrak{u},t)\right)^{\frac{1}{2}}\Delta_p\left(\Psi(\pi,\mathfrak{v},t)\right)^{\frac{1}{2}}.$$

Then ζ has a unique SCFP.

Proof. Take $\Omega = \Sigma = \Xi$ in Theorem 3.1 and the result follows.

4. Generations of Fractals

Definition 4.1 [19]. Assume that two non-empty compact subsets Ω and Σ of an FMS $(\Xi,\Psi,*)$. $\mathcal{H}_\Psi$ is a Hausdorff fuzzy metric on $\kappa(\Xi)$ defined as

$$\mathcal{H}_\Psi(\Omega,\Sigma,t) = \max\{\omega(\Omega,\Sigma,t), \overline{\omega}(\Omega,\Sigma,t)\}.$$

where

$$\omega(\Omega, \Sigma, \mathrm{t}) = \sup_{\vartheta\in\Omega} \inf_{b\in\Sigma} \Psi(\vartheta, b, \mathrm{t}),$$

and

$$\overline{w}\ (\Omega, \Sigma, \mathrm{t}) = \sup_{b\in\Sigma} \inf_{\vartheta\in\Omega} \Psi(\vartheta, b, \mathrm{t})\,, \qquad \mathrm{t} > 0.$$

Definition 4.2 [27]. *Let $(\Xi, \Phi, \Psi, *, \Delta)$ be an IFMS and Ω, Σ be two non-empty compact subsets of Ξ. Then we define the* functions H_Φ and H_Ψ on $\mathrm{K}(\Xi) \times \mathrm{K}(\Xi) \times (0,1)$ by

$$\mathcal{H}_\Phi(\Omega, \Sigma, \mathrm{t}) = \min\{g(\Omega, \Sigma, \mathrm{t}), \bar{g}\ (\Omega, \Sigma, \mathrm{t})\}.$$

where

$$g(\Omega, \Sigma, \mathrm{t}) = \inf_{\vartheta\in\Omega} \sup_{b\in\Sigma} \Phi(\vartheta, b, \mathrm{t})$$

and

$$\bar{g}\ (\Omega, \Sigma, \mathrm{t}) = \inf_{b\in\Sigma} \sup_{\vartheta\in\Omega} \Phi(\vartheta, b, \mathrm{t}),$$

and H_Ψ is defined as in Definition 4.1.

Definition 4.3. *Let $(\Xi, \Phi, \Psi, *, \Delta)$ be an IFMS and Ω, Σ be two non-empty subsets of Ξ. Then a mapping* $\mathrm{F}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ is a "coupling concerning $\kappa(\Omega)$ *and* $\kappa(\Sigma)$" *if* for all $C \in \kappa(\Omega)$ and $D \in \kappa(\Sigma)$, $F(C, D) \in \kappa(\Sigma)$ and $F(D, C) \in \kappa(\Omega)$.

Definition 4.4. *Let $(\Xi, \Phi, \Psi, *, \Delta)$ be an IFMS, Ω, Σ be two non-empty compact subsets of Ξ and a mapping* $\mathrm{F}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ be a coupling concerning $\kappa(\Omega)$ *and* $\kappa(\Sigma)$. *Then F is called an IFCC with respect to $\kappa(\Omega)$ and $\kappa(\Sigma)$ in the IFMS $(\kappa(\Xi), \mathcal{H}_\Phi, \mathcal{H}_\Psi, *, \Delta)$ if there exists a $\sigma \in (0,1)$ such that*

$$\mathcal{H}_\Phi(F(C_1, D_1), F(C_2, D_2)\,, \sigma\mathrm{t})\ \geq \left(\mathcal{H}_\Phi\ (C_1, C_2, \mathrm{t})\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Phi\ (D_1, D_2, \mathrm{t})\right)^{\frac{1}{2}},$$

$$\mathcal{H}_\Psi(F(C_1, D_1), F(C_2, D_2)\,, \sigma\mathrm{t})\ \leq \left(\mathcal{H}_\Psi\ (C_1, C_2, \mathrm{t})\right)^{\frac{1}{2}} \Delta \left(\mathcal{H}_\Psi\ (D_1, D_2, \mathrm{t})\right)^{\frac{1}{2}}$$

for all $C_1, C_2 \in \kappa(\Omega)$ and $D_1, D_2 \in \kappa(\Sigma)$

Theorem 4.1: *Let $(\Xi, \Phi, \Psi, *, \Delta)$ be an IFMS and Ω, Σ be two non-empty compact subsets of $\Xi, \zeta: \Xi \times \Xi \to \Xi$ be a IFCC with respect to Ω and Σ with CF σ. Then $\hat{\zeta}$ is a IFCC with respect to $\kappa(\Omega)$ and $\kappa(\Sigma)$ in the IFMS $(\kappa(\Xi), \mathcal{H}_\Phi, \mathcal{H}_\Psi, *, \Delta)$ with the same CF.*

Proof. From the construction of $\hat{\zeta}(\Omega, \Sigma)$, it follows that for all $C \in \kappa(\Omega)$ and $D \in \kappa(\Sigma), \hat{\zeta}\ (C, D) \in \kappa(\Sigma)$ and $\hat{\zeta}\ (D, C) \in \kappa(\Omega)$. Let $C_1, C_2 \in \kappa(\Omega)$ and $D_1, D_2 \in \kappa(\Sigma)$. Then

$$\omega\left(\hat{\zeta}(C_1\,, D_1), \hat{\zeta}(C_2\,, D_2), \sigma\mathrm{t}\right) = \omega\begin{pmatrix} \{\zeta(c_1, d_1): c_1 \in C_1\,,\ d_1 \in D_1\}, \\ \{\zeta(c_2, d_2): c_2 \in C_2\,,\ d_2 \in D_2\}, \sigma\mathrm{t} \end{pmatrix}$$

$$= \sup_{\substack{c_1\in C_1\\ d_1\in D_1}} \inf_{\substack{c_2\in C_2\\ d_2\in D_2}} \Phi(\zeta(c_1,d_1),\zeta(c_2,d_2),\sigma t)$$

$$\geq \sup_{\substack{c_1\in C_1\\ d_1\in D_1}} \inf_{\substack{c_2\in C_2\\ d_2\in D_2}} \left(\Phi(c_1,c_2,t)\right)^{\frac{1}{2}} * \left(\Phi(d_1,d_2,t)\right)^{\frac{1}{2}},$$

$$= \left(\sup_{c_1\in C_1} \inf_{c_2\in C_2} \Phi(c_1,c_2,t)\right)^{\frac{1}{2}} * \left(\inf_{d_1\in D_1} \sup_{d_2\in D_2} \Phi(d_1,d_2,t)\right)^{\frac{1}{2}}$$

$$= (\omega\,(C_1,C_2,t))^{\frac{1}{2}} * \left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}$$

and

$$\mathcal{H}_\Psi\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right) \leq \omega\left(\hat{\zeta}(C_1\,,D_1),\hat{\zeta}(C_2\,,D_2),\sigma t\right) = \omega\begin{pmatrix}\{\zeta(c_1,d_1): c_1\in C_1\,,\ d_1\in D_1\},\\ \{\zeta(c_2,d_2): c_2\in C_2\,,\ d_2\in D_2\},\sigma t\end{pmatrix}$$

$$= \sup_{\substack{c_1\in C_1\\ d_1\in D_1}} \inf_{\substack{c_2\in C_2\\ d_2\in D_2}} \Psi(\zeta(c_1,d_1),\zeta(c_2,d_2),\sigma t)$$

$$\leq \sup_{\substack{c_1\in C_1\\ d_1\in D_1}} \inf_{\substack{c_2\in C_2\\ d_2\in D_2}} \left(\Psi(c_1,c_2,t)\right)^{\frac{1}{2}} \Delta \left(\Psi(d_1,d_2,t)\right)^{\frac{1}{2}}$$

$$= \left(\sup_{c_1\in C_1} \inf_{c_2\in C_2} \Psi(c_1,c_2,t)\right)^{\frac{1}{2}} \Delta \left(\sup_{d_1\in D_1} \inf_{d_2\in D_2} \Psi(d_1,d_2,t)\right)^{\frac{1}{2}}$$

$$= (\omega\,(C_1,C_2,t))^{\frac{1}{2}}\Delta\left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}.$$

Similarly,

$$\overline{\omega}\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right) \geq \left(\mathcal{H}_\Phi\,(C_1,C_2,t)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Phi\,(D_1,D_2,t)\right)^{\frac{1}{2}},$$

$$\mathcal{H}_\Psi\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right) \leq (\overline{\omega}\,(C_1,C_2,t))^{\frac{1}{2}}\Delta\left(\overline{\omega}\,(D_1,D_2,t)\right)^{\frac{1}{2}}.$$

Therefore,

$$\mathcal{H}_\Phi\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right) = \max\begin{Bmatrix}\omega\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right),\\ \overline{\omega}\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right)\end{Bmatrix}$$

$$\geq \left(\mathcal{H}_\Phi\,(C_1,C_2,t)\right)^{\frac{1}{2}} * \left(\mathcal{H}_\Phi\,(D_1,D_2,t)\right)^{\frac{1}{2}},$$

and

$$\mathcal{H}_\Psi\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right) = \max\begin{Bmatrix}\omega\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right),\\ \overline{\omega}\left(\hat{\zeta}(C_1,D_1),\hat{\zeta}(C_2,D_2)\,,\sigma t\right)\end{Bmatrix} \leq$$

$$\max\{(\omega\,(C_1,C_2,t))^{\frac{1}{2}}\Delta\left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}, (\overline{\omega}\,(C_1,C_2,t))^{\frac{1}{2}}\Delta\left(\overline{\omega}\,(D_1,D_2,t)\right)^{\frac{1}{2}}\} \leq$$

$$\max\{\{\max\{\left(\omega\,(C_1,C_2,t)\right)^{\frac{1}{2}}, \left(\overline{\omega}\,(C_1,C_2,t)\right)^{\frac{1}{2}}\}\Delta \max\{\left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}, \left(\overline{\omega}\,(D_1,D_2,t)\right)^{\frac{1}{2}}\},$$

$$\max\{(\overline{\omega}\,(C_1,C_2,t))^{\frac{1}{2}},\ \left(\omega\,(C_1,C_2,t)\right)^{\frac{1}{2}})\}\Delta \max\{\left(\overline{\omega}\,(D_1,D_2,t)\right)^{\frac{1}{2}}, \left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}\}\}=$$

$$\max\{\left(\omega\,(C_1,C_2,t)\right)^{\frac{1}{2}}, \left(\overline{\omega}\,(C_1,C_2,t)\right)^{\frac{1}{2}}\}\Delta \max\{\left(\omega\,(D_1,D_2,t)\right)^{\frac{1}{2}}, \left(\overline{\omega}\,(D_1,D_2,t)\right)^{\frac{1}{2}}\},$$

$$= \left(\mathcal{H}_{\Psi}\,(C_1, C_2, \mathrm{t})\right)^{\frac{1}{2}} \Delta \left(\mathcal{H}_{\Psi}\,(D_1, D_2, \mathrm{t})\right)^{\frac{1}{2}}$$

being Δ nondecreasing.

Hence, $\hat{\zeta}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ is an IFCC with respect to $\kappa(\Omega)$ and $\kappa(\Sigma)$ in the IFMS $(\kappa(\Xi), \mathcal{H}_{\Phi}, \mathcal{H}_{\Psi}, *, \Delta)$ with contractivity factor σ.

Definition 4.5 [1]. Let $(\Xi, \Phi, *)$ be an FMS and Ξ has two non-empty compact subsets Ω, Σ and finite collection of continuous coupling $\{\zeta_i : i \in \mathbb{N}_{\mathfrak{n}}\}$ concerning Ω, Σ. Then the Hutchinson operator, corresponding to $\hat{G}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$, is defined as

$$\hat{G}(C, D) = \bigcup_{i=1}^{\mathfrak{n}} \hat{\zeta}_{\iota}\,(C, D)$$

$for\ any\ C, D \in \kappa(\Xi)$

The following lemma is evident:

Lemma 4.1. Let be given a FMS $(\Xi, \Phi, *)$, a finite collection of couplings $\zeta_i : \Xi \times \Xi \to \Xi$ for all $i \in \mathbb{N}_{\mathfrak{n}}$, concerning Ω, Σ, where Ω, Σ are closed compact subsets of Ξ. Then the Hutchinson operator is a coupling concerning $\kappa(\Omega), \kappa(\Sigma)$ and, $for\ all\ C_1, C_2 \in \kappa(\Omega)$ and $D_1, D_2 \in \kappa(\Sigma), holds$:

$$\mathcal{H}_{\Phi}\left(\,\hat{G}(C_1, D_1), \hat{G}(C_2, D_2), \mathrm{t}\right) = \max_{1 \le i \le \mathfrak{n}} \mathcal{H}_{\Phi}\left(\hat{\zeta}_i(C_1, D_1), \hat{\zeta}_i(C_2, D_2), \mathrm{t}\right),$$
$$\mathcal{H}_{\Psi}\left(\,\hat{G}(C_1, D_1), \hat{G}(C_2, D_2), \mathrm{t}\right) = \max_{1 \le i \le \mathfrak{n}} \mathcal{H}_{\Psi}\left(\hat{\zeta}_i(C_1, D_1), \hat{\zeta}_i(C_2, D_2), \mathrm{t}\right),$$

Definition 4.6 [1]. Let $(\Xi, \Phi, *)$ be a FMS and $\hat{\eta}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ be a mapping. $\Omega \in \kappa(\Xi)$ called a strong coupling fractal (SCF) of $\hat{\eta}$ if $\hat{\eta}(\Omega, \Omega) = \Omega$.

Lemma 4.2: *Let $(\Xi, \Phi, \Psi, *, \Delta)$ be an IFMS and Ω, Σ be two non-empty compact subsets of $\Xi, \zeta_i: \Xi \times \Xi \to \Xi$ be a continuous IFCC with respect to Ω and Σ with CF $\sigma_i, i \in \mathbb{N}_{\mathfrak{n}}$. Then the Hutchinson operator* $\hat{G}: \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ is an IFCC *in the IFMS $(\kappa(\Xi), \mathcal{H}_{\Phi}, \mathcal{H}_{\Psi}, *, \Delta)$ with respect to $\kappa(\Omega)$ and $\kappa(\Sigma)$ with CF $\sigma = \sigma_h = max\{\sigma_{\mathfrak{n}}\,; \mathfrak{n} \in \mathbb{N}_{\mathfrak{n}}\}$.*

Proof. By the definition of $\hat{G}$, it satisfies that for all $C \in \kappa(\Omega)$ and $D \in \kappa(\Sigma)$, $\hat{G}(C, D) \in \kappa(\Sigma)$ and $\hat{G}(D, C) \in \kappa(\Omega)$. Let $C_1, C_2 \in \kappa(\Omega)$ and $D_1, D_2 \in \kappa(\Sigma)$:

$$\mathcal{H}_{\Phi}\left(\,\hat{G}(C_1, D_1), \hat{G}(C_2, D_2), \sigma_h \mathrm{t}\right) = \max_{1 \le i \le \mathfrak{n}} \mathcal{H}_{\Phi}\left(\hat{\zeta}_i(C_1, D_1), \hat{\zeta}_i(C_2, D_2), \sigma \mathrm{t}\right) = \mathcal{H}_{\Phi}\left(\hat{\zeta}_h(C_1, D_1), \hat{\zeta}_h(C_2, D_2), \sigma_h \mathrm{t}\right),$$
$$\mathcal{H}_{\Psi}\left(\,\hat{G}(C_1, D_1), \hat{G}(C_2, D_2), \sigma_h \mathrm{t}\right)$$
$$= \max_{1 \le i \le \mathfrak{n}} \mathcal{H}_{\Psi}\left(\hat{\zeta}_i(C_1, D_1), \hat{\zeta}_i(C_2, D_2), \sigma \mathrm{t}\right) = \mathcal{H}_{\Psi}\left(\hat{\zeta}_h(C_1, D_1), \hat{\zeta}_h(C_2, D_2), \sigma_h \mathrm{t}\right),$$

by Lemma 4.1. Since $\hat{\zeta}_h$ is an IFCC *with respect to $\kappa(\Omega)$ and $\kappa(\Sigma)$ in the IFMS $(\kappa(\Xi), \mathcal{H}_{\Phi}, \mathcal{H}_{\Psi}, *, \Delta)$ with the same CFσ_h by $Theorem$* 4.1, we have that

$$\mathcal{H}_{\Phi}\left(\hat{\zeta}_h(C_1,D_1),\hat{\zeta}_h(C_2,D_2),\sigma_h \mathrm{t}\right) \geq \left(\mathcal{H}_{\Phi}\left(C_1,C_2,\mathrm{t}\right)\right)^{\frac{1}{2}} * \left(\mathcal{H}_{\Phi}\left(D_1,D_2,\mathrm{t}\right)\right)^{\frac{1}{2}},$$

and

$$\mathcal{H}_{\Psi}\left(\hat{\zeta}_h(C_1,D_1),\hat{\zeta}_h(C_2,D_2),\sigma_h \mathrm{t}\right) \leq \left(\mathcal{H}_{\Psi}\left(C_1,C_2,\mathrm{t}\right)\right)^{\frac{1}{2}} \Delta \left(\mathcal{H}_{\Psi}\left(D_1,D_2,\mathrm{t}\right)\right)^{\frac{1}{2}},$$

This completes the proof.

Theorem 4.2. *Suppose $(\Xi,\Phi,\Psi,*,\Delta)$ be a complete IFMS such that $a * b \geq a *_p b$ for any $a,b \in [0,1]$. Consider an IFICS $\langle(\Xi,\Phi,\Psi,*,\Delta);\Omega,\Sigma,\ \zeta_i, i \in \mathbb{N}_{\mathfrak{n}}\rangle$ has a finite number of continuous IFCC on $\Xi \times \Xi$ for two subsets Ω,Σ of Ξ, which are compact and assume that $\hat{G}\colon \kappa(\Xi) \times \kappa(\Xi) \to \kappa(\Xi)$ be the corresponding Hutchinson operator. Then there exists a unique SCF for $\hat{G}$, that is, there exists a $P \in \kappa(\Omega) \cap \kappa(\Sigma)$, then $\hat{G}(P,P) = P$. Further, both the iterations $\{\Omega_{\mathfrak{n}}\}$ and $\{\Sigma_{\mathfrak{n}}\}$ constructed as*

$$\Sigma_{\mathfrak{n}+1} = \hat{G}\left(\Omega_{\mathfrak{n}},\Sigma_{\mathfrak{n}}\right), \qquad \Omega_{\mathfrak{n}+1} = \hat{G}\left(\Sigma_{\mathfrak{n}},\Omega_{\mathfrak{n}}\right), \mathfrak{n} \geq 0,$$

with $\Omega_0 = \kappa(\Omega)$ and $\Sigma_0 = \kappa(\Sigma)$ being arbitrary chosen, converge to the unique SCF.

Proof. By Lemma 4.2, $\hat{G}$ is an IFCC with CF $\sigma = \max\{\sigma_{\mathfrak{n}} : \mathfrak{n} \in \mathbb{N}_{\mathfrak{n}}\}$. Again, since $(\Xi,\Phi,\Psi,*,\Delta)$ is complete, so, $(\kappa(\Xi),\mathcal{H}_{\Phi},\mathcal{H}_{\Psi},*,\Delta)$ is complete. Then the thesis comes from an application of Theorem 3.1 since Ω,Σ are compact subsets of Ξ *which is complete and* $\kappa(\Omega)$, $\kappa(\Sigma)$ are compact subsets of the IFMS $(\kappa(\Xi),\mathcal{H}_{\Phi},\mathcal{H}_{\Psi},*,\Delta)$.

Example 4.1. Suppose $\Xi = \mathbb{R}$ and $\Omega = [-2,2]$, $\Sigma = [-1,2]$, and the IFMS $(\Xi,\Phi,\Psi,*,\Delta)$ with CTN $\vartheta * b = \vartheta \cdot b$, and CTCN $\vartheta \diamond b = \max\{\vartheta, b\}$. Φ and Ψ are FSs on $\Xi^2 \times (0,\infty)$ defined by,

$$\Phi(\pi,\mathfrak{p},\mathrm{t}) = e^{-\frac{|\pi-\mathfrak{p}|}{\mathrm{t}}},$$

$$\Psi(\pi,\mathfrak{p},\mathrm{t}) = 1 - e^{-\frac{|\pi-\mathfrak{p}|}{\mathrm{t}}}.$$

Let $\zeta_1,\zeta_2 : \Xi \times \Xi \to \Xi$ be given by $\zeta_1(\pi,\mathfrak{p}) = \frac{\mathfrak{p}-\pi}{9}$, $\zeta_2(\pi,\mathfrak{p}) = 1 + \frac{\mathfrak{p}-\pi}{9}$. For $\pi \in \Omega = [-2,2]$ and $\mathfrak{p} \in \Sigma = [-1,2]$, $\zeta_1(\pi,\mathfrak{p}),\zeta_2(\pi,\mathfrak{p}) \in \Sigma$ and $\zeta_1(\pi,\mathfrak{p}),\zeta_2(\pi,\mathfrak{p}) \in \Omega$. Then ζ_1, ζ_2 are couplings with respect to Ω,Σ. Then the IFICS $\langle(\Xi,\Phi,\Psi,*,\Delta);\Omega,\Sigma,\zeta_i,\ i \in \mathbb{N}_2\rangle$ generates a SCF.

Let $\Omega_0 = \Sigma_0 = \left[-\frac{1}{2},\frac{3}{2}\right]$. The subsequent list contains the first five steps of the iteration that lead to the SCF (cfr. Figure 4):

$$\Omega_1 = \Sigma_1 = \left[-\frac{1}{2},\frac{3}{2}\right].$$

$$\Omega_2 = \zeta(\Omega_1,\Omega_1) = \left[-\frac{2}{9},\frac{2}{9}\right] \cup \left[\frac{7}{9},\frac{11}{9}\right].$$

$$\Omega_3 = \zeta(\Omega_2,\Omega_2) = \left[-\frac{13}{81},-\frac{5}{81}\right] \cup \left[-\frac{4}{81},\frac{4}{81}\right] \cup \left[\frac{5}{81},\frac{13}{81}\right] \cup \left[\frac{68}{81},\frac{76}{81}\right] \cup \left[\frac{77}{81},\frac{85}{81}\right] \cup \left[\frac{86}{81},\frac{94}{81}\right].$$

$$\Omega_4 = \zeta(\Omega_3,\Omega_3) = \left[-\frac{107}{729},-\frac{55}{729}\right] \cup \left[-\frac{26}{729},\frac{26}{729}\right] \cup \left[\frac{55}{729},\frac{107}{729}\right]$$
$$\cup \left[\frac{622}{729},\frac{674}{729}\right] \cup \left[\frac{703}{729},\frac{755}{729}\right] \cup \left[\frac{784}{729},\frac{836}{729}\right].$$

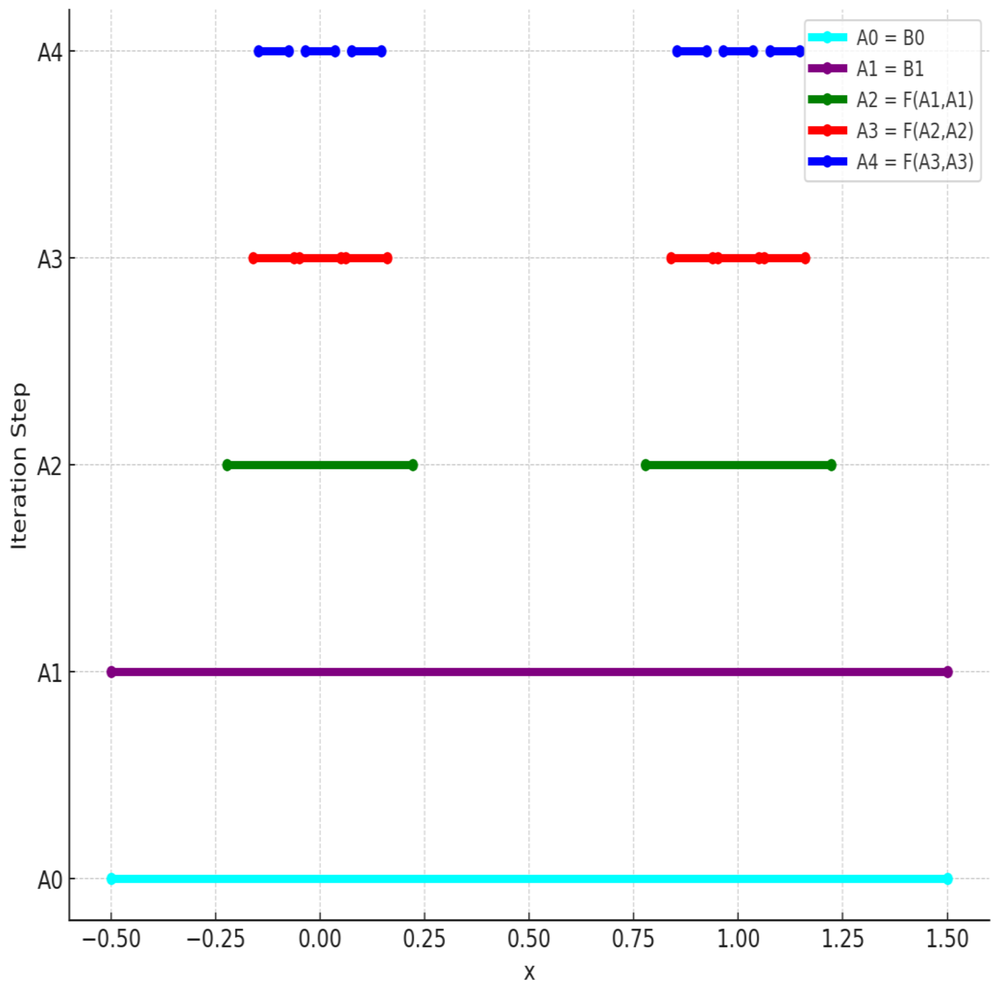

Figure 3. First five iterations of Example 4.1

Example 4.2. Suppose that $\Xi = \mathbb{R}$ and $\Omega = [-2,2]$, $\Sigma = [-1,2]$, and an IFMS $(\Xi, \Phi, \Psi, *, \Delta)$ with CTN $\vartheta * b = \vartheta \cdot b$, and CTCN $\vartheta \diamond b = \max\{\vartheta, b\}$. Φ and Ψ are FSs on $\Xi^2 \times (0, \infty)$ defined by,

$$\Phi(\pi, \mathfrak{p}, \mathrm{t}) = \frac{\min\{\pi, \mathfrak{p}\} + t}{\max\{\pi, \mathfrak{p}\} + t},$$

$$\Psi(\pi, \mathfrak{p}, \mathrm{t}) = 1 - \frac{\min\{\pi, \mathfrak{p}\} + t}{\max\{\pi, \mathfrak{p}\} + t}.$$

Let $\zeta_1, \zeta_2 : \Xi \times \Xi \to \Xi$ be given by $\zeta_1(\pi, \mathfrak{p}) = \frac{\mathfrak{p}-\pi}{16}$, $\zeta_2(\pi, \mathfrak{p}) = 1 + \frac{\mathfrak{p}-\pi}{16}$. Let $\Omega_1 = \Sigma_1 = \left[-\frac{1}{2}, \frac{3}{2}\right]$. Then the first four iterations of the same are as follows (cfr. Fig. 4):

$$\Omega_1 = \Sigma_1 = \left[-\frac{1}{2}, \frac{3}{2}\right].$$

$$\Omega_2 = \zeta(\Omega_1, \Omega_1) = \left[-\frac{1}{8}, \frac{1}{8}\right] \cup \left[\frac{7}{8}, \frac{9}{8}\right].$$

$$\Omega_3 = \zeta(\Omega_2, \Omega_2) = \left[-\frac{1}{16}, -\frac{3}{64}\right] \cup \left[-\frac{1}{64}, \frac{1}{64}\right] \cup \left[\frac{3}{64}, \frac{1}{64}\right] \cup \left[\frac{15}{16}, \frac{61}{64}\right] \cup \left[\frac{63}{64}, \frac{65}{64}\right] \cup \left[\frac{67}{64}, \frac{17}{16}\right].$$

$$\Omega_4 = \zeta(\Omega_3,\Omega_3) = \left[-\frac{9}{128},-\frac{35}{512}\right] \cup \left[-\frac{69}{1024},-\frac{59}{1024}\right] \cup \left[-\frac{29}{512},-\frac{7}{128}\right] \cup \left[-\frac{1}{128},-\frac{3}{512}\right]$$
$$\cup \left[-\frac{5}{1024},\frac{5}{1024}\right] \cup \left[\frac{3}{512},\frac{1}{128}\right] \cup \left[\frac{7}{128},\frac{29}{512}\right] \cup \left[\frac{59}{1024},\frac{69}{1024}\right] \cup \left[\frac{35}{512},\frac{9}{128}\right]$$
$$\cup \left[\frac{119}{128},\frac{477}{512}\right] \cup \left[\frac{959}{1024},\frac{965}{1024}\right] \cup \left[\frac{483}{512},\frac{121}{128}\right] \cup \left[\frac{127}{128},\frac{509}{512}\right] \cup \left[\frac{1019}{1024},\frac{1029}{1024}\right]$$
$$\cup \left[\frac{515}{512},\frac{129}{128}\right] \cup \left[\frac{135}{128},\frac{541}{512}\right] \cup \left[\frac{1083}{1024},\frac{1093}{1024}\right] \cup \left[\frac{547}{512},\frac{137}{128}\right].$$

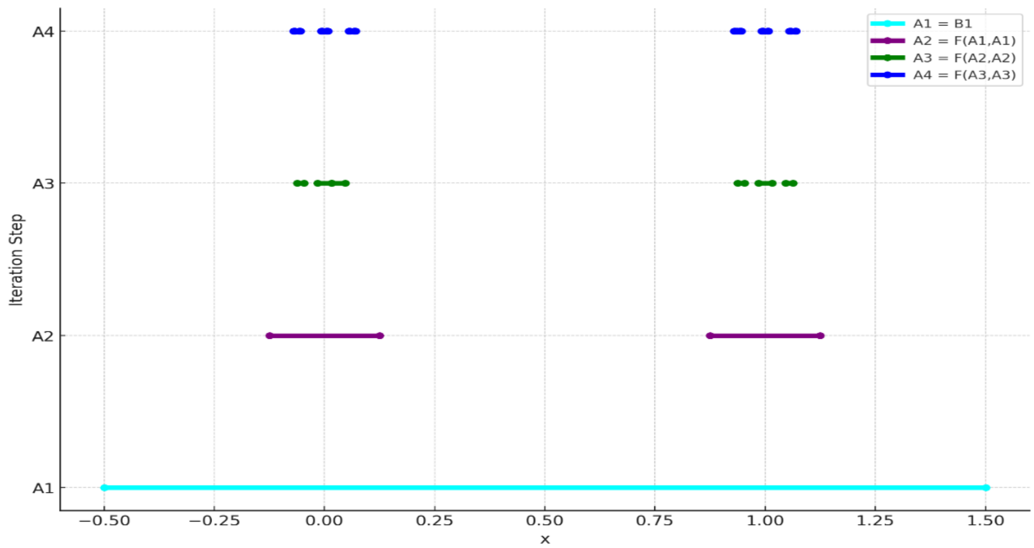

Figure 4. First four iterations of Example 4.2

5. Conclusion

In this paper, we provided a new framework of fixed-point theory and fractal generation in IFMS by means of a SCFP with the concept of an IFCC. We rigorously prove the existence and uniqueness of SCFPs, which is complemented by a Corollary and a non-trivial example. This work generalizes existing fuzzy CCFP results with an aim to make them more useful and significant. In addition, the manuscript illustrates the effectiveness of SCFPs for fractal generation based on an IFICS and Hutchinson-Barnsley operator to create strong-coupled fractal sets with respect to an invented intuitionistic fuzzy Hausdorff distance for compact sets. Future work may extend this framework to other generalized fuzzy metric spaces such as L-fuzzy or probabilistic metric spaces and study dynamic systems in an intuitionistic fuzzy context to explain stability and attractor behaviors. We can also generalize this work in intuitionistic fuzzy controlled metric spaces, neutrosophic metric spaces, and neutrosophic controlled metric spaces.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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