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Determination of Sub-Diffusion Process and Source Term from Nonlocal Data: Applications to Microwave Radiations

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Abstract. We investigate two inverse problems (IPs) for the time fractional diffusion equation (TFDE) with an involution. The determination of a space varying source term, along with solution of a diffusion equation containing n fractional order derivatives in Caputo's sense from extra data at a specific time, constitutes the first IP. The second IP investigates the extracting of a time varying source term as well as the solution of the TFDE from non-local type extra condition. The second IP has applications to microwave radiations. The existence and uniqueness results for the solutions of both IPs are presented.

1. Introduction and formulation of the problems

We will define two IPs for the n terms TFDE with involution in domain $(z,t) \in \Omega$, where, $\Omega = \Big\{(z,t) : -\pi < z < \pi, \quad 0 < t < T\Big\}$. The IPs are described for the following equation,

$$ {}^{C}D^{\alpha_0}_{0|t}\phi(z,t) + \sum_{m=1}^{n} \mu_m \, {}^{C}D^{\alpha_m}_{0|t}\phi(z,t) - \phi_{zz}(z,t) + \epsilon\phi_{zz}(-z,t) = H(z,t), \tag{1.1} $$

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where, ${}^{C}D^{\alpha_m}_{0|t}\phi(z,t)$ stands for Caputo fractional derivatives (CFDs) of order $0 < \alpha_m < 1$ and $\mu_m \geq 0$ where $m = 0, 1, 2, ..., n$ are real constants, subject to the boundary conditions,

$$\phi(-\pi, t) = \phi(\pi, t), \quad t \in [0, T], \tag{1.2}$$

and initial condition,

$$\phi(z, 0) = \psi(z), \quad z \in [-\pi, \pi], \tag{1.3}$$

Physically, the mathematical method presented (1.1)-(1.4) describe the anomalous thermal diffusion cycle in a tight copper wire wrapped around a thin sheet of insulating material. It is assumed that the insulation layer is highly permeable. The temperature on one side has an influence on the diffusion process from the other. The standard diffusion equation is modified for this reason, and we add a third term with an involution. Such process leads to the consideration of an IP for an equation of n parameters TFDE and periodic boundary conditions with respect to a spatial variable, for more detail see [1]- [3]. The non-integer order derivatives play an important role in explaining physical process abnormalities. In this article, we consider multi-term fractional derivative in time because there are anomalies in diffusion/transport process. This non-standard behavior in diffusion/transport process can be described by many ways. One way is to describe these anomalies through many techniques already available in literature e.g. non-chaotic slicer map, continuous time random walk (CTRW), stochastic process etc [4]- [7]. Another way is to introduce the fractional derivative in corresponding equation. So our motivation to consider multi-terms fractional derivatives in time is that we will have more parameters to explain the anomalous behavior in diffusion or transport model. By considering more parameters we are able to fix these anomalies. If the order of fractional derivative lies between 0 and 1, then our model describe the sub-diffusion process.

We have another boundary condition by considering a process in such a way that temperature at one end for all the time t is proportional to the rate of fluctuation of the average speed value of temperature across the wire. Then,

$$\phi(-\pi, t) = \rho\left({}^{c}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n} \mu_m {}^{c}D^{\alpha_m}_{0|t}\right)\int_{-\pi}^{\pi} \phi(\zeta, t)d\zeta, \quad t \in [0, T]. \tag{1.4}$$

Here, ρ denotes proportionality constant.

From nonlocal boundary condition (1.4), and also by using (1.1), we obtain

$$\phi(-\pi, t) = \rho\int_{-\pi}^{\pi}\left\{\phi_{\zeta\zeta}(\zeta, t) - \epsilon\phi_{\zeta\zeta}(-\zeta, t) + H(\zeta, t)\right\}d\zeta, \quad t \in [0, T],$$

which implies that

$$\phi(-\pi, t) = \rho(1-\epsilon)\Big[\phi_\zeta(\pi, t) - \phi_\zeta(-\pi, t)\Big] + \rho\int_{-\pi}^{\pi} H(\zeta, t)d\zeta, \qquad t \in [0, T].$$

We define a new function

$$U(x, t) = \phi(x, t) - \rho\int_{-\pi}^{\pi} H(\zeta, t)d(\zeta),$$

Consequently, in terms of $U(x,t)$, we have the following IP,

$$^{C}D_{0|t}^{\alpha_0}U(z,t)+\sum_{m=1}^{n}\mu_m\,{}^{C}D_{0|t}^{\alpha_m}U(z,t)-U_{zz}(z,t)+\epsilon U_{zz}(-z,t)=H(z,t), \tag{1.5}$$

and the boundary conditions become,

$$\left.\begin{array}{l} U_z(-\pi,t)-U_z(\pi,t)-bU(\pi,t)=0,\\ U(-\pi,t)-U(\pi,t)=0. \end{array}\right\} \tag{1.6}$$

Physically the background of TFDE with involution which is given in 1.5 is a variation of the diffusion equation that demonstrates non-local behavior and describes transport processes with extended memory. It is obtained by substituting the first-order time derivative in the standard diffusion equation with a fractional derivative of order β, where β belongs to the interval (0, 1) given in [8]. The physical interpretation of this equation involves the emergence of self-similarity combined with the long-time limit, which is connected to experimental observations [9]. The equation is employed to examine anomalous diffusion in intricate environments, taking into consideration the impact of a uniform external field on the dynamics of a particle [10]. Numerical analysis and simulations of the equation have been performed to comprehend sub-diffusive transport processes and verify the stability and convergence of numerical schemes [11].The time fractional diffusion equation with involution is a mathematical model used to describe anomalous diffusion in various physical systems. It exhibits non-local behavior and is characterized by FDs. The physical interpretation of this equation is still an active area of research. Several papers have explored the physical background of the TFDE and its applications. Baeumer et al. discuss the derivation of physically meaningful boundary conditions for fractional diffusion equations and highlight the unsuitability of the CFD for modeling fractional diffusion [8]. Nadal et al. emphasize the non-local behavior of the fractional diffusion equation and its implications for physical understanding [13]. Bakalis and Zerbetto study the influence of a random moving particle in a complex environment using the TFDE and provide analytical solutions for various properties of the particle's motion [14].

The second condition in (1.6) refers to the inclusivity of the transmission intensity at the ends of the interval in a physical sense. While the first condition of (1.6) refers to the difference of flow rates along with opposite boundaries to the density value at the boundary in a physical sense. This Dirichlet type condition $U(\pi,t)=U(-\pi,t)=0$, was used instead of condition (1.6) in [15] alongside initial and final conditions

$$U(z,0)=\psi(z)-\rho\int_{-\pi}^{\pi}H(\zeta,t)d(\zeta),\quad z\in[-\pi,\pi], \tag{1.7}$$

$$U(z,T)=\Psi(z)-\rho\int_{-\pi}^{\pi}H(\zeta,t)d(\zeta),\quad z\in[-\pi,\pi], \tag{1.8}$$

and $U(z,t)$ as a solution of Eq (1.5) satisfies (1.7)-(1.8), where $\psi(z)$ and $\Psi(z)$ are given smooth functions; such that $|\epsilon|<1$; and $b=\frac{1}{\rho(\epsilon-1)}$.

Physically IPs arise in almost all areas of science and technology, in modeling of problems motivated by various physical and social processes. Most of these models are governed by differential and integral equations. If all the necessary inputs in these models are known, then the solution can be computed and behavior of the physical system under various conditions can be predicted. In terms of differential problems, the necessary inputs include such information as initial or boundary data, coefficients and force term, also shape and size of the domain. If all these data are enough to describe the system adequately, then it is possible to use the mathematical model for studying the physical system [16].

The investigation into boundary-value problems that are not localized is prompted by the realization that, in numerous instances, a nonlocal condition is a more accurate approach to addressing physical problems in comparison to the conventional local conditions. The examination of IPs with nonlocal boundary conditions is explored, for instance, in the works of [17]- [19].

In [20]- [22], for space TFDE by using a novel modified quasi-reversibility regularization method, conjugate gradient method and non-stationary iterative Tikhonov regularization method have been used for simultaneous identification of source term and initial data. IPs for time fractional diffusion-wave equation have been considered in [23]- [25]. In the articles [26]- [27], IPs of stochastic as well as distributed order diffusion equation are discussed.

Prior research has looked into the well-posedness of direct and IPs for parabolic equations with involution [28]- [30]. The solvability of many IPs for parabolic equations was investigated in articles by Anikonov, Y. E., and Belov, Y. Y., Bubnov, B. A., Prilepko, A. I., and Kostin, A. B., Monakhov, V. N., Kozhanov, A. I., Kaliev, I. A., Sabitov, K. B., and several others [31]- [32]. In Ahmad et al, [15] there are good references to publications on these types of problem. The article [33]- [45] from literature are close to our article's theme. In these articles, various types of direct and inverse initial-boundary value problems for evolutionary equations are considered, including problems with nonlocal boundary conditions and problems for equations with fractional derivatives. Some of these articles use the concept of smart microgrids to solve inverse problems to understand, predict, and control their internal dynamics under uncertainty. For equation (1.1), we define two IPs.

Inverse Source Problem-I for TFDE (ISP-I): In the ISP-I the right hand side of the TFDE will be considered a space dependent source term. In ISP-I we define $H(z,t) := h(z)$ along with $U(z,t)$ which is the solution of the system (1.5)-(1.8) along with over-specified condition $\phi(z,T) = \Psi(z)$. The strong solution of the ISP-I means a couple of functions $\left\{U(z,t), h(z)\right\}$ that satisfies the system $(1.5)-(1.8)$, such that $t^{\alpha_0+\alpha_m}U(z,t) \in C^{2,1}_{z,t}(\overline{\Omega}), t^{\alpha_0+\alpha_m}h(z) \in C([-\pi,\pi])$, and $t^{\alpha_0+\alpha_m}\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U(z,t) \in C((0,T])$, when $0 < \alpha_m < 1$ and μ_m is a real number. We proved that under certain regularity conditions on the given data the ISP-I has a unique strong solution.

Inverse Source Problem-II for TFDE (ISP-II): In the ISP-II, we want to recover time varying source term $s(t)$ and $U(z,t)$ for the system $(1.5)-(1.8)$, with $H(z,t) = s(t)h(z,t)$ for ISP-II. This structure of the source term arise in microwave heating process, in which the external energy is supplied to a target at a controlled level represented $s(t)$ and $h(z,t)$ is the local conversion rate of the microwave energy. The ISP-II does not give us unique solution, so for the uniqueness result of ISP-II we introduce an over-determination condition. Then, with the help of over-determination condition we are able to find the unique solution for the ISP-II. As, the over-determination condition is followed by

$$\int_{-\pi}^{\pi} U(z,t)dz = E(t). \tag{1.9}$$

The condition (1.9) is consider for the unique solution of the IP-II. ISP-II has a strong solution under some assumptions, that is, there exist a pair $\{U(z,t), s(t)\}$ with $t^{\alpha_0+\alpha_m}U(z,t) \in C^{2,1}_{z,t}(\overline{\Omega})$, $s(t) \in C([0,T])$, and $t^{\alpha_0+\alpha_m}\left({}^C D^{\alpha_0}_{0|t} + \sum_{m=1}^{n} \mu_m {}^C D^{\alpha_m}_{0|t}\right)U(z,t) \in C((0,T])$.

The rest of the article is organized as fallows: Section 2 presents some basic concepts from Fractional Calculus (FC) and representation of multinomial Mittag-Leffler (ML) type functions and some its basic results are provided. The spectral problem, properties of eigenvalues and eigenfunctions for the IBVP (1.5)-(1.8) are described in Section 2.2. Section 3 provides the main results about existence and uniqueness of the solution of the ISP-I. In Section 4 we discuss the existence and uniqueness of the solution of ISP-II. Finally, in last section we conclude the paper and present some future perspectives.

2. Preliminaries

This section provides some preliminary results from FC which will be used in the forthcoming section. Indeed after providing definition of fractional order integrals and derivatives, we define the spectral problem (in space variable) of IBVP (1.5)-(1.8).

Let $g \in AC([a,b])$,i.e the space of absolutely continuous function then CFDs left and right sided of order $0 < \beta < 1$, are defined as

$${}^C D^{\beta}_{a_+|t} g(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} \frac{d}{ds} g(s) ds,$$

$${}^C D^{\beta}_{b_-|t} g(t) = \frac{-1}{\Gamma(1-\beta)} \int_t^b (s-t)^{-\beta} \frac{d}{ds} g(s) ds.$$

Where, for the convince of the reader, we provide the definition of absolutely continuous function Let $I \subset \mathbb{R}$. A function $g : I \to \mathbb{R}$ is absolutely continuous on I if for every $\epsilon > 0$, there is a positive number δ in a manner that whenever a sequence of disjoint sub-intervals (x_k, y_k) on interval I is in a manner that $x_k < y_k \in I$ satisfies the following relation

$$\sum_k (y_k - x_k) < \delta$$

implies

$$\sum_k \left|g(y_k) - g(x_k)\right| < \epsilon.$$

2.1. **Multinomial Mittag-Leffler Function.** In this section, we are going to define multinomial Mittag-Leffler function and present some important results related to it which are used in the forthcoming sections.
The multinomial ML function, [48] for $\eta > 0$, $\xi_i > 0$, $z_i \in \mathbb{C}$, $i = 0, 1, ..., m$, $m \in \mathbb{N}$, is defined as where $(k; l_0, ..., l_m) = \dfrac{k!}{l_0! \times ... \times l_m!}$.
Moreover, notice that

$$E_{(\xi_0,\xi_1,...,\xi_m),\eta}(z_0, z_1, ..., z_m) = E_{(\xi_m,...,\xi_1,\xi_0),\eta}(z_m, ..., z_1, z_0), \tag{2.1}$$

and

$$= \sum_{k=0}^{\infty}\sum_{l_0=0}^{k}\sum_{l_1=0}^{l_0}...\sum_{l_m=0}^{l_{m-1}} \frac{k!}{l_1! l_2! ... \Big(k - l_0 - l_1 - ... - l_m\Big)!} \times \frac{z_0^{l_0} z_1^{l_1} ... z_m^{l_m}}{\Gamma\Big(\eta + \xi_0 l_0 + \xi_1 l_1 + ... + \xi_m l_m\Big)}.$$

Remark 2.1. *For $\xi_0 \neq 0$, and $z_0 \neq 0$, $z_1 = ... = z_m = 0$, $m \in \mathbb{N}$ the multinomial ML function become $E_{\xi_0,\eta}(z_0, 0, 0, ..., 0) = \sum_{k=0}^{\infty} \dfrac{z_0^k}{\Gamma\Big(\eta + \xi_0 k\Big)} := E_{\xi_0,\eta}(z_0)$, where $E_{\xi_0,\eta}(z_0)$ is the ML function of two parameters.*

For $n = 2$ the multinomial ML written in series form

$$\begin{aligned} E_{(\xi_0,\xi_1),\eta}(z_0, z_1) &= \sum_{k=0}^{\infty} \sum_{\substack{l_0+l_1=k \\ l_0\geq 0,\, l_1\geq 0}} \frac{k!}{l_0!\, l_1!} \frac{z_0^{l_0} z_1^{l_1}}{\Gamma(\eta + \xi_0 l_0 + \xi_1 l_1)} \\ &= \sum_{k=0}^{\infty}\sum_{l_0=0}^{k} \frac{k!}{l_0!\, (k - l_0)!} \frac{z_0^{l_0} z_1^{k-l_0}}{\Gamma(\eta + \xi_0 l_0 + \xi_1 (k - l_0))}. \end{aligned} \tag{2.2}$$

For $z_1 = 0, \eta = 1$ the multinomial ML function takes form $E_{\xi_0,1}(z_0, 0) = E_{\xi_0,1}(z_0)$, i.e., the one parameter ML function. Following Lemma 3.2 of, [49] and from 2.1

$$E_{(\xi_0,\xi_0-\xi_1,...,\xi_0-\xi_m),\eta}(z_0, z_1, ..., z_m) = E_{(\xi_0-\xi_m,,...,\xi_0-\xi_1,\xi_0),\eta}(z_m, ..., z_1, z_0).$$

we can have the following Lemmata

Lemma 2.1. *[50] For $0 < \eta < 1$ and $0 < \xi_m < ... < \xi_0 < 1$ be given. Assume that $\xi_0\pi/2 < \mu < \xi_0\pi$, $\mu \leq |arg(z_m)| \leq \pi$ and $z_i > 0, i = 0, 1, ..., m$. Then, there exists a constant depending only on $\mu, \xi_i, i = 0, 1, ..., m$ such that*

$$\left|E_{(\xi_0-\xi_m,...,\xi_0-\xi_1,\xi_0),\eta}(z_m, ..., z_1, z_0)\right| \leq \frac{C_1}{1+|z_m|}.$$

We use the following notations throughout the paper

$$\Xi := (\xi_0 - \xi_1, \xi_0 - \xi_2 ..., \xi_0 - \xi_n, \xi_0)$$

and

$$\mathcal{E}_{\Xi,\eta}(t; m_1, m_2, ..., m_n) := t^{\eta-1} E_{\Xi,\eta}(-m_1 t^{\xi_0-\xi_1}, -m_2 t^{\xi_0-\xi_2}, ..., -m_n t^{\xi_0}),$$

where m_i are positive real constants.

Lemma 2.2. *[50] For η, $m_i > 0$, $\xi_i > 0$ and $0 < \xi_m < ... < \xi_0 < 1$ for $i = 0, 1, ..., n$, $n \in \mathbb{N}$ the Laplace transform of the multinomial ML function is given by*

$$\mathcal{L}\left\{\mathcal{E}_{\Xi,\eta}\left(t; m_1, m_2, ...m_{n-1}, m_n\right); s\right\} = \frac{s^{\xi_0-\eta}}{\left(s^{\xi_0} + m_1 s^{\xi_1} + ... + m_{n-1} s^{\xi_n} + m_n\right)}, \tag{2.3}$$

if $\left|m_1 s^{\xi_0-\xi_1} + m_2 s^{\xi_0-\xi_2} + ... + m_{n-1} s^{\xi_0-\xi_n} + m_n s^{\xi_0}\right| < 1.$

Lemma 2.3. *[50] For $g \in C^1([a, b])$ and $m_i > 0, \lambda_i > 0$, for $i = 0, 1, ..., n$, we have*

$$\left|g(t) * \mathcal{E}_{\Xi,\xi_0+1}(t; m_0, m_1, ..., m_n)\right| \leq \frac{C_1}{m_n} \|g\|_{C^1([0,T])},$$

where $\|.\|$ represents a norm given by $\|g\|_{C^1([0,T])} := \max_{0\leq t\leq T} |g(t)| + \max_{0\leq t\leq T} |g'(t)|$ and $C^1([0, T])$ denotes the space of a continuous first-order differentiable function.

2.2. **The spectral problem.** A spectral problem emerges when the Fourier method is used to address the problem (1.5)-(1.8). The spectral problem for the operator £ given by the following equation

$$£X(z) \equiv -X''(z) + \epsilon X''(-z) = \lambda X(z), \quad -\pi < z < \pi, \tag{2.4}$$

where, λ is a spectral parameter, and the boundary conditions are

$$\left.\begin{array}{l} X'(-\pi) - X'(\pi) = bX(\pi), \\ X(-\pi) = X(\pi). \end{array}\right\} \tag{2.5}$$

The equation (2.4) and equation (2.5) has two series of the eigenvalues

$$\lambda_{k1} = (1+\epsilon)k^2, \quad k \in \mathbb{N},$$

and

$$\lambda_{k2} = (1-\epsilon)(k+\sigma_k)^2, \quad \sigma_k = \frac{b}{k+1} O(1) > 0, \ k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\},$$

with associating normalized eigenfunctions given by

$$X_{k1} = \frac{1}{\sqrt{\pi}} \sin(kx),\ k \in \mathbb{N},\ X_{k2} = w_k \cos((k+\sigma_k)x),\ k \in \mathbb{N}_0. \tag{2.6}$$

Here, w_k denote the normalization coefficient:

$$w_k^{-2} = \|\cos((k+\sigma_k)x)\|^2 = \pi + \frac{b^2}{(k+\sigma_k)\,[b^2+(k+\sigma_k)^2\pi^2]}.$$

3. The main results

In this section, we will provide our main results about both IPs defined for TFDE with involution. Indeed, this section has been divided into two subsections corresponding to the results of ISP-I and ISP-II.

3.1. **Inverse source problem-I (ISP-I).** In ISP-I, we consider $H(z,t) = h(z)$. By eigenfunction expansion method the solution of the ISP-I can be written as

$$U(z,t) = U_{02}(t)X_{02}(z) + \sum_{k=1}^{\infty}\Bigg[\sum_{i=1}^{2} U_{ki}(t)X_{ki}(z)\Bigg], \tag{3.1}$$

where, $U_{02}(t) = \langle U(.,t), X_{02}(.)\rangle$, $X_{02}(z) = \left\{\pi + \frac{1}{b(1+\pi^2)}\right\}\cos(bz)$ and $U_{ki}(t) = \langle U(.,t), X_{ki}(.)\rangle$.

Expression of $h(z)$ is given by

$$h(z) = h_{02}X_{02}(z) + \sum_{k=1}^{\infty}\Bigg[\sum_{i=1}^{2} h_{ki}X_{ki}(z)\Bigg], \tag{3.2}$$

where, $h_{02} = \langle h(.), X_{02}(.)\rangle$ and $h_{ki} = \langle h(.), X_{ki}(.)\rangle$.

Consider

$$U_{ki}(t) = \langle U(.,t), X_{ki}(.)\rangle, \qquad i = 1,2. \tag{3.3}$$

We apply the operator $\left({}^{C}D_{0|t}^{\alpha_0} + \sum_{m=1}^{n} \mu_m {}^{C}D_{0|t}^{\alpha_m}\right)$ to (3.3). Then, using Equation (1.1) and integration by parts, we obtain a n term TFDE

$$\left({}^{C}D_{0|t}^{\alpha_0} + \sum_{m=1}^{n} \mu_m {}^{C}D_{0|t}^{\alpha_m}\right)U_{ki}(t) + \lambda_{ki}U_{ki}(t) = h_{ki}, \quad i = 1,2, \tag{3.4}$$

$$U_{ki}(0) = \psi_{ki} - \delta_{ki}, \quad i = 1,2, \tag{3.5}$$

$$U_{ki}(T) = \Psi_{ki} - \delta_{ki}, \quad i = 1,2. \tag{3.6}$$

In above equations, we use notations

$$\psi_{ki} = \int_{-\pi}^{\pi} \psi(x) X_{ki}(z) d(z), \qquad \Psi_{ki} = \int_{-\pi}^{\pi} \Psi(z) X_{ki}(z) d(z),$$
$$\delta_{ki} = \rho \int_{-\pi}^{\pi} h(\zeta) d(\zeta) \int_{-\pi}^{\pi} X_{ki}(z) d(z).$$

After applying Laplace transformation on equation (3.4), we obtain

$$U_{ki}(s) = \frac{s^{\alpha_0 - 1}}{s^{\alpha_0} + \sum_{m=1}^{n} \mu_m s^{\alpha_m} + \lambda_{ki}} U_{ki}(0) + \sum_{m=1}^{n} \mu_m \frac{s^{\alpha_m - 1}}{s^{\alpha_0} + \sum_{m=1}^{n} \mu_m s^{\alpha_m} + \lambda_{ki}} U_{ki}(0) + h_{ki} \frac{1}{s\left(s^{\alpha_0} + \sum_{m=1}^{n} \mu_m s^{\alpha_m} + \lambda_{ki}\right)}. \tag{3.7}$$

Now apply inverse Laplace transformation on equation (3.7) and by virtue of Lemma 2.2, we have

$$\begin{aligned} U_{ki}(t) = {} & \mathcal{E}_{\Xi,1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) U_{ki}(0) \\ & + \sum_{m=1}^{n} \mu_m \mathcal{E}_{\Xi,\alpha_0 - \alpha_m + 1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) U_{ki}(0) \\ & + h_{ki} \mathcal{E}_{\Xi,\alpha_0 + 1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n), \end{aligned} \tag{3.8}$$

where, $U_{ki}(0)$ and h_{ki} are unknowns. By using initial condition we obtain

$$\begin{aligned} U_{ki}(t) = {} & \psi_{ki} \Big\{ \mathcal{E}_{\Xi,1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \\ & + \sum_{m=1}^{n} \mu_m \mathcal{E}_{\Xi,\alpha_0 - \alpha_m + 1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \Big\} \\ & - \delta_{ki} \Big\{ \mathcal{E}_{\Xi,1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \\ & + \sum_{m=1}^{n} \mu_m \mathcal{E}_{\Xi,\alpha_0 - \alpha_m + 1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \Big\} \\ & + h_{ki} \mathcal{E}_{\Xi,\alpha_0 + 1}(t; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n). \end{aligned} \tag{3.9}$$

By, using the value of (3.6), we obtain

$$\begin{aligned} h_{ki} = {} & \frac{1}{\mathcal{E}_{\Xi,\alpha_0 + 1}(T; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n)} \times \\ & \Big[(\Psi_{ki} - \delta_{ki}) - \psi_{ki} \Big\{ \mathcal{E}_{\Xi,1}(T; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \\ & + \sum_{m=1}^{n} \mu_m \mathcal{E}_{\Xi,\alpha_0 - \alpha_m + 1}(T; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \Big\} \\ & - \delta_{ki} \Big\{ \mathcal{E}_{\Xi,1}(T; \lambda_{ki}, \mu_1, \mu_2, ..., \mu_n) \end{aligned}$$

$$+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg\}\Bigg]. \tag{3.10}$$

By putting values of (3.9) and (3.10) in (3.1) and (3.2), we obtain

$$\begin{aligned} U(z,t) &= U_{02}(t)X_{02}(z)+\sum_{k=1}^{\infty}\Bigg[\sum_{i=1}^{2}\Bigg(\psi_{ki}\Bigg\{\mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\\ &\quad+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg\}\\ &\quad-\delta_{ki}\Bigg\{\mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\\ &\quad+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg\}\\ &\quad+h_{ki}\mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{ki}(z)\Bigg]. \end{aligned} \tag{3.11}$$

$$\begin{aligned} h(z) &= h_{02}X_{02}(z)+\sum_{k=1}^{\infty}\Bigg[\sum_{i=1}^{2}\frac{1}{\mathcal{E}_{\Xi,\alpha_0+1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)}\times\\ &\quad\Bigg[(\Psi_{ki}-\delta_{ki})-\psi_{ki}\Bigg\{\mathcal{E}_{\Xi,1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\\ &\quad+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg\}\\ &\quad-\delta_{ki}\Bigg\{\mathcal{E}_{\Xi,1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\\ &\quad+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg\}X_{ki}(z)\Bigg]. \end{aligned} \tag{3.12}$$

Hence, the couple of functions $\Big\{U(z,t),h(z)\Big\}$ is required solution for the ISP-I.

3.2. **Existence of the Solution of the ISP-I.** In this subsection, we will prove a classical solution for the ISP-I, from the following theorem.

Theorem 3.1. *If $\psi(z)$ and $\varphi(z)\in C^4([-\pi,\pi])$ and $\psi(z)$, $\varphi(z)$ and $\psi''(z)$, $\varphi''(z)$ satisfy the boundary conditions (1.6), then the ISP-I has a unique classical solution.*

Proof : We need to show that $t^{\alpha_0+\alpha_m}h(z)\in C([-\pi,\pi])$, $t^{\alpha_0+\alpha_m}U(z,t)\in {C_{z,t}}^{4,1}(\overline{\Omega})$, $t^{\alpha_0+\alpha_m}$ $\left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)U(z,t)\in C((0,T])$ represents a continuous functions. Then, the ISP-1 have a strong solution.

From the analysis of (3.10), it's easy to see that the solution (3.1) of the ISP-I, will form a convergent series iff

$$\lim_{k\to\infty}\delta_{ki}\Big\{\mathcal{E}_{\Xi,1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n) + \sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(T;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Big\} = 0. \qquad i=1,2. \tag{3.13}$$

As, we know that

$$\lim_{k\to\infty}\delta_{ki} = \lim_{k\to\infty}\Big(\rho\int_{-\pi}^{\pi}h(\zeta)d\zeta\Big)\int_{-\pi}^{\pi}X_{ki}(z)dz, \quad i=1,2.$$

which implies that

$$\lim_{k\to\infty}\delta_{ki} = \lim_{k\to\infty}\frac{d}{\lambda_{ki}}\Big(\rho\int_{-\pi}^{\pi}h(\zeta)d\zeta\Big)X_{ki}(\pi)dz = 0,\ i=1,2,3,...,m. \tag{3.14}$$

Now, we shall show of $U(z,t)$ given by (3.11) represents a continuous function. By using Lemma 2.1, we obtained

$$\begin{aligned} T^{\alpha_0+\alpha_m}|h(z)| \;\leq\; & \frac{|\lambda_{02}|}{c_1}T^{\alpha_0+\alpha_m}|\varphi_{02}| - |\psi_{02}|\Big\{T^{\alpha_m} + \sum_{m=1}^{n}\mu_m T^{\alpha_0}\Big\} \\ & + \sum_{k=1}^{\infty}\sum_{i=1}^{2}\Big[\frac{|\lambda_{ki}|}{c_1}T^{\alpha_0+\alpha_m}|\varphi_{ki}| \\ & -|\psi_{ki}|\Big\{T^{\alpha_m} + \sum_{m=1}^{n}\mu_m T^{\alpha_0}\Big\}\Big], \end{aligned}$$

implies that

$$\begin{aligned} T^{\alpha_0+\alpha_m}|h(z)| \;\leq\; & \frac{|\lambda_{02}|}{c_1}T^{\alpha_0+\alpha_m}|\varphi_{02}| - |\psi_{02}|\Big\{T^{\alpha_m} + \sum_{m=1}^{n}\mu_m T^{\alpha_0}\Big\} \\ & + \sum_{k=1}^{\infty}\Big[T^{\alpha_0+\alpha_m}|\varphi^4(z)|\Big\{\frac{1+\epsilon}{k^2} + \frac{1-\epsilon}{(k+\sigma_k)^2}\Big\} \\ & - \sum_{i=1}^{2}|\psi_{ki}|\Big\{T^{\alpha_m} + \sum_{m=1}^{n}\mu_m T^{\alpha_0}\Big\}\Big]. \end{aligned}$$

Hence, $T^{\alpha_0+\alpha_m}h(z)$ is uniformly convergent. Now we are going to prove that the series related to the operator $\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U(z,t)$ also uniformly convergent.

Consider

$$\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U(z,t) \;=\; \left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U_{02}(t)X_{02}(z)$$

$$+\left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)$$
$$\sum_{k=1}^{\infty}\sum_{i=1}^{2}\Big[U_{ki}(t)X_{ki}(z)\Big], \tag{3.15}$$

by using Lemma 15.2 (page 278, see [47])

$$\begin{aligned}\left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)U(z,t) &= \left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)U_{02}(t)X_{02}(z)\\ &\quad+\sum_{k=1}^{\infty}\sum_{i=1}^{2}\left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)\\ &\quad U_{ki}(t)X_{ki}(z),\\ &= h(z)-\Bigg[\lambda_{02}U_{02}(t)X_{02}(z)\\ &\quad+\sum_{k=1}^{\infty}\sum_{i=1}^{2}\lambda_{ki}\Big\{U_{ki}(t)X_{ki}(z)\Big\}\Bigg].\end{aligned}$$

First, we will prove that the series $\left[\lambda_{02}U_{02}(t)X_{02}(z)+\sum\limits_{k=1}^{\infty}\sum\limits_{i=1}^{2}\lambda_{ki}\Big\{U_{ki}(t)X_{ki}(z)\Big\}\right]$ is uniformly convergent, by using Lemma 2.1, we obtain

$$\begin{aligned}t^{\alpha_0+\alpha_m}\left|\sum_{k=1}^{\infty}\sum_{i=1}^{2}\lambda_{ki}U_{ki}(t)X_{ki}(z)\right| &\leq \sum_{k=1}^{\infty}c_1|\psi^4(z)|\left(\frac{1+\epsilon}{k^2}+\frac{1-\epsilon}{(k+\sigma_k)^2}\right)\\ &\quad\left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right\}+\sum_{i=1}^{2}t^{\alpha_0+\alpha_m}|h_{ki}|,\end{aligned}$$

and

$$t^{\alpha_0+\alpha_m}\Big|\lambda_{02}U_{02}(t)X_{02}(z)\Big|\leq c_1|\psi_{02}|\left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu t^{\alpha_0}\right\}+t^{\alpha_0+\alpha_m}|h_{02}|.$$

By virtue of Weierstrass M-test the series $t^{\alpha_0+\alpha_m}\left[\lambda_{02}U_{02}(t)X_{02}(z)+\sum\limits_{k=1}^{\infty}\sum\limits_{i=1}^{2}\right.$ $\left.\lambda_{ki}\{U_{ki}(t)X_{ki}(z)\}\right]$ represents a continuous function. Furthermore, the uniform convergence of $T^{\alpha_0+\alpha_m}h(z)$ have been proved already. First, we will prove the uniqueness result for the ISP-I.

Theorem 3.2. *Consider a couple of functions $\Big\{U_1(z,t),h_1(z)\Big\}$ and $\Big\{U_2(z,t),h_2(z)\Big\}$ be the two classical solution sets for the ISP-I and $z_0\in(-\pi,\pi)$, such that $U_1(z_0,t)=U_2(z_0,t)$, then $U_1(z,t)=U_2(z,t)$ implies $h_1(z)=h_2(z)$ for all $z\in(-\pi,\pi)$.*

Proof. The proof of theorem 3.2 can be proved by following the same strategy given in [51]. □

4. ISP-II

In this section, we will introduce a pair of function $\{U(z,t), s(t)h(z,t)\}$ for the system (1.5)-(1.8), for the ISP-II. Whenever, the over-determination condition (1.9) is given. Furthermore, we will be proved $s(t)$, $t^{\alpha_0+\alpha_m}U(z,t)$ and $t^{\alpha_0+\alpha_m}\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U(z,t)$ represents a continuous functions.

4.1. **Solution for the ISP-II.** By using eigen-function expansion method the solution for the ISP-II can be written by

$$U_{ki}(t) = \langle U(.,t), X_{ki}(.)\rangle \qquad i = 1,2. \tag{4.1}$$

where $U_{ki}(t)$ satisfy the following fractional differential equation

$$\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\right)U_{ki}(t) + \lambda_{ki}U_{ki}(t) = s(t)h_{ki}(t),\ i = 1,2. \tag{4.2}$$

By following the same strategy which we used for ISP-I, we find $U_{ki}(t)$, then we write the series solution for ISP-II

$$\begin{aligned} U_{ki}(t) \;=\;& \psi_{ki}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n) \\ &+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Big\} \\ &-\delta_{ki}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n) \\ &+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Big\} \\ &+\Big\{s(t)h_{ki}(t)\Big\} * \mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n). \end{aligned} \tag{4.3}$$

$$\begin{aligned} U(z,t) \;=\;& \Bigg[\Bigg(\psi_{02}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n) \\ &+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Big\} \\ &+\Big\{s(t)h_{02}(t)\Big\} * \mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{02}(z)\Bigg] \\ &+\sum_{k=1}^{\infty}\sum_{i=1}^{2}\Bigg[\Bigg(\psi_{ki}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n) \\ &+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Big\} \\ &+\Big\{s(t)h_{ki}(t)\Big\} * \mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{ki}(z)\Bigg], \end{aligned} \tag{4.4}$$

where $s(t)$ is still to be determined.

4.2. **Existence of the Solution for the ISP-II.** Now, we proved the existence of solution for ISP-II in the given domain $\overline{\Omega}$, under the assumptions of following theorem:

Theorem 4.1. *The following conditions should be satisfied for the existence of solution for the ISP-II.*
(i) $\psi \in C^2([-\pi,\pi])$ *be such that* $\psi(-\pi) = 0 = \psi(\pi)$.
(ii) $h(.,t) \in C^2([-\pi,\pi])$ *be such that* $h(-\pi,t) = 0 = h(\pi,t)$.
Moreover,

$$\int_{-\pi}^{\pi} h(z,t)dz \neq 0$$

and

$$\left(\int_{-\pi}^{\pi} h(z,t)dz\right)^{-1} \leq M_1, \qquad \forall \quad 0 \leq t \leq T.$$

Where, the constant $M_1 > 0$.
(iii) $E \in AC([0,T])$ *and* $E(t)$ *satisfies the condition*

$$\int_{-\pi}^{\pi} \psi_m(z)dz = E(0). \qquad m = 0,1,2,...,n.$$

Then, there exist a unique classical solution for the ISP-II.

Proof : To find $s(t)$, we will use extra condition (1.9) and apply the operator $\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n} \mu_m {}^{C}D^{\alpha_m}_{0|t}\right)$ to (1.9), then we have

$$\int_{-\pi}^{\pi}\left({}^{C}D^{\alpha_0}_{0|t}U(z,t) + \sum_{m=1}^{n} \mu_m {}^{C}D^{\alpha_m}_{0|t}U(z,t)\right)dz \tag{4.5}$$
$$=\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n} \mu_m {}^{C}D^{\alpha_m}_{0|t}\right)E(t),$$

from equation (1.5), we obtain

$$\int_{-\pi}^{\pi}\Big(U_{zz}(z,t) - \epsilon U_{zz}(z,t)\Big)dz + s(t)\int_{-\pi}^{\pi} h(z,t)dz \tag{4.6}$$
$$=\left({}^{C}D^{\alpha_0}_{0|t} + \sum_{m=1}^{n} \mu_m {}^{C}D^{\alpha_m}_{0|t}\right)E(t),$$

this results in the following: Integral equation of the Volterra kind

$$s(t)=\frac{\left({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\right)E(t)-\int_{-\pi}^{\pi}\left(U_{zz}(z,t)-\epsilon U_{zz}(-z,t)\right)dz}{\int_{-\pi}^{\pi}h(z,t)dz}, \tag{4.7}$$

first, we solve it

$$\int_{-\pi}^{\pi}\left(U_{zz}(z,t)-\epsilon U_{zz}(-z,t)\right)dz, \tag{4.8}$$

after taking integration of (4.8) and then using boundary conditions, we obtain $d(1+\epsilon)U(\pi,t)$. Similarly, we have another way to solve equation (4.8), by using Fourier method ,

$$T(t)\int_{-\pi}^{\pi}\left\{X''(z)-X''(-z)\right\}dz,$$

by taking integration and boundary conditions (2.5), we obtain

$$d(1+\epsilon)T(t)X(\pi) \qquad \Rightarrow \qquad d(1+\epsilon)U(\pi,t).$$

which, is same in both cases.
Then, we have

$$\begin{aligned} & d(1+\epsilon)U(\pi,t) \\ = \;& d(1+\epsilon)\Bigg[\Bigg(\psi_{02}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n) \\ & +\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Big\} \\ & +\Big\{s(t)h_{02}(t)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{02}(\pi)\Bigg] \\ & +\sum_{k=1}^{\infty}\Bigg[\Bigg(\psi_{k2}\Big\{\mathcal{E}_{\Xi,1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n) \\ & +\sum_{m=1}^{n}\mu_n\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\Big\} \\ & +\Big\{s(t)h_{k2}(t)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{k2}(\pi)\Bigg], \end{aligned} \tag{4.9}$$

which implies that

$$s(t) \;=\; \left(\int_{-\pi}^{\pi}g(z,t)dz\right)^{-1}\Bigg[\Bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m{}^{C}D^{\alpha_m}_{0|t}\Bigg)E(t)$$

$$
\begin{aligned}
&+d(1+\epsilon)\Bigg(\psi_{02}\Bigg\{\mathcal{E}_{\Xi,1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\\
&+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Bigg\}\\
&+d(1+\epsilon)\sum_{k=1}^{\infty}\Bigg[\Bigg(\psi_{k2}\Bigg\{\mathcal{E}_{\Xi,1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\\
&+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\Bigg\}\\
&+\Big\{h_{k2}(t)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{k2}(\pi)\Bigg]. \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
F(t) &= d(1+\epsilon)\Bigg(\psi_{02}\Bigg\{\mathcal{E}_{\Xi,1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\\
&+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Bigg\}\Bigg)X_{02}(\pi)\\
&+d(1+\epsilon)\sum_{k=1}^{\infty}\Bigg(\psi_{k2}\{\mathcal{E}_{\Xi,1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\\
&+\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\}\Bigg)\\
&\cos((k+\delta_k)\pi), \qquad (4.11)
\end{aligned}
$$

and

$$
\begin{aligned}
k(t,\tau) &= d(1+\epsilon)\Bigg(\Big\{h_{02}(\tau)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t-\tau;\lambda_{02},\mu_1,\mu_2,...,\mu_n)\Bigg)X_{02}(\pi)\\
&+d(1+\epsilon)\sum_{k=1}^{\infty}\Bigg(\Big\{h_{k2}(\tau)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t-\tau;\lambda_{k2},\mu_1,\mu_2,...,\mu_n)\Bigg)\\
&\cos((k+\delta_k)\pi). \qquad (4.12)
\end{aligned}
$$

Then equation (4.10) can be written as,

$$
\begin{aligned}
s(t) &= \Bigg(\int_{-\pi}^{\pi}g(z,t)dx\Bigg)^{-1}\Bigg[\Bigg({}^{C}D_{0|t}^{\alpha_0}+\sum_{m=1}^{n}\mu_m{}^{C}D_{0|t}^{\alpha_m}\Bigg)E(t)+\mathcal{F}(t)\\
&+\int_{0}^{t}s(\tau)k(t,\tau)d\tau\Bigg].
\end{aligned}
$$

Now, we consider a space of continuous functions $C([0,T])$, with the Chebyshev norm. Then, we are going to define a map $M : C([0,T]) \to C([0,T])$ by

$$M(s(t)) := s(t),$$

where, $s(t)$ is given by (4.7).

$$\begin{aligned} t^{\alpha_0+\alpha_m}|F(t)| &\leqslant d(1+\epsilon)\frac{c_1|\psi_{02}|}{|\lambda_{02}|}\left[\left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right\}\right] \\ &\quad +d(1+\epsilon)\frac{c_1|\psi_{k2}|}{|\lambda_{k2}|}\left[\left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right\}\right]. \end{aligned} \tag{4.13}$$

The uniform convergence of the series involves in (4.13) is ensured by using Lemma 2.1 and continuity of $|\psi_{k2}|$. Hence, by Weierstrass M-test $F(t)$ presents a continuous function. Now, we shall show that equation (4.12), represents a continuous function.

$$|k(t,\tau)| \leqslant d(1+\epsilon)\left[\frac{c_1}{|\lambda_{02}|}\|h_{02}\| + \frac{c_1}{|\lambda_{k2}|}\|h_{k2}\|\right]. \tag{4.14}$$

Then, the uniform convergence of the series involves in (4.14) is ensured by using Lemma 2.3 and continuity of $g(z,t)$. Hence, by Weierstrass M-test $k(t,\tau)$, represents a continuous function such that

$$|K(t,\tau)| \quad \leq M_2. \tag{4.15}$$

Hence, the mapping is well defined. Now, we will show that $\mathbb{M}(s(t)) := s(t)$, is a contraction mapping under the assumption $T < \frac{1}{M_1M_2}$. For this we consider

$$|\mathbb{M}(s(t)) - \mathbb{M}(c(t))| \leq \left(\int_{-\pi}^{\pi} h(z,t)dz\right)^{-1}\int_0^t k(t,\tau)|s(\tau)-c(\tau)|d\tau,$$

By assumptions of the Theorem 4.1, we obtain

$$\max_{0\leq t\leq T}\Big|M(s(t)) - M(c(t))\Big| \leq M_1M_2T\max_{0\leq t\leq T}|s(\tau)-c(\tau)|.$$

which implies

$$\Big\|M(s(t)) - M(c(t))\Big\|_{C([0,T])} \leq \quad M_1M_2T\|s-c\|_{C([0,T])}.$$

Where, M_1 and M_2 are positive constants independents of n and M_2 is given by (4.15) which implies that the mapping $M(\cdot)$ is contraction map. Hence, by Banach fixed point theorem unique existence of $s(t)$ is proved.

By using Lemma 2.1 and Lemma 2.3 in equation (4.3), we have the following relation

$$\begin{aligned} t^{\alpha_0+\alpha_m}|U_{ki}(t)| &\leqslant \frac{c_1}{|\lambda_{ki}|}\left\{|\psi_{ki}|\left(t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right)\right. \\ &\quad \left. +t^{\alpha_0+\alpha_m}\|sh_{ki}\|_{C([0,T])}\right\} \end{aligned} \tag{4.16}$$

Since $s \in C([0,T]), \|s\|_{C([0,T])} < M$ where $M > 0$, and by using Lemma 2.3

$$t^{\alpha_0+\alpha_m}|U_{ki}(t)| \leqslant \frac{c_1}{|\lambda_{ki}|}\Bigg\{|\psi^4(z)|\bigg(\frac{1+\epsilon}{k^2}+\frac{1-\epsilon}{(k+\sigma_k)^2}\bigg)\bigg(t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\bigg) + t^{\alpha_0+\alpha_m}M\|h_{ki}\|_{C([0,T])}\Bigg\}$$

Next, we will prove the continuity of solution $U(z,t)$, which is given by(4.4) and each term of (1.5), that is $U_{zz}(z,t), U_{zz}(-z,t)$.

By using Lemma 2.1 and Lemma 2.3 in equation (4.4), we obtain

$$\begin{aligned} t^{\alpha_0+\alpha_m}|U(z,t)| \leqslant{} & \frac{c_1}{\lambda_{02}}\Bigg[|\psi_{02}|\bigg\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\bigg\} + t^{\alpha_0+\alpha_m}M\|h_{02}\|_{C([0,T])}\Bigg] \\ & + \sum_{k=1}^{\infty}\sum_{i=1}^{2}\frac{c_1}{|\lambda_{ki}|}|\psi^4(z)|\bigg(\frac{1+\epsilon}{k^2}+\frac{1-\epsilon}{(k+\sigma_k)^2}\bigg) \\ & \bigg(t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\bigg) + M\|h_{ki}\|_{C([0,T])}t^{\alpha_0+\alpha_m}\Bigg]. \end{aligned}$$

By Weierstrass M-test the above series represent a continuous function.

Now, we are going to prove that the series related to the operator $t^{\alpha_0+\alpha_m}\bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg)U(z,t)$ also uniformly convergent, for this we consider the following relation

$$\begin{aligned} \bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg)U(z,t) ={} & \bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg)U_{02}(t)X_{02}(z) \\ & + \bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg) \\ & \sum_{k=1}^{\infty}\sum_{i=1}^{2}\Big[U_{ki}(t)X_{ki}(z)\Big], \end{aligned} \tag{4.17}$$

by using Lemma 15.2 (page 278, see [47]), we have

$$\begin{aligned} \bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg)U(z,t) ={} & \bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg)U_{02}(t)X_{02}(z) \\ & + \sum_{k=1}^{\infty}\sum_{i=1}^{2}\bigg({}^{C}D^{\alpha_0}_{0|t}+\sum_{m=1}^{n}\mu_m {}^{C}D^{\alpha_m}_{0|t}\bigg) \\ & \Big[U_{ki}(t)X_{ki}(z)\Big] \\ ={} & s(t)h(z,t) - \Bigg[\lambda_{02}U_{02}(t)X_{02}(z) + \\ & \sum_{k=1}^{\infty}\sum_{i=1}^{2}\lambda_{ki}\Big\{U_{ki}(t)X_{ki}(z)\Big\}\Bigg], \end{aligned}$$

we already prove the local well-posedness of $s(t)$, we will prove that the series $\Big[\lambda_{02}U_{02}(t)X_{02}(z) + \sum_{k=1}^{\infty}\sum_{i=1}^{2}\lambda_{ki}\Big\{U_{ki}(t)X_{ki}(z)\Big\}\Big]$ is uniformly convergent, by using Lemma 2.1 and 2.3, we obtain

$$\begin{aligned} t^{\alpha_0+\alpha_m}\left|\sum_{k=1}^{\infty}\sum_{i=1}^{2}\lambda_{ki}U_{ki}(t)X_{ki}(z)\right| \quad \leqslant \quad & \sum_{k=1}^{\infty}c_1|\psi^4(z)|\left(\frac{1+\epsilon}{k^2}+\frac{1-\epsilon}{(k+\sigma_k)^2}\right) \\ & \left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right\}+\sum_{i=1}^{2}t^{\alpha_0+\alpha_m} \\ & M\|h_{ki}\|_{C([0,T])}, \end{aligned}$$

and

$$\begin{aligned} t^{\alpha_0+\alpha_m}\Big|\lambda_{02}U_{02}(t)X_{02}(z)\Big| \quad \leqslant \quad & c_1|\psi_{02}|\left\{t^{\alpha_m}+\sum_{m=1}^{n}\mu_m t^{\alpha_0}\right\} \\ & +t^{\alpha_0+\alpha_m}M\|h_{02}\|_{C([0,T])}. \end{aligned}$$

Hence, the above series converge uniformly. Similarly we proved $U_{zz}(z,t), U_{zz}(-z,t)$ represent a continuous function.

4.3. **Uniqueness of the Solution of ISP-II.** In this Subsection, we will consider the uniqueness of the ISP-II solution.

Theorem 4.2. *Under the assumptions of Theorem 4.1, solution of the ISP-II is unique.*

Proof : With the help of Banach fixed point theorem, we prove the uniqueness of the source term $s(t)$. Now, we will present uniqueness of $U(z,t)$.

Let $U(z,t)$ and $V(z,t)$ be the two regular solution sets of the ISP-II, and let $\widetilde{U}(z,t) = U(z,t) - V(z,t)$. Then, the function $\widetilde{U}(z,t)$ satisfy the following equation

$$^{C}D_{0|t}^{\alpha_0}\widetilde{U}(z,t) + \sum_{m=1}^{n}\mu_m {}^{C}D_{0|t}^{\alpha_m}\widetilde{U}(z,t) - \widetilde{U}_{zz}(z,t) \quad + \quad \epsilon\widetilde{U}_{zz}(-z,t) = 0,$$

with the boundary conditions (1.6) and the initial condition

$$\widetilde{U}(z,0) = 0, \tag{4.18}$$

Consider the functions

$$\widetilde{U}_{ki}(t) = \langle\widetilde{U}(z,t), X_{ki}(z)\rangle, \qquad i = 1,2. \tag{4.19}$$

Applying the fractional operator to both sides of the first relation in (4.19), we have

$$\left(^{C}D_{0_+,t}^{\alpha_0} + \sum_{i=1}^{m}\mu_i {}^{C}D_{0_+,t}^{\alpha_i}\right)\tilde{U}_{ki}(t) \quad = \quad \left\langle\left(^{C}D_{0_+,t}^{\alpha_0} + \sum_{i=1}^{m}\mu_i {}^{C}D_{0_+,t}^{\alpha_i}\right)\widetilde{U}(z,t), X_{ki}(z)\right\rangle,$$

By virtue of (1.5), we have the following equation

$$ {}^{C}D^{\alpha_0}_{0_+,t}\widetilde{U}_{k,1}(t)+\sum_{i=1}^{m}\mu_i {}^{C}D^{\alpha_i}_{0_+,t}\widetilde{U}_{k,1}(t) \quad = \quad -\lambda_{ki}\widetilde{U}_{ki}(t)+\widetilde{s}(t)h_{ki}(t), $$

The solution of the above equation is

$$ \begin{aligned} \widetilde{U}_{ki}(t) \quad = \quad & \mathcal{E}_{\Xi,1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n)\widetilde{U}_{ki}(0) \\ & +\sum_{m=1}^{n}\mu_m\mathcal{E}_{\Xi,\alpha_0-\alpha_m+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n) \\ & \widetilde{U}_{ki}(0)+\Big\{s(t)h_{ki}(t)\Big\}*\mathcal{E}_{\Xi,\alpha_0+1}(t;\lambda_{ki},\mu_1,\mu_2,...,\mu_n). \end{aligned} $$

By equation (4.18), uniqueness of $\widetilde{s}(t)$ and completeness of $X_{ki}(z)$, we obtain

$$ \widetilde{U}_{ki}(t)=0, \quad \forall\, n\in\mathbb{N}\cup\{0\}. $$

and hence $U(z,t)=V(z,t)$.

5. Conclusions

We considered two IPs for TFDE with involution along with one Dirichlet type and second boundary condition of non-local type. The eigenfunction expansion method is used to have series representation of the solutions of both IPs. The unique existence of the time dependent source term is obtained by using Banach fixed point theorem in IP-2. We checked existence and uniqueness results for both IPs by considering suitable over-specified and certain regularity conditions on given datum. In future, we consider an another interesting problem by considering multi-time Hilfer fractional derivatives in time with same spectral problem.

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