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Optimal Solutions of the Time-Fractional Wave Models

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Abstract. In this article, we provide an update on the optimal auxiliary function method (OAFM) in this work. This semi-analytical approach uses the Caputo fractional derivative operator (FOAFM) to solve fractional order differential equations. The efficiency and reliability of the method are shown by using modified equal-width model (MEW),

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equal-width model (EW), and regularized long wave model (RLW). Hydromagnetics waves in cold plasma are largely dependent on the aforementioned models. Our objective is to study the nonlinear behavior of the plasma system and determine its important features. The results show that even at the first iteration, our proposed method is simple, less expensive computationally, and rapidly converges to accurate results. The presence of the proper auxiliary constants allows for the achievement of convergence and stability. The technique has a remarkable ability to solve many different scientific and technical problems.

1. Introduction

A large area of research adopted on fractional assessment equations during the last 10 years, because of how many different applications there are in today's scientific and technological industries. Time-fractional equations have been demonstrated to explain a number of physical phenomena and to solve a wide range of problems. In this field, more creative uses of fractional calculus are needed ([1], [2], [3], [4], [5], [6]). Advances in fractional differential equations have attracted a lot of attention recently due to their various applications in a variety of nonlinear complex systems that arise in viscoelastic, mathematical biology, life sciences, fluid mechanics ([7], [8], [9]) existence and approximate controllability see ([10], [11], [12]), dengue transmission analysis is covered in [13], which is one of the main point of fractional calculus popularity.

Ford and Simpson found that the most efficient way to find time-fractional problems is to use the fractional Caputo derivative [14], as it always contains the initial parameters that are not available in different models [15]. Fractional derivatives and integrals, according to Spanier and Oldham, can be utilized to depict synthetic models that are substantially more meaningful than those produced using traditional techniques [16]. Eventually, there may be literature-based discussions about the promises and uses of fractional theory, similar to fractal mathematics. It is recommended that interested readers consult ([17], [18], [19], [20], [21], [22], [23], [24], [25]).

These methods are used to address nonlinear issues, including fluid problems, nonlinear boundary conditions problems, nonlinear wave equations, and other topics ([26], [27], [28], [29], [30]). Non-linear dispersive media allow wave pulses to propagate because of their dispersive effects as well as dynamical balance between non-linearity. There is a persistent waveform that holds these equations. A solitons is a kind of solitary wave that, when it collides with other solitons, continues to propagate at the same speed and shape.

Many complex nonlinear processes in the sciences are described by the fractional-order nonlinear (EW) equations; these are most commonly found in the domains of fluid mechanics, solid-state physics, plasma physics, and plasma waves. water waves, waves in enharmonic crystals, surface waves in compressible fluids, cold plasma, and other nonlinear systems were among the nonlinear systems whose behavior was explained by the (EW) equations ([31], [32], [33], [34]). Xu analyzed the fractional wave equations in [35] and reduced the principal equation to two fractional ODE. To evaluate partial differential equations, for example, Dehghan et al. in [36] employed the HAM; in this work, fractional derivatives are expressed in the Liouville-Caputo sense.

Various academics have approached the problem of solving nonlinear fractional differential equations in different ways. Many researchers have employed various techniques in the past to address a range of issues, including the Adomian decomposition strategy, the perturbation method, and other techniques refer to ([37], [38], [39], [40], [41], [42], [43], [44], [45].

In our study, we use the fractional optimal auxiliary function method, in this context Marinca et al.([46], [47], [48]), present a new procedure called the Optimal Homotopy Asymptotic Method(OHAM). The OHAM convergence is fast to the actual solution in comparison of other strategies and is most versatile. In continuation to OHAM, the auxiliary function technique (OAFM), i.e further extended by Hakeem Ullah et al. ([49], [50], [51]), which uses a fractional complex transformation to solve higher dimensional problems, is presented in a similar way. This process uses an auxiliary parameter for convergence control and axiomatic functions that control and speed up the method's convergence. If we desire to improve the method's accuracy, we just make a change and increase the amount of auxiliary convergence or the auxiliary function Controlling parameters to get precise results. Recent research on the thin-film flow of a fourth-grade fluid down a vertical cylinder using the proposed approach was conducted by Marinca and colleagues. Later, Laiq Zada et al. [52] employed the presented approach for the seventh order of partial differential equations.

The rest of the manuscript is organized as: the preliminary fractional computation is included in the second section. The fundamental theory of FAOFM is presented in the third section, while the suggested method is evaluated for a better understanding of the time-fractional models in section three, while the last section includes the conclusion.

2. Definitions

Here, we provide some definitions of the Caputo fractional derivative.

Definition 2.1. *([13], [18]) The Riemann-Liouville operator of $\Im \in C_\mu$ having order $\alpha \geq 0$ is*

$$L_\tau^\alpha \Im(\tau) = \Im(\tau) \ \ if \alpha = 0, \tag{2.1}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \Im(s) ds, \quad if \alpha > 0, \tag{2.2}$$

Definition 2.2. *([13], [18]) Fractional derivative of $\Im \in C_\mu$ have $\alpha \geq 0$, $m \in N \cup 0$ in Caputo-sense is*

$$D_\tau^\alpha \Im(\tau) = L_\tau^{m-\alpha} D_\tau^m \Im, \quad if \ \alpha = n \in N, \tag{2.3}$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau - s)^{m-\alpha-1} \Im^m(s) ds, \quad if \ m-1 < \alpha < n, n \in N. \tag{2.4}$$

3. Theory of optimal auxiliary function method

Let us consider

$$\frac{\partial^\alpha \Im(\upsilon, \tau)}{\partial \tau^\alpha} = A\Im(\upsilon, \tau) + s(\upsilon), \alpha > 0, \tag{3.1}$$

where $\frac{\partial^\alpha}{\partial\tau^\alpha}$ the fractional operator, A is the differential operator comprising of two parts linear and non linear, "s" be the analytic function, υ and τ are the spatial and temporal variable, while α is the fractional derivatives parameter.
With the given initial conditions

$$\begin{aligned} D_0^{\alpha-r}(\upsilon,0) &= g_r(\upsilon), r = 0,1,2,...,s-1, \\ D_0^{\alpha-s}(\upsilon,0) &= 0, s = [\alpha], \\ D_0^{r}(\upsilon,0) &= h_r(\upsilon), \\ D_0^{s}(\upsilon,0) &= 0, s = [\alpha]. \end{aligned} \tag{3.2}$$

Taking

$$\widetilde{\mathfrak{I}}(\upsilon,\tau,G_l) = \mathfrak{I}_0(\upsilon,\tau) + \mathfrak{I}_1(\upsilon,\tau,G_l), l = 1,2,...,s, \tag{3.3}$$

substituting Equation 3.2 in Equation 3.1, we obtain ordered approximation.
The zero-order approximation is

$$\begin{aligned} \frac{\partial^\alpha \mathfrak{I}_0(\upsilon,\tau)}{\partial\tau^\alpha} - s(\upsilon) &= 0, \\ \mathfrak{I}_0(\upsilon,0) &= g_r(\upsilon), \end{aligned} \tag{3.4}$$

The 1st approximation is

$$\begin{aligned} \frac{\partial^\alpha \mathfrak{I}_1(\upsilon,\tau,G_l)}{\partial\tau^\alpha} + N(\mathfrak{I}_0(\upsilon,\tau) + \mathfrak{I}_1(\upsilon,\tau,G_l)) = 0, \\ \mathfrak{I}_1(\upsilon,0) = h_r(\upsilon), \end{aligned} \tag{3.5}$$

Equations 3.4-3.5, having fractional derivatives, thus by utilizing I^α operator, we obtained

$$(\mathfrak{I}_0(\upsilon,\tau) = I^\alpha[s(\upsilon)] = 0, \tag{3.6}$$

and

$$\mathfrak{I}_1(\upsilon,\tau,G_l) = I^\alpha[N(\mathfrak{I}_0(\upsilon,\tau) + \mathfrak{I}_1(\upsilon,\tau,G_l))] = 0, \tag{3.7}$$

The nonlinear term is determined as

$$N(\mathfrak{I}_0(\upsilon,\tau) + \mathfrak{I}_1(\upsilon,\tau,G_l)) = N(\mathfrak{I}_0(\upsilon,\tau)) + \sum_{l=1}^{\infty} \mathfrak{I}_1^l(\upsilon,\tau,G_l) N^l(\mathfrak{I}_0(\upsilon,\tau)), \tag{3.8}$$

Equation 3.8 can be re-written as

$$\begin{aligned} L(\mathfrak{I}_1(\upsilon,\tau,G_l)) + D_1((\mathfrak{I}_0(\upsilon,\tau),G_m)F(N(\mathfrak{I}_0(\upsilon,\tau)))) + D_2((\mathfrak{I}_0(\upsilon,\tau),G_n)) = 0, \\ \left(B(\mathfrak{I}_1(\upsilon,\tau,G_l), \frac{d(\mathfrak{I}_1(\upsilon,\tau,G_l))}{d\upsilon}\right) = 0, n = 1,2,3,...,q, m = q+1, q+2, q+3,...,s, \end{aligned} \tag{3.9}$$

Lemma:
For the convergence of the method we have

$$K(G_s) = \int_I R^2(\mathfrak{I},G_s)d\upsilon, \tag{3.10}$$

The auxiliary/optimal constants are obtained as

$$\partial_{G_1}K = 0, \partial_{G_2}K = 0, \partial_{G_3}K = 0, ..., \partial_{G_s}K = 0. \tag{3.11}$$

Using the G_s values list, we determined the approximation as

$$\tilde{\Im}(\upsilon,\tau) = \Im_0(\upsilon,\tau) + \Im_1(\upsilon,\tau). \tag{3.12}$$

Problem 3.1. Consider the Equal Width Wave equation

$$D_\tau^\alpha \Im + \Im\Im_\upsilon + \Im_{\upsilon\upsilon\tau} \quad , \qquad \tau > 0, \qquad 0 < \alpha \leq 1, \tag{3.13}$$

with the given initial value

$$\Im(\upsilon,0) = 3sech^2(\frac{\upsilon-15}{2}), \tag{3.14}$$

the exact solution is

$$\Im(\upsilon,\tau) = 3sech^2(\frac{\upsilon-15-\tau}{2}), \tag{3.15}$$

where D_τ^α, are the fractional operators for fractional value α, υ and τ are the spatial and time independent variable for $\Im(\upsilon,\tau)$. We consider the auxiliary function as

$$\begin{aligned} D_1 &= G_1 3sech^2(\frac{\upsilon-15}{2}) + G_2 3sech^4(\frac{\upsilon-15}{2}), \\ D_2 &= G_3 3sech^8(\frac{\upsilon-15}{2}) + G_4 3sech^{12}(\frac{\upsilon-15}{2}), \end{aligned} \tag{3.16}$$

the linear and non-linear terms can be written as

$$\begin{aligned} L &= D_\tau^\alpha \Im(\upsilon,\tau), \\ N &= \Im\Im_\upsilon - \Im_{\upsilon\upsilon\tau}, \end{aligned} \tag{3.17}$$

Zeorth order system:

$$D_\tau^\alpha \Im_0(\upsilon,\tau) = 0, \tag{3.18}$$

with initial condition

$$\Im_0(\upsilon,0) = 3sech^2\Big(\frac{\upsilon-15}{2}\Big),$$

its solution is

$$\Im_0(\upsilon,\tau) = 3sech^2\Big(\frac{\upsilon-15}{2}\Big), \tag{3.19}$$

First order system:

$$D_\tau^\alpha \Im(\upsilon,\tau) + D_1(\Im_0(\upsilon,\tau),G_m)N(\Im_0(\upsilon,\tau)) + D_2(\Im_0(\upsilon,\tau),G_n) = 0, \tag{3.20}$$

with given initial condition, using Eqs.(20),(22) in Eq.(24) its solution is obtain as

$$\begin{aligned} \Im_1(\upsilon,\tau) = -\frac{1}{\alpha\Gamma(\alpha)} 3\tau^\alpha sech^2\Big(\frac{\upsilon-15}{2}\Big)tanh\Big(\frac{\upsilon-15}{2}\Big), \\ \Big(9sech^4\Big(\frac{\upsilon-15}{2}\Big)G_4 + (9sech^4\Big(\frac{\upsilon-15}{2}\Big)G_2 - G_3\Big)tanh\Big(\frac{\upsilon-15}{2}\Big)\Big), \end{aligned} \tag{3.21}$$

the first-order solution is given by

$$\Im(\upsilon,\tau) = \Im_0(\upsilon,\tau) + \Im_1(\upsilon,\tau),$$

$$\mathfrak{I}(\upsilon,\tau) = 3sech^2\Big(\frac{\upsilon-15}{2}\Big) - \frac{1}{\alpha\Gamma(\alpha)}3\tau^{\alpha}sech^2\Big(\frac{\upsilon-15}{2}\Big)tanh\Big(\frac{\upsilon-15}{2}\Big), \tag{3.22}$$

$$\Big(9sech^4\Big(\frac{\upsilon-15}{2}\Big)G_4 + (9sech^4\Big(\frac{\upsilon-15}{2}\Big)G_2 - G_3\Big)tanh\Big(\frac{\upsilon-15}{2}\Big)\Big),$$

Thus by using least square method, we get the optimal constants

$$R_{\mathfrak{I}} \quad = \quad \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau}(\tau-r)^{-\alpha}D_{\tau}^{\alpha}\mathfrak{I}(\upsilon,\tau)dr - \frac{\mathfrak{I}\partial\mathfrak{I}}{\partial\upsilon} + \frac{\partial^3\mathfrak{I}}{\partial\upsilon\partial\upsilon\partial\tau}, \tag{3.23}$$

the optimal constants are $G1 = -1.8238166409253036' *^{-} 10, G2 = 2.6918543610880453' *^{-} 12,$ $G3 = 0., G4 = 4.4999999999999964',$

by the above optimal/auxiliary constant, we obtained the approximated equation

$$\tilde{\mathfrak{I}} = \mathfrak{I}_0 + \mathfrak{I}_1, \tag{3.24}$$

the exact solution to the problem is

$$\mathfrak{I}(\upsilon,\tau) = 3sech^2\Big(\frac{\upsilon-15+\tau}{2}\Big), \tag{3.25}$$

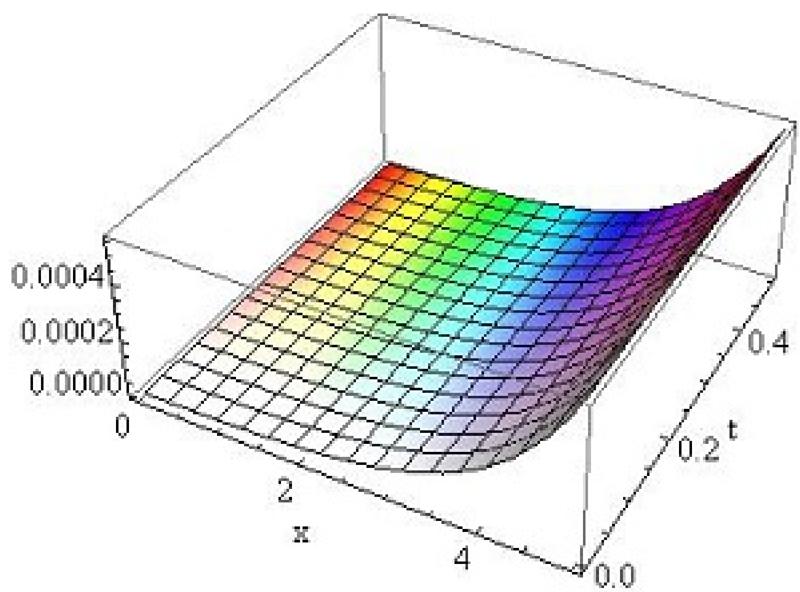

Figure 1. OAFM solution of $\mathfrak{I}(\upsilon,\tau)$ of problem 3.1.

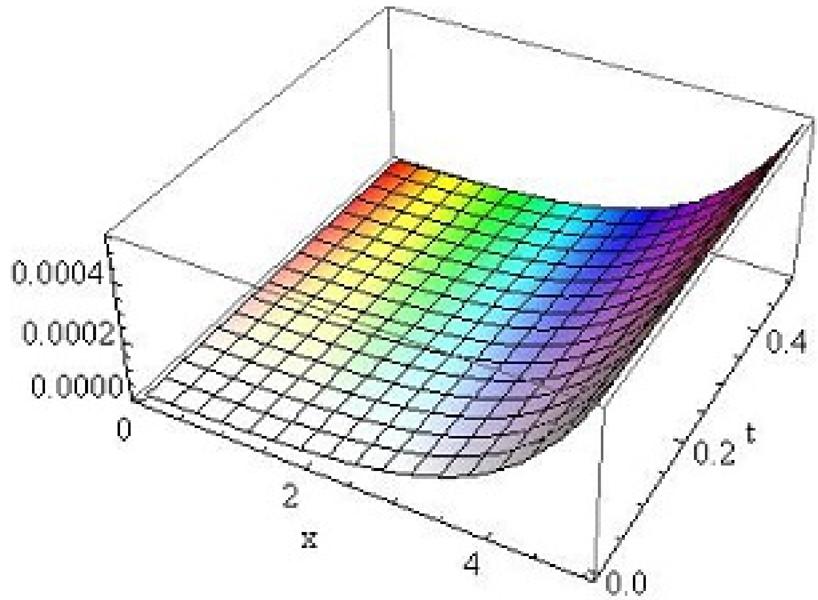

FIGURE 2. Actual solution of $\Im(\upsilon, \tau)$ of problem 3.1.

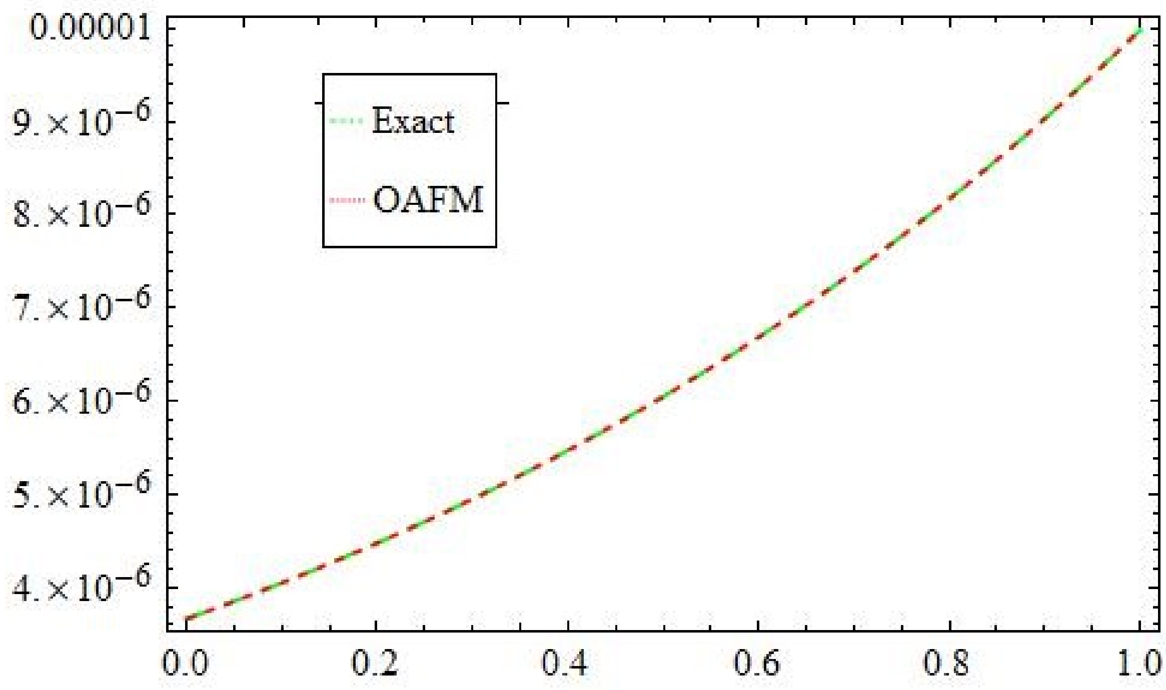

FIGURE 3. OAFM and actual solution of $\Im(\upsilon, \tau)$ of problem 3.1.

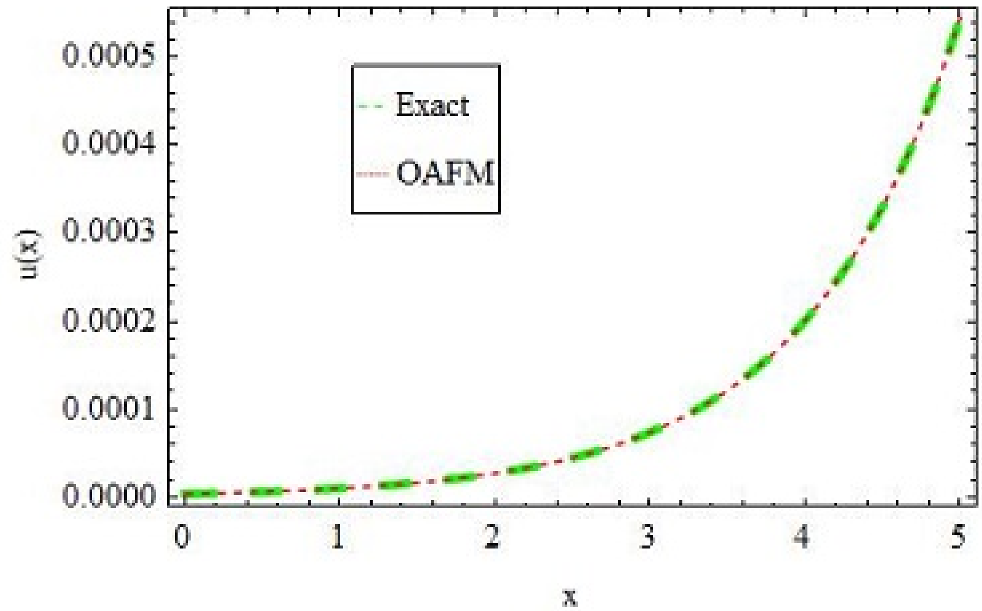

Figure 4. OAFM and actual solution of $\Im(\upsilon, \tau)$ of problem 3.1.

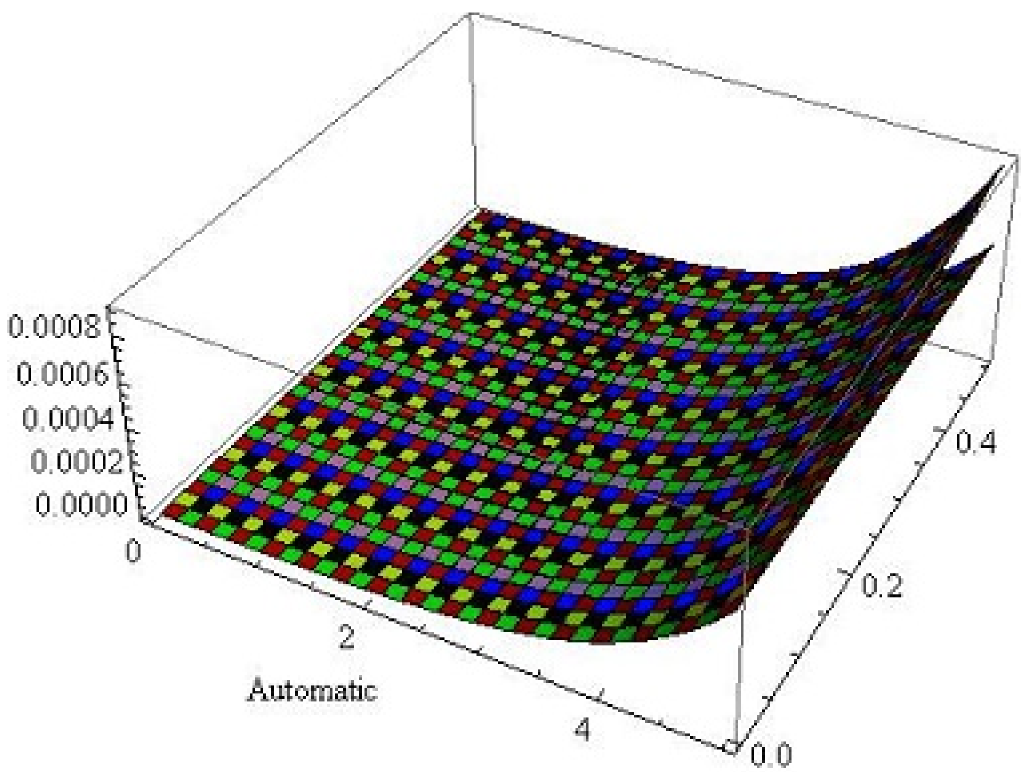

Figure 5. 3D OAFM solution of $\Im(\upsilon, \tau)$ for different value of α of problem 3.1.

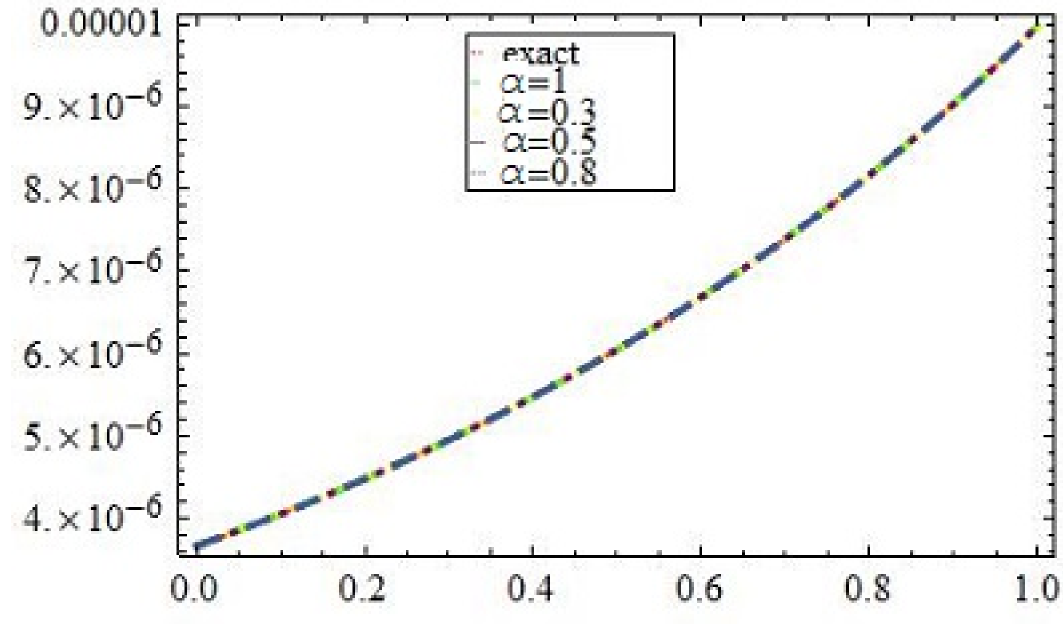

Figure 6. OAFM solution of $\Im(\upsilon,\tau)$ for different value of α of problem 3.1.

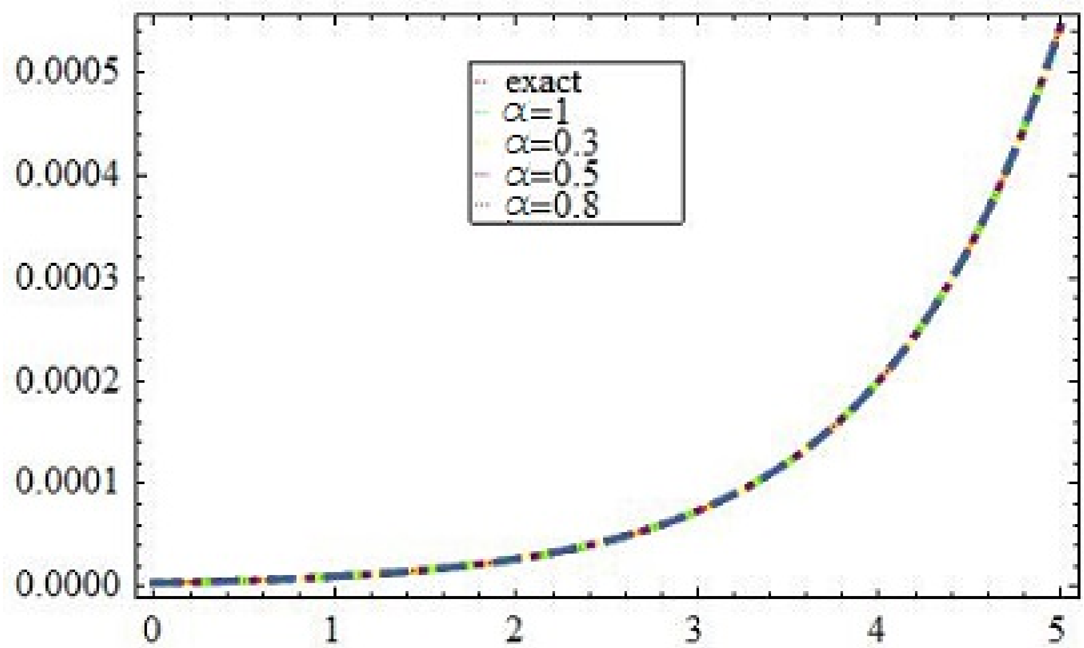

Figure 7. OAFM solution of $\Im(\upsilon,\tau)$ for different value of α of problem 3.1.

Problem 3.2. Consider the modified equal width (MEW) wave equation

$$D_\tau^\alpha \mathfrak{I} + \epsilon \mathfrak{I}^2 \mathfrak{I}_\upsilon - \mu \mathfrak{I}_{\upsilon\upsilon\tau} = 0, \quad , \qquad \tau > 0, \qquad \epsilon > 0, \qquad \mu > 0, \qquad 0 < \alpha \leq 1, \tag{3.26}$$

with the given initial value

$$\mathfrak{I}(\upsilon, 0) = Asech(\upsilon - 30), \tag{3.27}$$

the exact solution is

$$\mathfrak{I}(\upsilon, \tau) = Asech(\upsilon - 30 - p\tau), \tag{3.28}$$

where D_τ^α, is the fractional operator for fractional value α as given in Eq. (30), υ, τ represents the spatial and time respectively independent variable of $\mathfrak{I}(\upsilon, \tau)$, $A = \sqrt{(6p/\epsilon)}$, is the wave amplitude, p is the velocity of the wave, for parameter $\epsilon = 3$ and $\mu = 1$. We consider the auxiliary function as

$$\begin{aligned} D_1 &= G_1\Big(A^3 \epsilon sech^3(\upsilon - 30) tanh(\upsilon - 30)\Big)^2 + G_2\Big(A^3 \epsilon sech^3(\upsilon - 30) tanh(\upsilon - 30)\Big)^4, \\ D_2 &= G_3\Big(A^3 \epsilon sech^3(\upsilon - 30) tanh(\upsilon - 30)\Big)^6 + G_4\Big(A^3 \epsilon sech^3(\upsilon - 30) tanh(\upsilon - 30)\Big)^8, \end{aligned} \tag{3.29}$$

consider the linear part as well as non-linear part of the proposed model as

$$\begin{aligned} L &= D_\tau^\alpha \mathfrak{I}(\upsilon, \tau), \\ N &= \epsilon \mathfrak{I}^2 \mathfrak{I}_\upsilon - \mathfrak{I}_{\upsilon\upsilon\tau}, \end{aligned} \tag{3.30}$$

Zeorth order system:

$$D_\tau^\alpha \mathfrak{I}_0(\upsilon, \tau) = 0, \tag{3.31}$$

with initial condition the solution is obtained as

$$\mathfrak{I}_0(\upsilon, \tau) = Asech(\upsilon - 30), \tag{3.32}$$

First order system:

$$D_\tau^\alpha \mathfrak{I}(\upsilon, \tau) + D_1(\mathfrak{I}_0(\upsilon, \tau), G_m) N(\mathfrak{I}_0(\upsilon, \tau)) + D_2(\mathfrak{I}_0(\upsilon, \tau), G_n) = 0, \tag{3.33}$$

with the given initial condition, using Eqs. (33),(35) in Eq. (37) its solution is obtained as

$$\begin{aligned} \mathfrak{I}_1(\upsilon, \tau) &= -\frac{1}{\alpha \Gamma(\alpha)} A^9 \tau^\alpha \epsilon^3 sech^9(\upsilon - 30) tanh^3(\upsilon - 30) G_1 \\ &+ A^6 \epsilon^2 sech^6(\upsilon - 30) tanh^2(\upsilon - 30) G_2 + A^3 \epsilon sech^3(\upsilon - 30) tanh(\upsilon - 30) G_3 \\ &+ A^9 \epsilon^3 sech^9(\upsilon - 30) tanh^3(\upsilon - 30) G_4, \end{aligned} \tag{3.34}$$

the first-order solution is given by

$$\mathfrak{I}(\upsilon, \tau) = \mathfrak{I}_0(\upsilon, \tau) + \mathfrak{I}_1(\upsilon, \tau),$$

$$\begin{aligned}\Im(v,\tau) &= Asech(v-30)-\frac{1}{\alpha\Gamma(\alpha)}A^9\tau^\alpha\epsilon^3 sech^9(v-30)tanh^3(v-30)G_1 \\ &+ A^6\epsilon^2 sech^6(v-30)tanh^2(v-30)G_2+A^3\epsilon sech^3(v-30)tanh(v-30)G_3 \\ &+ A^9\epsilon^3 sech^9(v-30)tanh^3(v-30)G_4,\end{aligned} \tag{3.35}$$

Thus by the least square approach as in Eqs. (14)-(16), we get the optimal constants, the residual of the model is

$$R_\Im = \frac{1}{\Gamma(1-\alpha)}\int_0^\tau(\tau-r)^{-\alpha}D_\tau^\alpha\Im(v,\tau)dr-\epsilon\frac{\Im^2\partial v}{\partial v}+\frac{\partial^3\Im}{\partial v\partial v\partial\tau}, \tag{3.36}$$

the optimal constant are $G1 = 2045.8611187338465'$, $G2 = -6.5790852371332375'{*}^6$,
$G3 = 2.66595946703424'$, $G4 = 0.$,
by the above optimal/auxiliary constant, we obtained the approximated equation

$$\tilde{\Im} = \Im_0 + \Im_1, \tag{3.37}$$

the actual solution of the model is

$$\Im(v,\tau) = Asech(v-30-p\tau), \tag{3.38}$$

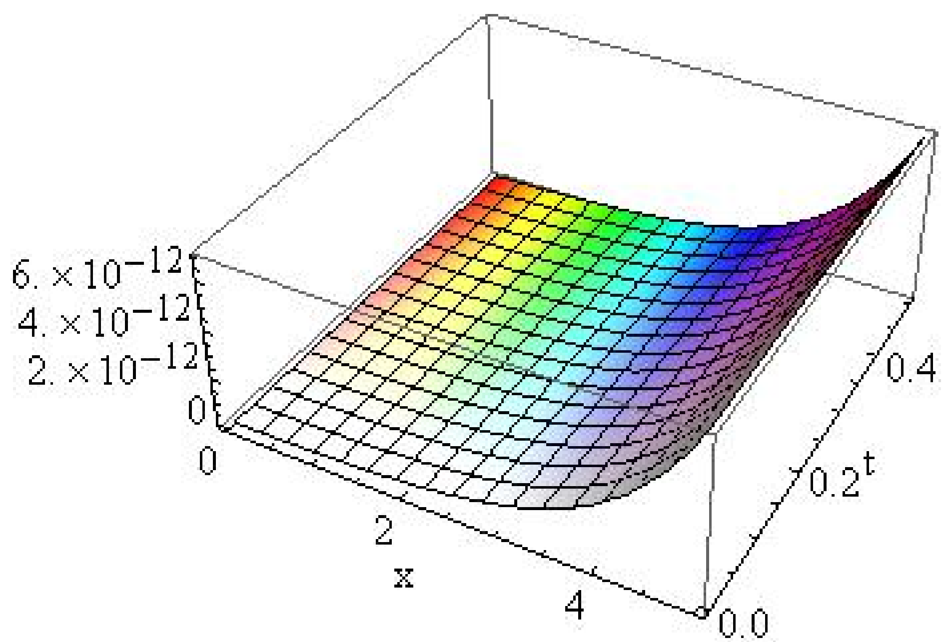

FIGURE 8. OAFM solution of $\Im(v,\tau)$ of problem 3.2.

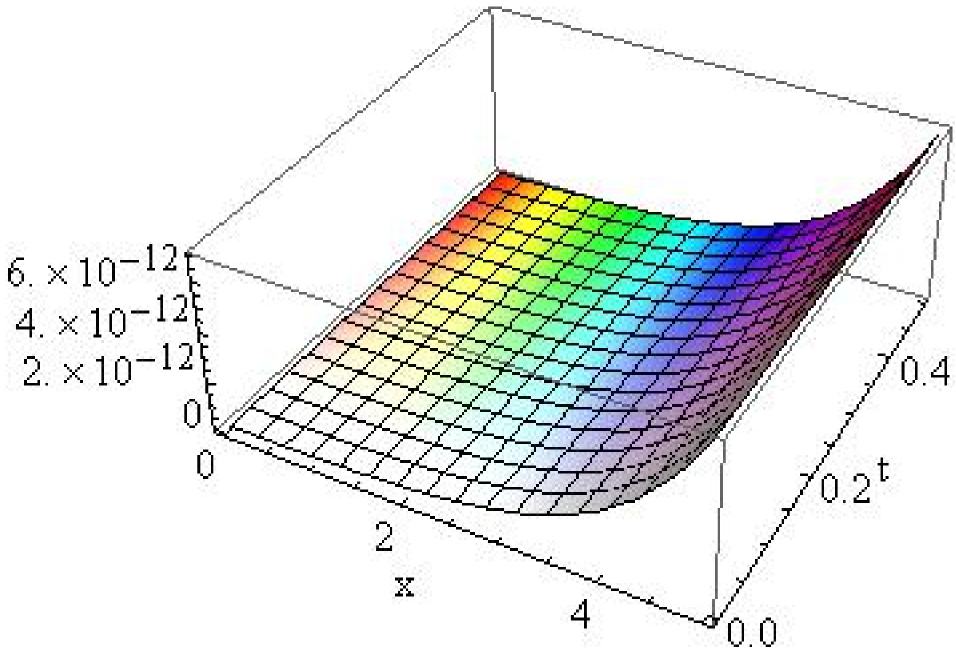

Figure 9. Actual solution of $\Im(\upsilon,\tau)$ of problem 3.2.

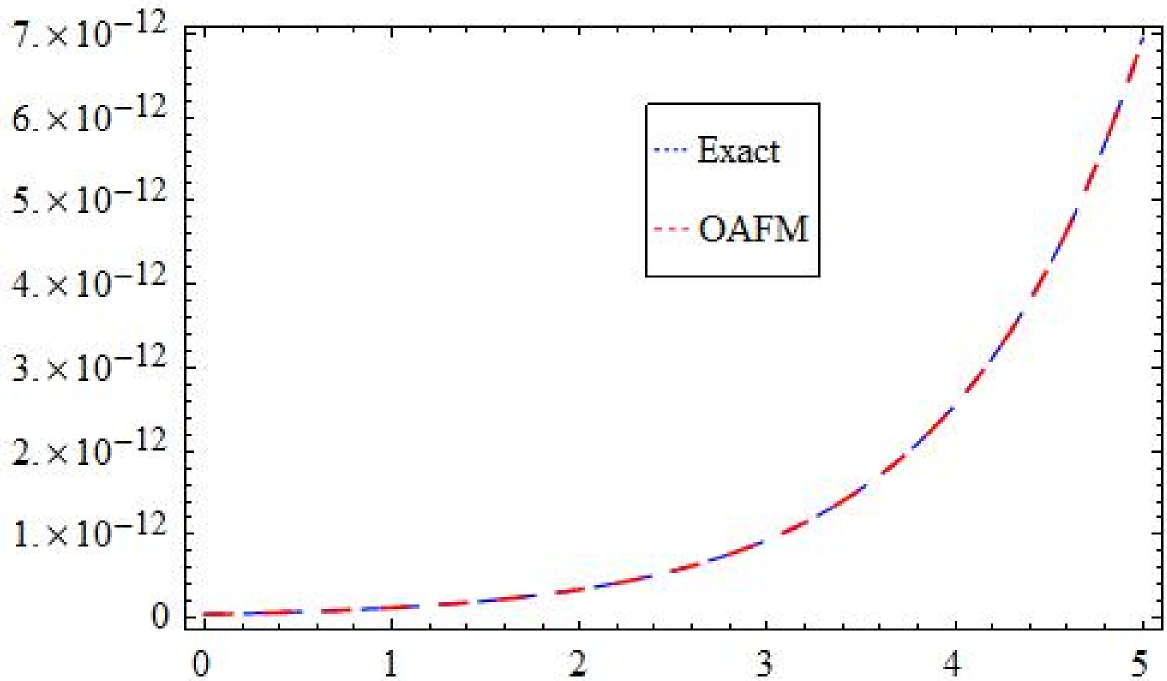

Figure 10. OAFM and actual solution of $\Im(\upsilon,\tau)$ of problem 3.2.

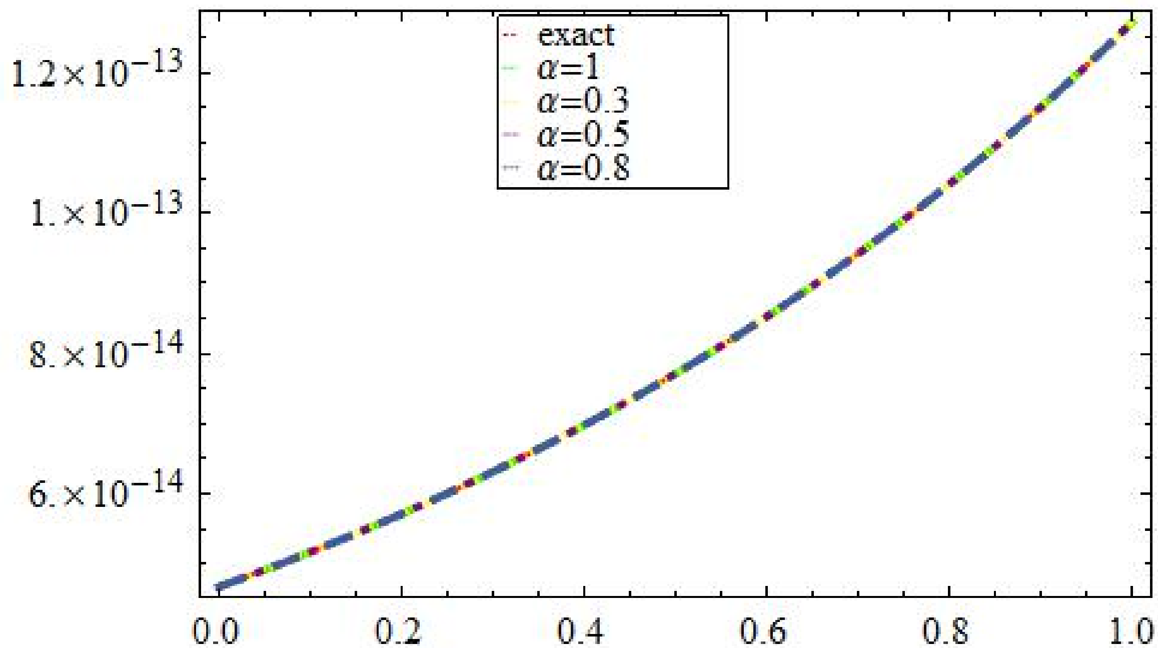

Figure 11. OAFM solution of $\Im(\upsilon, \tau)$ for different value of α of problem 3.1.

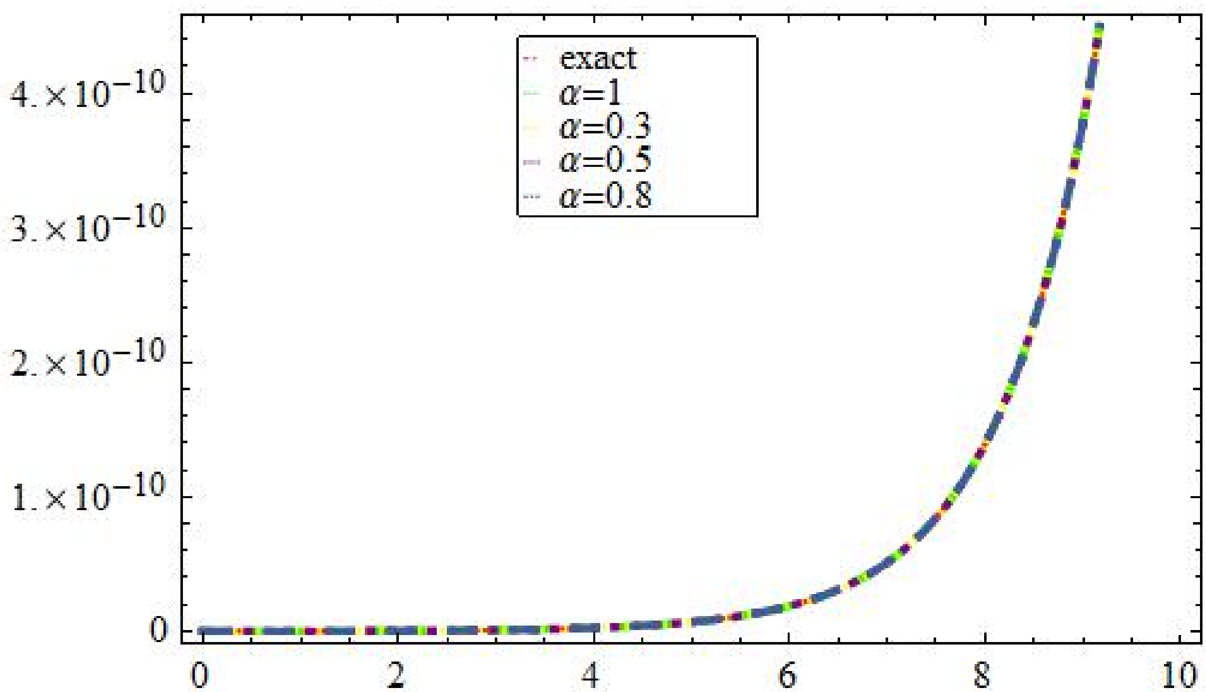

Figure 12. OAFM solution of $\Im(\upsilon, \tau)$ for different value of α of problem 3.1.

Problem 3.3. Consider the regularized long Wave equation

$$D_\tau^\alpha \Im + \Im_v + \epsilon \Im \Im_v - \mu \Im_{vv\tau} = 0, \qquad \tau > 0, \qquad 0 < \alpha \leq 1, \tag{3.39}$$

with the given initial value

$$\Im(v,0) = 3dsech^2(k(v - v_0)), \tag{3.40}$$

the exact solution is

$$\Im(v,\tau) = 3dsech^2(k(v - v_0 - s\tau)), \tag{3.41}$$

where D_τ^α, are the fractional operators for fractional value α, v and τ are the spatial and time variable for $\Im(v,\tau)$. The boundary conditions are obtained form the actual solution. The parameters used in the above equations are: $v_o = 0, d = 0.1, \epsilon = \mu = 1$, while $s = 1 + \epsilon d$ and for $k = \frac{1}{2}\sqrt{\frac{\epsilon d}{s\mu}}$. We consider the auxiliary function as

$$\begin{aligned} D_1 &= G_1\Big(sech(0.15v) + tanh(0.15v)\Big) + G_2\Big(sech(0.15v) + tanh(0.15v)\Big)^2, \\ D_2 &= G_3\Big(sech(0.15v) + tanh(0.15v)\Big)^3 + G_4\Big(sech(0.15v) + tanh(0.15v)\Big)^4, \end{aligned} \tag{3.42}$$

consider the linear part as well as non-linear part of the proposed model as

$$\begin{aligned} L &= D_\tau^\alpha \Im(v,\tau), \\ N &= \Im_v + \epsilon \Im \Im_v - \mu \Im_{vv\tau}, \end{aligned} \tag{3.43}$$

Zeorth order system:

$$D_\tau^\alpha \Im_0(v,\tau) = 0, \tag{3.44}$$

with initial condition the solution is obtained as

$$\Im_0(v,\tau) = 0.3sech^2(0.15v), \tag{3.45}$$

First order system:

$$D_\tau^\alpha \Im(v,\tau) + D_1(\Im_0(v,\tau), G_m)N(\Im_0(v,\tau)) + D_2(\Im_0(v,\tau), G_n) = 0, \tag{3.46}$$

with the given initial condition, using Eqs. (46) and (48) in Eq. (50) its solution with the help of Caputo operator is obtained as

$$\begin{aligned} \Im_1(v,\tau) &= \frac{1}{\alpha\Gamma(\alpha)}\tau^\alpha\Big(sech[0.15'v] + tanh[0.15'v])^3 G_3 + (sech[0.15'v] \\ &+ tanh[0.15'v])^4 G_4 + sech^2[0.15'v](0.09' + 0.027'sech^2[0.15'v])tanh[0.15'v](sech[0.15'v] \\ &+ tanh[0.15'v]) + (G_1 + G_2(sech[0.15'v] + tanh[0.15'v]))\Big), \end{aligned} \tag{3.47}$$

the first-order solution is given by

$$\Im(v,\tau) = \Im_0(v,\tau) + \Im_1(v,\tau),$$

$$\begin{aligned}\Im(v,\tau) &= 0.3sech(0.15v)+\frac{1}{\alpha\Gamma(\alpha)}\tau^{\alpha}\Big(sech[0.15'v]+tanh[0.15'v])^{3}G_{3}+(sech[0.15'v] \\ &+ tanh[0.15'v])^{4}G_{4}+sech^{2}[0.15'v](0.09'+0.027'sech^{2}[0.15'v])tanh[0.15'v](sech[0.15'v] \\ &+ tanh[0.15'v])+(G_{1}+G_{2}(sech[0.15'v]+tanh[0.15'v]))\Big),\end{aligned} \tag{3.48}$$

Thus by the least square approach as in Eqs. (14)-(16), we get the optimal constants, the residual of the model is

$$R_{\Im} = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{\tau}(\tau-r)^{-\alpha}D_{\tau}^{\alpha}\Im(v,\tau)dr-\Im_{v}-\epsilon\Im\Im_{v}+\mu\Im_{vv\tau}, \tag{3.49}$$

the optimal constant are
$G_1 = -3.0055387673507363', G_2 = 1.6343992448416826'$,
$G_3 = 0.01641193013563335', G_4 = -0.016363057960769745'$,
by the above optimal/auxiliary constant, we obtained the approximated equation

$$\tilde{\Im} = \Im_0 + \Im_1, \tag{3.50}$$

the actual solution of the model is

$$\Im(v,\tau) = Asech^{2}(v - v_0 - s\tau), \tag{3.51}$$

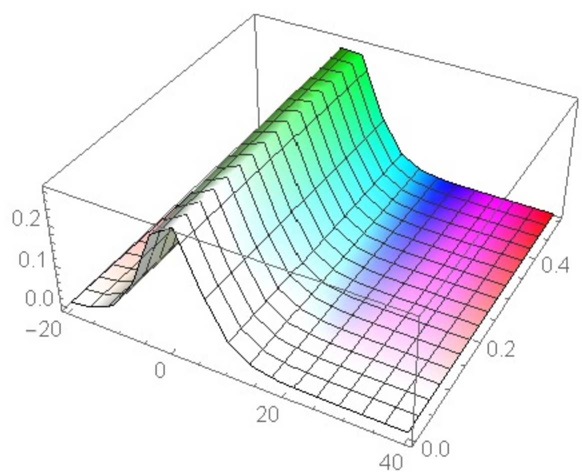

Figure 13. OAFM solution of $\Im(v,\tau)$ of problem 3.3.

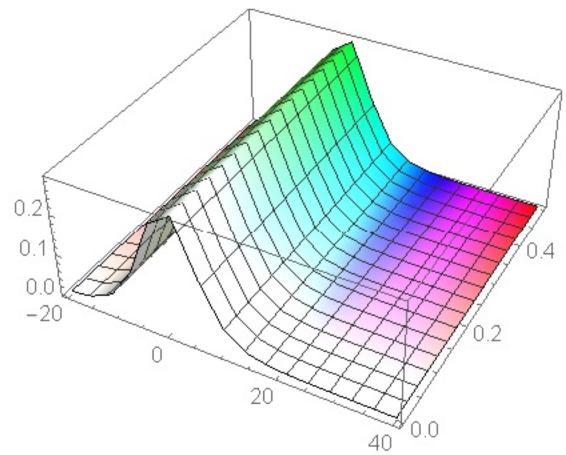

Figure 14. Actual solution of $\Im(\upsilon,\tau)$ of problem 3.3.

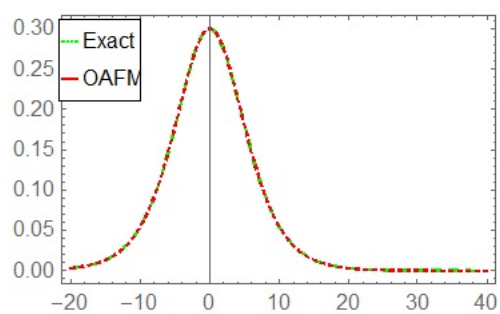

Figure 15. OAFM and actual solution of $\Im(\upsilon,\tau)$ of problem 3.3.

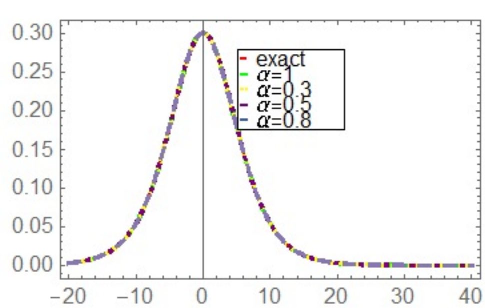

Figure 16. OAFM solution of $\Im(\upsilon,\tau)$ for different value of α of problem 3.3.

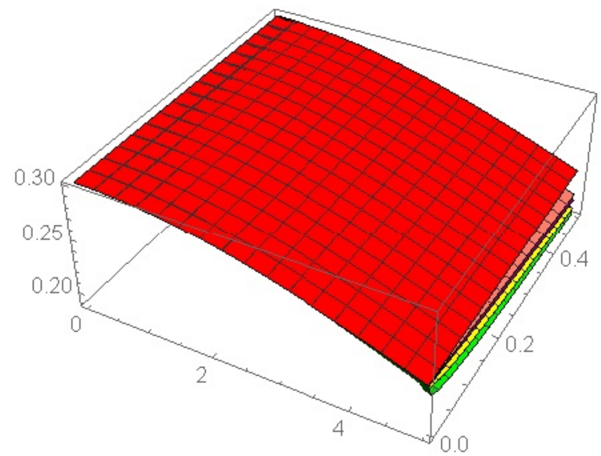

Figure 17. OAFM solution of $\Im(\upsilon, \tau)$ for different value of α of problem 3.3.

4. Discussion

In the present paper, we take the (WE) and (MEW) wave equations using the methodology of OAFM. By the proposed method, we obtained the first-ordered solution, which shows a close resemblance to the exact solution of the models. In addition, even at the first iteration, the best solution is obtained as compared to the series solution methods.
Figures 1 and 2 show the 3D approximate and exact solution of $\Im(\upsilon, \tau)$ respectively at $\upsilon \in [0, 5]$, $\alpha = 0.5$, and $\tau \in [0, 1]$ of problem 3.1. Figures 3 and 4 represent the approximated and exact solution of $\Im(\upsilon, \tau)$ problem 3.1 at $\Im \in [0, 1]$, $\upsilon \in [0, 5]$, $\alpha = 0.5$, and $\tau \in [0, 1]$. Figures 5-7 represent the 3D and 2D exact solution and approximated solution respectively for different values of fractional order α, which enhance that the proposed method is very effective and versatile in fast convergence as the fractional value approaches integer value. Figures 8 and 9 represent the 3D approximate and exact solution of $\Im(\upsilon, \tau)$ respectively of problem 3.2 at $\upsilon \in [0, 5]$, $\alpha = 0.5$, and $\tau \in [0, 1]$. Figure 10 shows the comparison of the OAFM and exact solution of $\Im(\upsilon, \tau)$ problem 3.2 at $\upsilon \in [0, 1]$, $\alpha = 0.5$, and $\tau \in [0, 1]$. Fig.11 and Fig.12 show the exact solution and OAFM solution respectively for different values of fractional order α, which validate that the method has fast convergence as the fractional order approach to the integer order.

5. Conclusion

The time-fractional model of the (EW) and (MEW) wave equations is investigated in the current research using the OAFM with the Caputo operator. It was demonstrated that the content of the suggested strategy was very interesting, straightforward, and easy to understand. To confirm the effectiveness of this technique, two test problems have been solved in this context. The results of the current solution for the fractional-order alternatives to the problem's integer order were also seen through simulations. Each problem's solution graphs showed that the method came quite close to the ideal solution. Finally, the acquired results and solution, demonstrate the method's applicability to both the scheme and analytical solutions of fractional partial differential equations, thus the method can be applied to a variety of science and engineering problems.

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