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Interpolative Hardy-Rogers-Type Proximal $\mathcal{Z}$-Contraction Maps in b-Metric Spaces with Some Applications

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Abstract. Several known types of contractions involving the combination of $d(fx, fy)$ and $d(x, y)$ are unified by the simulation function and the concept of $\mathcal{Z}$-contraction concerns ζ, which generalizes the Banach contraction principle. Our findings build upon or generalize a number of findings in the literature. In this study, we develop interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction maps and demonstrate the existence and uniqueness of the best proximity points in complete b-metric spaces. We illustrate our findings with examples. We provide some pertinent applications.

1. Introduction

The successive approximation methods that were first developed by a number of prior mathematicians, including well-known figures like Cauchy, Liouville, Picard, Lipschitz, and others, are successfully encapsulated and reinterpreted by the Banach contraction principle. Regarding the Hardy-Rogers fixed point theorem's generalization to the interpolative Hardy-Rogers type contractive mapping. Interestingly, this new kind of mapping was first developed by E. Karapınar, who integrated the interpolation notion into the Hardy-Rogers framework. This method probably broadens the original theorem's usefulness by generating intermediate points between known data points. It is true that interpolation is frequently used in mathematical study to generalize different types of contractions. Researchers can broaden the application of current theorems and offer a more adaptable framework for examining fixed points in metric spaces by including interpolation techniques into contraction mappings. It appears that the interpolative approach has been used to generalize various contraction types and in other studies. This illustrates the interpolation

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approach's adaptability and efficiency in extending the notion of fixed points and offering fresh perspectives on the existence and uniqueness of solutions. For more details [15,16,18,24–30,45].

In 2015, Khojasteh et al. [32] established the concept of simulation functions intending to consider a new class of contractions, called $\mathcal{Z}$-contractions. Such family generalized, extended and improved several results that had been obtained in previous years. The simplicity and usefulness of these contractions have inspired many researchers to diversify them further see [19,20,33,36,38]. The concept of b-metric space or metric type space was introduced by Czerwik [13] as a generalization of metric space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions, for instance [22,23,40,41].

However, it is not always possible to solve the equation $Tx = x$, especially for the non-self-mappings. Then, it becomes important to identify a x for which $d(x, Tx)$ is the least, which subsequently leads to the development of a new idea, which is the best proximity point. The study of the best proximity point theory has been an interesting area of research since last few decades [10,12,14,34,35,42–44].

Definition 1.1 ([13]). *Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$*

(i) $0 \leq d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space. In this paper, we denote $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{N}$ is the set of all natural numbers.

The following lemmas are useful in proving our main results.

Lemma 1.1 ([41]). *Suppose (X,d) is a b-metric space with coefficient $s \geq 1$ and $\{a_n\}$ be a sequence in X such that $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. If $\{a_n\}$ is a not Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(a_{m_k}, a_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(a_{m_k}, a_{n_k}) \geq \epsilon, d(a_{m_k}, a_{n_k-1}) < \epsilon$ and*

(i) $\epsilon \leq \liminf_{k\to\infty} d(a_{m_k}, a_{n_k}) \leq \limsup_{k\to\infty} d(a_{m_k}, a_{n_k}) \leq s\epsilon$,

(ii) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(a_{m_k+1}, a_{n_k}) \leq \limsup_{k\to\infty} d(a_{m_k+1}, a_{n_k}) \leq s^2\epsilon$,

(iii) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} d(a_{m_k}, a_{n_k+1}) \leq \limsup_{k\to\infty} d(a_{m_k}, a_{n_k+1}) \leq s^2\epsilon$,

(iv) $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} d(a_{m_k+1}, a_{n_k+1}) \leq \limsup_{k\to\infty} d(a_{m_k+1}, a_{n_k+1}) \leq s^3\epsilon$.

Lemma 1.2 ([2]). *Let (X,d) be a b-metric space with coefficient $s \geq 1$.*
Suppose that $\{a_n\}$ and $\{b_n\}$ are b-convergent to x and y respectively. Then we have

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(a_n, b_n) \leq \limsup_{n\to\infty} d(a_n, b_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(a_n, b_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(a_n, z) \leq \limsup_{n\to\infty} d(a_n, z) \leq sd(x,z).$$

Recently, Mohamed Edraoui [17] proved the following theorem using interpolative Hardy-Rogers pair contraction in complete metric spaces.

Definition 1.2. [17] *Let (E, d) be a metric space. A pair of mappings $T, S : E \to E$ is said to be interpolative Hardy-Rogers pair contraction if there exist $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Ta, Sb) \leq k[d(a,b)^{\beta}][d(Ta,a)]^{\gamma}[d(Sb,b)]^{\alpha}\left[\frac{d(Ta,b)+d(a,Sb)}{2}\right]^{1-\alpha-\beta-\gamma}$$

for all $a, b \in X$ such that $Ta \neq a$ whenever $Sb \neq b$.

Theorem 1.1. [17] *Suppose that (E, d) be a complete metric space, and (T, S) is a interpolative Hardy-Rogers pair contraction. Then, S and T have a unique common fixed point.*

Definition 1.3 ([32]). *A simulation function is a mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ satisfying the following conditions:*

(1) $\zeta(0, 0) = 0$;
(2) $\zeta(t, s) < s - t$ *for all* $s, t > 0$;
(3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, \infty)$$

then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Using the simulation function approach, the notion of $\mathcal{Z}$-contraction was introduced in [32] which is a generalization of Banach contraction. It also unified various existing types of contraction mappings. The advantage of this notion is in providing a unique point of view for several fixed point problems.

Definition 1.4 ([32]). *Let (X, d) be a metric space. A self mapping f on X is called a $\mathcal{Z}$-contraction if for some simulation function $\zeta \in \mathcal{Z}$, T satisfies $\zeta(d(fx, fy), d(x, y)) \geq 0$, for all $x, y \in X$.*

It should be observed that all $\mathcal{Z}$-contraction mappings are continuous and contractive. Olgun et al. [37] relaxed this continuity, defining a generalized $\mathcal{Z}$-contraction mapping which is not necessarily continuous.

Definition 1.5 ([37]). *Let (X, d) be a metric space. A self mapping f on X is called a generalized $\mathcal{Z}$-contraction if for some simulation function $\zeta \in \mathcal{Z}$, and for all $x, y \in X$, T satisfies $\zeta(d(fx, fy), M(x, y)) \geq 0$, where*

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

A novel kind of mapping was defined by Kumam et al. [33] by combining $\mathcal{Z}$-contraction and Suzuki type contraction, as explained below.

Definition 1.6 ([33]). *Let (X,d) be a metric space. A self mapping f on X is called a Suzuki type $\mathcal{Z}$-contraction if for some simulation function $\zeta \in \mathcal{Z}$, T satisfies $\frac{1}{2}d(x,fx) < d(x,y)$ then*

$$\zeta(d(fx,fy),d(x,y)) \geq 0, \quad \text{for all } x,y \in X.$$

In 2018, Padcharoen et al. [39] proved the following theorem in complete metric spaces.

Theorem 1.2 ([39]). *Let (X,d) be a complete metric space and $f : X \to X$ be a self-map on X. If there exists a simulation function ζ such that $\frac{1}{2}d(x,fx) < d(x,y)$ then $\zeta(d(fx,fy),M(x,y)) \geq 0$, for all $x,y \in X$, where*

$$M(x,y) = \max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

Then f has a unique fixed point in X.

Remark 1.1. *It is clear that all Suzuki type $\mathcal{Z}$-contraction is generalized Suzuki type $\mathcal{Z}$-contraction for $M(x,y) = d(x,y)$.*

Remark 1.2 ([8]). *Let ζ be a simulation function. If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(kt_n,s_n) < 0$ for any $k > 1$.*

The following are examples of simulation functions.

Example 1.1 ([8]). *Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ be defined by*

(i) $\zeta(t,s) = \lambda s - t$ for all $t,s \in \mathbb{R}^+,$ where $\lambda \in [0,1)$;
(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s,t \in \mathbb{R}^+$;
(iii) $\zeta(t,s) = s - kt$ for all $t,s \in \mathbb{R}^+,$ where $k > 1$;
(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s,t \in \mathbb{R}^+$;
(v) $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s,t \in \mathbb{R}^+$ where $k > 1$.

The following theorem is due to Babu et al. [3] in complete b-metric spaces.

Theorem 1.3 ([3]). Let (X,d) be a complete b-metric space and $f : X \to X$ be a self-map on X. If there exists a simulation function ζ such that $\frac{1}{2s}d(x,fx) < d(x,y)$ then $\zeta(s^4 d(fx,fy),M(x,y)) \geq 0$ for all $x,y \in X$, where

$$M(x,y) = \max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2s}\right\}.$$

Then f has a unique fixed point in X.

In 2019, E. Karapınar et al. [26] established the following.

Definition 1.7 ([26]). *Let T be a self-mapping defined on a metric space (Xd). If there exist $\zeta \in \mathcal{Z}, \psi \in \Psi, \gamma, \beta \in (0,1)$ with $\gamma + \beta < 1$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\zeta(\alpha(a,b)d(Ta,Tb), \psi(R(a,b))) \geq 0 \text{ for all } a, b \in X,$$

where $R(a,b) = [d(a,b)]^{\beta}.[d(a,Ta)]^{\alpha}.[d(b,Tb)]^{1-\alpha-\beta}$, then T is said to be an interpolative Rus-Reich-Ćirić type $\mathcal{Z}$-contraction with respect to ζ.

Theorem 1.4 ([26]). *Let (X,d) be a complete metric space, $\zeta \in \mathcal{Z}$. If a self-mapping $T : X \to X$ forms an interpolative Rus-Reich-Ćirić type $\mathcal{Z}$-contraction with respect to ζ and satisfies*

(1) *T is triangular α-orbital admissible,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(3) *T is continuous,*

then there exists $u \in X$ such that $Tu = u$.

The following is due to Khan et al. [31].

Definition 1.8 ([31]). *A mapping $T : X \to X$ is called an interpolative (ψ, ϕ)-Hardy-Rogers type $\mathcal{Z}$-contraction with respect to ζ if there exist $\theta : X \times X \to \mathbb{R}, \zeta \in \mathcal{Z}, \psi \in \Psi, \phi \in \Phi, \alpha_i \in (0,1)$, where $i = 1,2,3$, such that $\phi(t) > \psi(t), t > 0$ and $\sum_{i=1}^{3} \alpha_i < 1$ satisfying the inequality*

$$\zeta(\theta(a,b)\phi(d(Ta,Tb)), \psi(H(a,b))) \geq 0, \text{ for all} a, b \in X \setminus Fix(T),$$

where $H(a,b) = [d(a,b)]^{\alpha_1}.[d(a,Ta)]^{\alpha_2}.[d(b,Tb)]^{\alpha_3}.[\frac{d(a,Tb)+d(b,Ta)}{2}]^{1-\sum_{i=1}^{3}\alpha_i}$.

Theorem 1.5 ([31]). *Let $T : X \to X$ be a self mapping on a complete metric space (X,d). Suppose that T is quasi triangular α-orbital admissible and forms an interpolative (ψ, ϕ)-HR type $\mathcal{Z}$-contraction with respect to ζ. If there exists $x_0 \in X$ such that $\theta(x_0, Tx_0) \geq 1$ and T is continuous, then T has a fixed point in X.*

Let A and B be two non-empty subsets of a metric space (X,d). Define

$$d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}.$$
$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

In [1], Abbas et al. introduced the concept of proximal simulative contraction of the first kind and the second kind.

Definition 1.9 ([1]). *For two non-empty subsets A and B of a metric space (X,d), a mapping $T : A \to B$ is said to be a proximal simulative contraction of the first kind if there exists a simulation function ζ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ then $\zeta(d(a_1,a_2), d(b_1,b_2)) \geq 0$ for all $a_1, a_2, b_1, b_2 \in A$.*

Definition 1.10 ([1]). *For two non-empty subsets A and B of a metric space (X,d), a mapping $T: A \to B$ is said to be proximal simulative contraction of the second kind if there exists a simulation function ζ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ then $\zeta(d(Ta_1, Ta_2), d(Tb_1, Tb_2)) \geq 0$ for all $a_1, a_2, b_1, b_2 \in A$.*

In 2024, Goswami et al. [21] proved the following theorems in complete metric spaces.

Theorem 1.6 ([21]). *Let A and B be two non-empty subsets of a complete metric space (X,d). Suppose $T: A \to B$ be a map with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and A_0 is closed. If there exists a simulation function ζ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ then $\zeta(d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0$ for all $a_1, a_2, b_1, b_2 \in A$, where*

$$\begin{aligned} &M(b_1,b_2,a_1,a_2) \\ &\quad = \max\left\{d(b_1,b_2), d(b_1,a_1), d(b_2,a_2), \frac{d(b_1,a_2)+d(b_2,a_1)}{2}\right\} \end{aligned}$$

Then T has a unique best proximity point in A_0.

Theorem 1.7 ([21]). *Let A and B be two non-empty subsets of a complete metric space (X,d). Suppose $T: A \to B$ be a map with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and B_0 is closed subset of B. If there exists a simulation function ζ such that*

$$d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$$

then

$$\zeta(d(Ta_1,Ta_2), M(Tb_1,Tb_2,Ta_1,Ta_2)) \geq 0, \quad \forall a_1,a_2,b_1,b_2 \in A,$$

where

$$\begin{aligned} M(Tb_1,Tb_2,Ta_1,Ta_2) = \max\Bigg\{& d(Tb_1,Tb_2), d(Tb_1,Ta_1), d(Tb_2,Ta_2), \\ & \frac{d(Tb_1,Ta_2)+d(Tb_2,Ta_1)}{2}\Bigg\}. \end{aligned}$$

Then T has a unique best proximity point in A_0.

Recently, Babu [4] proved the following theorems in complete b-metric spaces.

Theorem 1.8 ([4]). *Let A and B be two non-empty subsets of a complete b-metric space (X,d). Suppose $T: A \to B$ be a map with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and A_0 is closed. If there exists a simulation function ζ such that $d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$ then $\zeta(d(a_1,a_2), M(b_1,b_2,a_1,a_2)) \geq 0$ for all $a_1, a_2, b_1, b_2 \in A$, where*

$$\begin{aligned} &M(b_1,b_2,a_1,a_2) \\ &\quad = \max\left\{d(b_1,b_2), d(b_1,a_1), d(b_2,a_2), \frac{d(b_1,a_2)+d(b_2,a_1)}{2s}\right\} \end{aligned}$$

Then T has a unique best proximity point in A_0.

Theorem 1.9 ([4]). *Let A and B be two non-empty subsets of a complete b-metric space (X,d). Suppose $T : A \to B$ be a map with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and B_0 is closed subset of B. If there exists a simulation function ζ such that*

$$d(a_1, Tb_1) = d(A,B) = d(a_2, Tb_2)$$

then

$$\zeta(d(Ta_1, Ta_2), M(Tb_1, Tb_2, Ta_1, Ta_2)) \geq 0, \quad \forall a_1, a_2, b_1, b_2 \in A,$$

where

$$M(Tb_1, Tb_2, Ta_1, Ta_2) = \max\left\{d(Tb_1, Tb_2), d(Tb_1, Ta_1), d(Tb_2, Ta_2), \frac{d(Tb_1, Ta_2) + d(Tb_2, Ta_1)}{2s}\right\}.$$

Then T has a unique best proximity point in A_0.

Motivated by all the above works we introduce interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the first kind and the second kind respectively and we prove the existence and uniqueness of the best proximity points in complete b-metric spaces.

2. Best Proximity Points for an Interpolative Hardy-Rogers-Type Proximal $\mathcal{Z}$-Contraction Maps

We introduce interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the first kind and the second kind as follows:

Definition 2.1. *For two non-empty subsets A and B of a b-metric space (X,d), a mapping $T : A \to B$ is said to be an interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the first kind if there exist a simulation function ζ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ such that $d(a_1, Tb_1) = d(a_2, Tb_2) = d(A,B)$ then*

$$\zeta\left(s^2 d(a_1, a_2), \prod_{\alpha,\beta,\gamma}(b_1, b_2, a_1, a_2)\right) \geq 0, \quad \forall b_1, b_2, a_1, a_2 \in A \tag{2.1}$$

where

$$\prod_{\alpha,\beta,\gamma}(b_1, b_2, a_1, a_2) = [d(b_1, b_2)]^{\beta} [d(b_1, a_1)]^{\alpha} [d(b_2, a_2)]^{\gamma} \left[\frac{d(b_2, a_1) + d(b_1, a_2)}{2s}\right]^{1-\alpha-\beta-\gamma}$$

Definition 2.2. *For two non-empty subsets A and B of a b-metric space (X,d), a mapping $T : A \to B$ is said to be an interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the second kind if there exist a* *simulation function ζ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$ such that $d(a_1, Tb_1) = d(a_2, Tb_2) = d(A, B)$ then*

$$\zeta\left(s^2 d(Ta_1, Ta_2), \prod_{\alpha,\beta,\gamma}(Tb_1, Tb_2, Ta_1, Ta_2)\right) \geq 0, \quad \forall b_1, b_2, a_1, a_2 \in A \tag{2.2}$$

where

$$\prod_{\alpha,\beta,\gamma}(Tb_1, Tb_2, Ta_1, Ta_2) = \max\left[d(Tb_1, Tb_2)\right]^{\beta}\left[d(Tb_1, Ta_1)\right]^{\alpha}\left[d(Tb_2, Ta_2)\right]^{\gamma}$$
$$\left[\frac{d(Tb_2, Ta_1) + d(Tb_1, Ta_2)}{2s}\right]^{1-\alpha-\beta-\gamma}.$$

Theorem 2.1. *Let A and B be two non-empty subsets of a complete b-metric space (X, d). Suppose $T: A \to B$ be an interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the first kind with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and A_0 is closed subset of A. Then T has a unique best proximity point in A_0.*

Proof. Suppose $a_0 \in A_0$. Since, $T(A_0) \subseteq B_0$, there exists $a_1 \in A_0$ such that

$$d(a_1, Ta_0) = d(A, B).$$

Again, since, $Ta_1 \in B_0$, there exists $a_2 \in A_0$ such that

$$d(a_2, Ta_1) = d(A, B).$$

Repeating this way, we get a sequence $\{a_n\}$ in A_0 such that

$$d(a_n, Ta_{n-1}) = d(A, B) = d(a_{n+1}, Ta_n), \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

If for some $n \in \mathbb{N}$, $a_n = a_{n-1}$, then

$$d(a_n, Ta_{n-1}) = d(a_n, Ta_n) = d(A, B),$$

and so a_n is a best proximity point of T. We assume that $d(a_n, Ta_{n-1}) > 0$. As T is an interpolative Hardy-Rogers proximal $\mathcal{Z}$-contraction of first kind, there exists ζ such that

$$\zeta\left(s^2 d(a_n, a_{n+1}), \prod_{\alpha,\beta,\gamma}(a_{n-1}, a_n, a_n, a_{n+1})\right) \geq 0, \quad \text{for all } n \in \mathbb{N}, \tag{2.4}$$

where

$$\begin{aligned}\prod_{\alpha,\beta,\gamma}(a_{n-1}, a_n, a_n, a_{n+1}) &= \left[d(a_{n-1}, a_n)\right]^{\beta}\left[d(a_{n-1}, a_n)\right]^{\alpha}\left[d(a_n, a_{n+1})\right]^{\gamma} \\ &\quad \left[\frac{d(a_{n-1}, a_{n+1}) + d(a_n, a_n)}{2s}\right]^{1-\alpha-\beta-\gamma} \\ &= \left[d(a_{n-1}, a_n)\right]^{\alpha+\beta}\left[d(a_n, a_{n+1})\right]^{\gamma} \\ &\quad \left[\frac{d(a_{n-1}, a_n) + d(a_n, a_{n+1})}{2}\right]^{1-\alpha-\beta-\gamma}.\end{aligned}$$

If $d(a_n, a_{n-1}) < d(a_n, a_{n+1})$ then

$$\prod_{\alpha,\beta,\gamma}(a_{n-1}, a_n, a_n, a_{n+1}) = d(a_n, a_{n+1})$$

. Now, from (2.4) using (ζ_2), we get,

$$\begin{aligned}0 &\leq \zeta\left(s^2 d(a_n, a_{n+1}), \prod_{\alpha,\beta,\gamma}(a_{n-1}, a_n, a_n, a_{n+1})\right)\\ &< d(a_n, a_{n+1}) - s^2 d(a_n, a_{n+1})\\ &< 0,\end{aligned}$$

which is a contradiction.

Therefore, $d(a_n, a_{n+1}) \leq d(a_{n-1}, a_n)$, for all $n \in \mathbb{N}$, i.e., $\{d(a_n, a_{n+1})\}$ is a decreasing sequence of positive real numbers and so there exists a real number $r \geq 0$ such that $\lim_{n\to\infty} d(a_n, a_{n+1}) = r$. If $r > 0$, then from (2.4) and using (ζ_3), we get,

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\left[\zeta\left(s^2 d(a_n, a_{n+1}), \prod_{\alpha,\beta,\gamma}(a_{n-1}, a_n, a_n, a_{n+1})\right)\right]\\ &= \limsup_{n\to\infty}\left(s^2 d(a_n, a_{n+1}), d(a_{n-1}, a_n)\right)\\ &< 0,\end{aligned}$$

a contradiction. Therefore, $r = 0$, i.e., $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$.
Now, we prove that $\{a_n\}$ is a b-Cauchy sequence. On the contrary, suppose that $\{a_n\}$ is not b-Cauchy. By Lemma 1.1, there exist an $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that

$$d\left(a_{m_k}, a_{n_k}\right) \geq \epsilon \text{ and } d(a_{m_k}, a_{n_k-1}) < \epsilon$$

satisfying $(i) - (iv)$ of Lemma 1.1.

Again, from (2.3), $d(a_{m_k}, Ta_{m_k-1}) = d(a_{n_k}, Ta_{n_k-1}) = d(A, B)$, and so,

$$\zeta\left(s^2 d\left(a_{m_k}, a_{n_k}\right), \prod_{\alpha,\beta,\gamma}\left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right)\right) \geq 0, \tag{2.5}$$

where

$$\begin{aligned}&\prod_{\alpha,\beta,\gamma}\left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right)\\ &\quad= \left[d(a_{m_k-1}, a_{n_k-1})\right]^{\beta}\left[d(a_{m_k-1}, a_{m_k})\right]^{\alpha}\\ &\qquad\left[d(a_{n_k-1}, a_{n_k})\right]^{\gamma}\left[\frac{d(a_{m_k-1}, a_{n_k}) + d(a_{m_k}, a_{n_k-1})}{2s}\right]^{1-\alpha-\beta-\gamma}\end{aligned}$$

On taking limit superior as $k \to \infty$, we get,

$$\limsup_{k\to\infty} \prod_{\alpha,\beta,\gamma} \left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right) = 0.$$

Now, from (2.5), Lemma 1.2 and using (ζ_2), we get,

$$\begin{aligned}
0 &\le \limsup_{k\to\infty} \left[\zeta\left(s^2 d\left(a_{m_k}, a_{n_k}\right), \prod_{\alpha,\beta,\gamma} \left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right)\right)\right] \\
&\le \limsup_{k\to\infty} \left[\prod_{\alpha,\beta,\gamma} \left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right) - s^2 d\left(a_{m_k}, a_{n_k}\right)\right] \\
&= \limsup_{k\to\infty} \prod_{\alpha,\beta,\gamma} \left(a_{m_k-1}, a_{n_k-1}, a_{m_k}, a_{n_k}\right) - s^2 \liminf_{k\to\infty} d\left(a_{m_k}, a_{n_k}\right) \\
&= -s^2\left(\frac{\epsilon}{s^2}\right) \\
&< 0,
\end{aligned}$$

which is a contradiction. Therefore, $\{a_n\}$ is a b-Cauchy sequence in A_0. Since, A_0 is closed, there exists some $x \in A_0$ such that $a_n \to x$ as $n \to \infty$.
Since, $Tx \in B_0$, there exists $z \in A_0$ such that

$$d(z, Tx) = d(A, B). \tag{2.6}$$

Now, we show that $d(x,z) = 0$. If possible, let $d(x,z) > 0$.

From (2.3), (2.6) and definition of T, we get,

$$\zeta\left(s^2 d(a_{n+1}, z), \prod_{\alpha,\beta,\gamma} (a_n, x, a_{n+1}, z)\right) \ge 0,$$

where

$$\prod_{\alpha,\beta,\gamma} (a_n, x, a_{n+1}, z) = [d(x, a_n)]^{\beta} \left[d(a_n, a_{n+1})\right]^{\alpha} \left[d(x,z)\right]^{\gamma} \left[\frac{d(a_n, z) + d(x, a_{n+1})}{2s}\right]^{1-\alpha-\beta-\gamma}$$

So, $\limsup_{n\to\infty} P(a_n, x, a_{n+1}, z) = 0$. Using condition (ζ_3), we get,

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \left[\zeta\left(s^2 d(a_{n+1}, z), \prod_{\alpha,\beta,\gamma} (a_n, x, a_{n+1}, z)\right)\right] \\
&\le \limsup_{n\to\infty} \left[\prod_{\alpha,\beta,\gamma} (a_n, x, a_{n+1}, z) - s^2 d(a_{n+1}, z)\right] \\
&= \limsup_{n\to\infty} \prod_{\alpha,\beta,\gamma} (a_n, x, a_{n+1}, z) - s^2 \liminf_{n\to\infty} d(a_{n+1}, z)
\end{aligned}$$

$$= -s^2\left(\frac{d(x,z)}{s}\right)$$
$$< 0,$$

it is a contradiction. Hence, $x = z$ and therefore $d(x,Tx) = d(A,B)$, i.e., x is a best proximity point of T.

Uniqueness. Suppose $y(\neq x) \in A_0$ be another best proximity point of T. Since, $d(x,Tx) = d(A,B)$ and $d(y,Ty) = d(A,B)$, so, by Definition 2.1, $0 \le \zeta\left(s^2 d(x,y), \prod_{\alpha,\beta,\gamma}(x,y,x,y)\right)$, where

$$\prod_{\alpha,\beta,\gamma}(x,y,x,y) = [d(x,y)]^{\beta}\,[d(x,x)]^{\alpha}\,[d(y,y)]^{\gamma}\left[\frac{d(x,y)+d(x,y)}{2s}\right]^{1-\alpha-\beta-\gamma}$$
$$= 0.$$

From the condition (ζ_2), we get,

$$0 \le \zeta\left(s^2 d(x,y), \prod_{\alpha,\beta,\gamma}(x,y,x,y)\right) < \prod_{\alpha,\beta,\gamma}(x,y,x,y) - s^2 d(x,y) < 0,$$

which is a contradiction.

Hence, the best proximity point of T is unique. □

Example 2.1. *Let $X = \mathbb{R}^2$ and*

$$A = [0,\infty)\times\{1\}, \quad A_0 = [0,1]\times\{1\}$$
$$B = [0,\infty)\times\{0\}, \quad B_0 = [0,1]\times\{0\}$$

We define $d : X \times X \to \mathbb{R}^+$ by $d((a_1,a_2),(b_1,b_2)) = |a_1 - b_1|^2 + |a_2 - b_2|^2$ for all $(a_1,a_2),(b_1,b_2) \in X$. Then, clearly (X,d) is a b-metric space with $s = 2$.

We define the map $T : A \to B$ by

$$T(x,1) = \begin{cases} \left(\frac{x^2}{5}, 0\right) & \text{if } x \in [0,1], \\ \left(\frac{3}{2}x^2 + 1, 0\right) & \text{if } x > 1 \end{cases}$$

and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{9}{10}s - t, s,t \in \mathbb{R}^+$. Clearly, $T(A_0) \subseteq B_0, d(A,B) = 1$ and ζ is a simulation function.

Now, let $(x,1),(y,1),(u,1),(v,1) \in A$ such that

$$\begin{aligned} d((u,1),T(x,1)) &= d(A,B) = 1 \\ d((v,1),T(y,1)) &= d(A,B) = 1. \end{aligned} \tag{2.7}$$

Without loss of generality, we assume that $u \ge v$. From (2.7), we have, $x \in [0,1], y \in [0,1], u = \frac{x^2}{5} \in \left[0,\frac{1}{5}\right], v = \frac{y^2}{5} \in \left[0,\frac{1}{5}\right]$.

Now, $s^2 d((u.1),(v,1)) = 4|u-v|^2 = 4(u-v)^2$ *and*

$$\begin{aligned}&\prod_{\alpha,\beta,\gamma}((x,1),(y,1),(u,1),(v,1))\\&=\left[|x-y|^2\right]\left[|x-u|^2\right]\left[|y-v|^2\right]\left[\frac{|x-v|^2+|y-u|^2}{4}\right]\\&=\left[5(\sqrt{u}-\sqrt{v})^2\right]\left[(\sqrt{5u}-u)^2\right]\left[(\sqrt{5v}-v)^2\right]\left[\frac{(\sqrt{5u}-v)^2+|\sqrt{5v}-u|^2}{4}\right].\end{aligned}$$

We consider

$$\begin{aligned}&\zeta\left(s^2 d((u.1),(v,1)),\prod_{\alpha,\beta,\gamma}((x,1),(y,1),(u,1),(v,1))\right)\\&=\frac{9}{10}\prod_{\alpha,\beta,\gamma}((x,1),(y,1),(u,1),(v,1))-s^4 d((u.1),(v,1))\\&=\frac{9}{10}\max\left\{25(u-v)^2,16u^2,16v^2,\frac{(5u-v)^2+|5v-u|^2}{4}\right\}\\&\quad-16(u-v)^2\\&\geq 0.\end{aligned}$$

Therefore T is an interpolative Hardy-Rogers proximal $\mathcal{Z}$-contraction of first kind. Hence, T satisfies all the hypotheses of Theorem 2.1 and $(0,1)$ is the unique best proximity point in A_0.

Remark 2.1. *Taking $A = B = X$ and $s = 1$ in Theorem 2.1, we get Theorem 2.2 of [27] as a particular case.*

Remark 2.2. *In Theorem 2.1, the mapping T is not necessarily continuous. Moreover, the sets A and B are not required to be closed. Thus, for $\prod_{\alpha,\beta,\gamma}(y_1,y_2,x_1,x_2) = [d(y_1,y_2)]^\alpha < d(y_1,y_2), \alpha < 1$, $d(y_1,y_2) > 1$, (when the mapping T reduces to proximal simulative contraction of first kind) Theorem 2.1 improves Theorem 1 of [1] in b-metric spaces.*

Corollary 2.1. *Let A and B be two non-empty subsets of a complete b-metric space (X,d). Suppose $T : A \to B$ be a mapping with $T(A_0) \subseteq B_0$ where A_0, B_0 are non-empty and A_0 is closed subset of A such that $d(a_1,Tb_1) = d(A,B) = d(a_2,Tb_2)$ then*

$$\zeta\left(s^4 d(a_1,a_2),\prod_{\alpha,\beta,\gamma}(b_1,b_2,a_1,a_2)\right)\geq 0,\quad \forall b_1,b_2,a_1,a_2\in A,$$

where

$$\begin{aligned}&\prod_{\alpha,\beta,\gamma}(b_1,b_2,a_1,a_2)\\&=[d(b_1,b_2)]^\beta\left[\frac{d(b_1,a_1)+d(b_2,a_2)}{2}\right]^\alpha\left[\frac{d(b_2,a_1)+d(b_1,a_2)}{2s}\right]^{1-\alpha-\beta}.\end{aligned}$$

Then T has a unique best proximity point.

Theorem 2.2. *Let A and B be two non-empty subsets of a complete b-metric space (X, d). Suppose $T : A \to B$ be an interpolative Hardy-Rogers-type proximal $\mathcal{Z}$-contraction of the second kind with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and B_0 is a closed subset of B. Then T has a unique best proximity point in A_0.*

Proof. Following the same procedure as Theorem 2.1 and using the condition of generalized proximal $\mathcal{Z}$-contraction of second kind, we can show that $\{d(Ta_n, Ta_{n+1})\}$ is a decreasing sequence and

$$\lim_{n\to\infty} d(Ta_n, Ta_{n+1}) = 0.$$

Proceeding the technique of Theorem 2.1, it can be proven that $\{Ta_n\}$ is a b-Cauchy sequence in B_0 and so it converges to some $y = Tu \in B_0$, where $u \in A_0$. Since, $Tu \in B_0$, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A, B).$$

Therefore,

$$\zeta\left(s^2 d(Ta_{n+1}, Tz), \prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz)\right) \geq 0,$$

where,

$$\prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz) = [d(Tu, Ta_n)]^{\beta}\,[d(Ta_n, Ta_{n+1})]^{\alpha}\,[d(Tu, Tz)]^{\gamma}$$
$$\left[\frac{d(Ta_n, Tz) + d(Tu, Ta_{n+1})}{2s}\right]^{1-\alpha-\beta-\gamma}$$

So,

$$\limsup_{n\to\infty} \prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz) = 0.$$

Also,

$$\limsup_{n\to\infty} d(Ta_{n+1}, Tz) = d(Tu, Tz).$$

If $d(Tu, Tz) > 0$ then using condition (ζ_3), we get,

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}\left[\zeta\left(s^2 d(Ta_{n+1}, Tz), \prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz)\right)\right] \\
&\leq \limsup_{n\to\infty}\left[\prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz) - s^2 d(Ta_{n+1}, Tz)\right] \\
&= \limsup_{n\to\infty}\prod_{\alpha,\beta,\gamma}(Ta_n, Tu, Ta_{n+1}, Tz) - s^2 \liminf_{n\to\infty} d(Ta_{n+1}, Tz) \\
&= -s^2\left(\frac{d(Tu, Tz)}{s}\right) < 0,
\end{aligned}$$

it is a contradiction. Hence, $Tu = Tz$ and therefore $d(z, Tz) = d(A, B)$,
i.e., z is a best proximity point of T.

Uniqueness:
Suppose $y(\neq z) \in A_0$ be another best proximity point of T.
Since, $d(z, Tz) = d(A, B)$ and $d(y, Ty) = d(A, B)$, so, by Definition 2.2,

$$0 \leq \zeta\left(s^2 d(Tz, Ty), \prod_{\alpha,\beta,\gamma}(Tz, Ty, Tz, Ty)\right),$$

where

$$\begin{aligned}\prod_{\alpha,\beta,\gamma}(Tz, Ty, Tz, Ty) &= [d(Tz, Ty)]^{\beta}\,[d(Tz, Tz)]^{\alpha}\,[d(Ty, Ty)]^{\gamma} \\ &\quad \left[\frac{d(Tz, Ty) + d(Tz, Ty)}{2s}\right]^{1-\alpha-\beta-\gamma} \\ &= 0.\end{aligned}$$

From the condition (ζ_2), we get,

$$\begin{aligned}0 &\leq \zeta\left(s^2 d(Tz, Ty), \prod_{\alpha,\beta,\gamma}(Tz, Ty, Tz, Ty)\right) \\ &< \prod_{\alpha,\beta,\gamma}(Tz, Ty, Tz, Ty) - s^2 d(Tz, Ty) \\ &< 0,\end{aligned}$$

which is a contradiction.

Hence, the best proximity point of T is unique. □

Example 2.2. *Let $X = \mathbb{R}^2$,*

$$\begin{aligned}A &= \{(x, 0) : x \geq 0\}, \quad A_0 = \{(x, 0) : x \in [0, 1]\}, \\ B &= \{(x, 1) : x \geq 0\}, \quad B_0 = \{(x, 1) : x \in [0, 1]\},\end{aligned}$$

We define $d : X \times X \to \mathbb{R}^+$ by $d((a_1, a_2), (b_1, b_2)) = |a_1 - b_1|^2 + |a_2 - b_2|^2$ for all $(a_1, a_2), (b_1, b_2) \in X$. Then, clearly (X, d) is a b-metric space with $s = 2$.
We define the map $T : A \to B$ by

$$T(x, 0) = \begin{cases} \left(\frac{x^2}{25}, 1\right) & \text{if } x \in [0, 1], \\ \left(x^2 - \frac{2}{3}, 1\right) & \text{if } x > 1 \end{cases}$$

and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ by $\zeta(t, s) = \frac{9}{10}s - t, s, t \in \mathbb{R}^+$. Clearly, $T(A_0) \subseteq B_0, d(A, B) = 1$ and ζ is a simulation function.

Now, let $(x, 0), (y, 0), (u, 0), (v, 0) \in A$ such that

$$\begin{aligned}d((u, 0), T(x, 0)) &= d(A, B) = 1 \\ d((v, 0), T(y, 0)) &= d(A, B) = 1.\end{aligned} \tag{2.8}$$

Without loss of generality, we assume that $u \geq v$. From (2.8), we have, $x \in [0,1], y \in [0,1], u = \frac{x^2}{25} \in \left[0, \frac{1}{25}\right], v = \frac{y^2}{25} \in \left[0, \frac{1}{25}\right]$.

Now, $s^2 d(T(u.0), T(v,0)) = 4|u-v|^2 = 4(u-v)^2$ and

$$\begin{aligned}
&\prod_{\alpha,\beta,\gamma}(T(x,0),T(y,0),T(u,0),T(v,0))\\
&= \left[|x-y|^2\right]\left[|x-u|^2\right]\left[|y-v|^2\right]\left[\frac{|x-v|^2+|y-u|^2}{4}\right]\\
&= \left[25(\sqrt{u}-\sqrt{v})^2\right]\left[(5\sqrt{u}-u)^2\right]\left[(5\sqrt{u}-u)^2\right]\left[\frac{(5\sqrt{u}-v)^2+|5\sqrt{v}-u|^2}{4}\right]
\end{aligned}$$

We consider

$$\begin{aligned}
&\zeta\left(s^2 d(T(u.0),T(v,0)), \prod_{\alpha,\beta,\gamma}(T(x,0),T(y,0),T(u,0),T(v,0))\right)\\
&= \frac{9}{10}\prod_{\alpha,\beta,\gamma}(T(x,0),T(y,0),T(u,0),T(v,0)) - s^2 d(T(u,0),T(v,0))\\
&= \frac{9}{10}\left[25(\sqrt{u}-\sqrt{v})^2\right]\left[(5\sqrt{u}-u)^2\right]\left[(5\sqrt{u}-u)^2\right]\left[\frac{(5\sqrt{u}-v)^2+|5\sqrt{v}-u|^2}{4}\right]\\
&\quad -4(u-v)^2\\
&\geq 0.
\end{aligned}$$

Therefore T is a generalized proximal $\mathcal{Z}$-contraction of second kind.
Hence, T satisfies all the hypotheses of Theorem 2.2 and $(0,0)$ is the unique best proximity point.

Remark 2.3. *Taking $A = B = X$ and $s = 1$ in Theorem 2.2, we get Theorem 2.2 of [17] as a particular case.*

Remark 2.4. *Taking $\prod_{\alpha,\beta,\gamma}(Ty_1, Ty_2, Tx_1, Tx_2) = [d(Ty_1, Ty_2)]^{\alpha} < d(Ty_1, Ty_2), 0 < \alpha < 1, d(Ty_1, Ty_2) > 1$ in Theorem 2.2, we get an improvement of Theorem 2 of [1].*

Corollary 2.2. *Let A and B be two non-empty subsets of a complete b-metric space (X, d). Suppose $T : A \to B$ be a mapping with $T(A_0) \subseteq B_0$ where A_0, B_0 are non-empty and B_0 is closed subset of B such that $d(a_1, Tb_1) = d(A, B) = d(a_2, Tb_2)$ implies*

$$\zeta\left(s^2 d(Ta_1, Ta_2), \prod_{\alpha,\beta,\gamma}(Tb_1, Tb_2, Ta_1, Ta_2)\right) \geq 0, \quad \text{for all } b_1, b_2, a_1, a_2 \in A,$$

where

$$\prod_{\alpha,\beta,\gamma}(Tb_1,Tb_2,Ta_1,Ta_2) = [d(Tb_1,Tb_2)]^{\beta}\left[\frac{d(Tb_1,Ta_1)+d(Tb_2,Ta_2)}{2}\right]^{\alpha}$$

$$\left[\frac{d(Tb_2,Ta_1)+d(Tb_1,Ta_2)}{2s}\right]^{1-\alpha-\beta}$$

Then T has a unique best proximity point.

3. Applications

3.1. Application to Nonlinear Integral Equations. Let $\Omega = C[a,b]$ be a set of real valued continuous functions on $[a,b]$, where $[a,b]$ is closed and bounded integral in $\mathbb{R}$. We define $d : \Omega \times \Omega \to \mathbb{R}^+$ by $d(\xi,\eta) = \max_{t\in[a,b]} |\xi(t) - \eta(t)|^p$, where $p > 1$ a real number, for all $\xi, \eta \in \Omega$. Therefore (Ω, d) is a complete b-metric space with $s = 2^{p-1}$. Many authors studied unique solution of a nonlinear integral equations [5–7,9]. If we take $A = B = \Omega$ in Theorem 2.1, we obtain the existence of unique solutions of nonlinear integral equation of Fredholm type defined by

$$\xi(t) = f(t) + \mu \int_a^b \mathcal{D}(t,r,\xi(r))dr \tag{3.1}$$

where $\xi \in C[a,b]$ is the unknown function, $\mu \in \mathbb{R}, t,r \in [a,b], \mathcal{D} : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are continuous functions. Let $\mathcal{F} : \Omega \to \Omega$ be a mapping defined by

$$\mathcal{F}(\xi(t)) = f(t) + \mu \int_a^b \mathcal{D}(t,r,\xi(r))dr. \tag{3.2}$$

Assume the following:

(1) there exists a continuous function $\gamma : [a,b]\times[a,b] \to \mathbb{R}^+$, such that

$$\max_{r\in[a,b]} \int_a^b \gamma(t,r)dr \le 1 \quad \text{and} \quad |\mu| \le 1;$$

(2) there exists a constant $K \in (0,1)$ such that for all $t, r \in [a,b]$ and $\xi, \zeta \in \mathbb{R}$, the following condition is satisfied:

$$|\mathcal{D}(t,r,\xi_1(r)) - \mathcal{D}(t,r,\xi_2(r)|^p \le \frac{K}{(b-a)^{p-1}2^{3p-3}}\gamma(t,r)\Delta(\eta_1,\eta_2,\xi_1,\xi_2),$$

where

$$\Delta(\eta_1,\eta_2,\xi_1,\xi_2) = [|\eta_1(t)-\eta_2(t)|^p]^\beta\,[|\eta_1(t)-\xi_1(t)|^p]^\alpha\,[|\eta_2(t)-\xi_2(t)|^p]^\gamma$$
$$\left[\frac{|\eta_2-\xi_1|^p + |\eta_1-\xi_2|^p}{2^p}\right]^{1-\alpha-\beta-\gamma}$$

Theorem 3.1. *Let $\mathcal{F} : \Omega \to \Omega$ be defined by (3.2) for which the conditions (i) and (ii) are hold. Then, the system of nonlinear integral equations (3.1) has a unique solution in Ω.*

Proof. Let $\xi_1, \xi_2 \in \Omega$ and let $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$ using Hölder's inequality and from the conditions (i) and (ii), for all t, we have

$$\begin{aligned} d(\xi_1,\xi_2) &= \max_{t\in[a,b]} |\xi_1(t)-\xi_2(t)|^p \\ &= |\mu|^p \max_{t\in[a,b]} \left|\int_a^b \mathcal{D}(t,r,\xi_1(r)) - \int_a^b \mathcal{D}(t,r,\xi_2(r))dr\right|^p \\ &= |\mu|^p \max_{t\in[a,b]} \left|\int_a^b (\mathcal{D}(t,r,\xi_1(r) - \mathcal{D}(t,r,\xi_2(r))dr\right|^p \end{aligned}$$

$$
\begin{aligned}
&\leq \left[|\mu|^p \max_{t\in[a,b]}\left(\int_a^b 1^p dr\right)^{\frac{1}{q}}\left(\int_a^b \left|(\mathcal{D}(t,r,\xi_1(r))-\mathcal{D}(t,r,\xi_2(r))\right|^p dr\right)^{\frac{1}{p}}\right]^p \\
&\leq (b-a)^{\frac{p}{q}} \max_{t\in[a,b]}\left(\int_a^b \left|(\mathcal{D}(t,r,\xi_1(r))-\mathcal{D}(t,r,\xi_2(r))\right|^p dr\right) \\
&= (b-a)^{p-1} \max_{t\in[a,b]}\left(\int_a^b \left|(\mathcal{D}(t,r,\xi_1(r))-\mathcal{D}(t,r,\xi_2(r))\right|^p dr\right) \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]}\int_a^b \frac{K}{(b-a)^{p-1}2^{3p-3}}\gamma(t,r)\Delta(\eta_1,\eta_2,\xi_1,\xi_2)dr
\end{aligned}
$$

which implies that

$$
\begin{aligned}
s^2 d(\xi_1,\xi_2) &\leq \frac{K}{s}\left[d(\eta_1,\eta_2)\right]^\beta\left[d(\eta_1,\xi_1)\right]^\alpha\left[d(\eta_2,\xi_2)\right]^\gamma\left[\frac{d(\eta_2,\xi_1)+d(\eta_1,\xi_2)}{2s}\right]^{1-\alpha-\beta-\gamma} \\
&\leq \lambda\left[d(\eta_1,\eta_2)\right]^\beta\left[d(\eta_1,\xi_1)\right]^\alpha\left[d(\eta_2,\xi_2)\right]^\gamma\left[\frac{d(\eta_2,\xi_1)+d(\eta_1,\xi_2)}{2s}\right]^{1-\alpha-\beta-\gamma}
\end{aligned}
$$

where $\lambda = \frac{K}{s} \in (0,1)$.

Therefore, by taking $\zeta(t,s) = \lambda s - t,\ \lambda \in (0,1)$, all the conditions of Theorem 2.1 are satisfied and hence $\mathcal{F}$ has a unique solution for nonlinear integral equations defined in (3.1). □

3.2. **Application to Dynamic Programming.** We discuss the following existence of bounded solutions for functional equations that arise in dynamic programming [11]. Let 'opt' represents inf or sup, $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are two Banach spaces; $\tilde{\mathcal{D}} \subseteq \tilde{\Theta}_1$ is the decision space; $\tilde{\mathcal{S}} \subseteq \tilde{\Theta}_2$ is the state space; $\mho(\tilde{\mathcal{S}})$, the set of all bounded real valued functions on $\tilde{\mathcal{S}}$ with b-metric is defined by:

$$d(p_x,p_y) = \sup_{t\in\tilde{\mathcal{S}}}|p_x(t)-p_y(t)|^r, \quad \text{for all } p_x,p_y \in \mho(\tilde{\mathcal{S}})$$

with parameter $s = 2^{r-1}$.

Now, we consider the following functional equations:

$$f(v_s) = \operatorname*{opt}_{v_d\in\tilde{\mathcal{D}}}\{\zeta(v_s,v_d)+C(v_s,v_d,f(\omega(v_s,v_d)))\}, \quad \text{for all } v_s \in \tilde{\mathcal{S}}, \tag{3.3}$$

where v_d is a decision vector, v_s is a state vector where as ω denotes the transformation of the process and f indicates the optimal return function.

Let $\aleph : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ be a mapping defined by:

$$\aleph f(v_s) = \operatorname*{opt}_{v_s\in\tilde{\mathcal{D}}}\{\zeta(v_s,v_d)+C(v_s,v_d,f(\omega(v_s,v_d)))\}, \quad \text{for all } v_s \in \tilde{\mathcal{S}}, \tag{3.4}$$

where $(v_s,f) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}})$.

Let $\xi : \mho(\tilde{\mathcal{S}}) \times \mathbb{R}$. Assume the following:

(1) for all $(v_s, v_d, g_1, g_2, f_1, f_2) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$, we have:

$$\begin{aligned}&\Big|C(v_s, v_d, f_1(\omega(v_s, v_d))) - C(v_s, v_d, f_2(\omega(v_s, v_d)))\Big| \\ &\quad\leq \left(\frac{2^{2-2r}}{l}\Delta_s(g_1, g_2, f_1, f_2)\right)^{\frac{1}{r}},\end{aligned}$$

where

$$\begin{aligned}\Delta_s(g_1, g_2, f_1, f_2) = {}& [|\aleph g_1(v_s) - \aleph g_2(v_s)|^r]^\beta\, [|\aleph g_1(v_s) - \aleph f_1(v_s)|^r]^\alpha \\ & [|\aleph g_2(v_s) - \aleph f_2(v_s)|^r]^\gamma \\ & \left[\frac{|\aleph g_1(v_s) - \aleph f_2(v_s)|^r + |\aleph g_2(v_s) - \aleph f_1(v_s)|^r}{2^r}\right]^{1-\alpha-\beta-\gamma}\end{aligned}$$

and $0 < \frac{1}{l} < 1$.

(2) ω, C are bounded.

Theorem 3.2. *Suppose $\aleph : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ is defined by (3.4) for which the conditions (i) and (ii) hold. Then (3.3) has a unique bounded common solution in $\mho(\tilde{\mathcal{S}})$.*

Proof. Take $\epsilon > 0$. Let $v_s \in \tilde{\mathcal{S}}, f_1, f_2 \in \mho(\tilde{\mathcal{S}})$. Since ω, C are bounded there exists $L \geq 0$ such that

$$\begin{aligned}&\sup\{\|\omega_1(v_s, v_d)\|, \|\omega_2(v_s, v_d)\|, \|C_1(v_s, v_d, t)\|, \|C_2(v_s, v_d, t)\| : \\ &\quad (v_s, v_d, t) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mathbb{R}\} \leq L.\end{aligned}$$

First, we assume that $\operatorname{opt}_{v_d \in \tilde{\mathcal{D}}} = \inf_{v_d \in \tilde{\mathcal{D}}}$.

By using (3.4), we can find $v_d \in \tilde{\mathcal{D}}$ and $(v_s, f_1, f_2) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$ such that

$$\aleph f_1(v_s) > C(v_s, v_d, f_1(\omega(v_s, v_d))) + \zeta(v_s, v_d) - \epsilon, \tag{3.5}$$

$$\aleph f_2(v_s) > C(v_s, v_d, f_2(\omega(v_s, v_d))) + \zeta(v_s, v_d) - \epsilon, \tag{3.6}$$

$$\aleph f_1(v_s) \leq C(v_s, v_d, f_1(\omega(v_s, v_d))) + \zeta(v_s, v_d), \tag{3.7}$$

$$\aleph f_2(v_s) \leq C(v_s, v_d, f_2(\omega(v_s, v_d))) + \zeta(v_s, v_d). \tag{3.8}$$

From (3.5) and (3.8), we get

$$\begin{aligned}&\aleph f_1(v_s) - \aleph f_2(v_s) \qquad (3.9)\\ &\quad > C(v_s, v_d, f_1(\omega(v_s, v_d))) - C(v_s, v_d, f_2(\omega(v_s, v_d))) - \epsilon \\ &\quad \geq -\Big\{\Big|C(v_s, v_d, f_1(\omega(v_s, v_d))) - C(v_s, v_d, f_2(\omega(v_s, v_d)))\Big| + \epsilon\Big\}.\end{aligned}$$

Also by using (3.6) and (3.7), we have

$$\begin{aligned}&\aleph f_1(v_s) - \aleph f_2(v_s) \qquad (3.10)\\ &\quad \leq C(v_s, v_d, f_1(\omega(v_s, v_d))) - C(v_s, v_d, f_2(\omega(v_s, v_d))) + \epsilon \\ &\quad \leq \Big|C(v_s, v_d, f_1(\omega(v_s, v_d))) - C(v_s, v_d, f_2(\omega(v_s, v_d)))\Big| + \epsilon.\end{aligned}$$

From (3.9) and (3.10), we have

$$\begin{aligned}&|\aleph f_1(v_s)-\aleph f_2(v_s)|\\&\quad< C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))+\epsilon\\&\quad\le \Big|C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))\Big|+\epsilon.\end{aligned}$$

Suppose that $\underset{v_s\in\tilde{\mathcal{D}}}{\text{opt}}=\underset{v_d\in\tilde{\mathcal{D}}}{\sup}$. Again by using the inequality (3.4), we can find $v_d\in\tilde{\mathcal{D}}$ and $(v_s,f_1,f_2)\in\tilde{\mathcal{S}}\times\mho(\tilde{\mathcal{S}})\times\mho(\tilde{\mathcal{S}})$ such that

$$\aleph f_1(v_s)< C(v_s,v_d,f_1(\omega(v_s,v_d)))+\zeta(v_s,v_d)+\epsilon, \tag{3.11}$$

$$\aleph f_2(v_s)< C(v_s,v_d,g(\omega(v_s,v_d)))+\zeta(v_s,v_d)+\epsilon, \tag{3.12}$$

$$\aleph f_1(v_s)\ge C(v_s,v_d,f_1(\omega(v_s,v_d)))+\zeta(v_s,v_d), \tag{3.13}$$

$$\aleph f_2(v_s)\ge C(v_s,v_d,f_2(\omega(v_s,v_d)))+\zeta(v_s,v_d). \tag{3.14}$$

From (3.11) and (3.14), we get

$$\begin{aligned}\aleph f_1(v_s)-\aleph f_2(v_s)&< C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))+\epsilon\\&\le \Big|C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,g(\omega(v_s,v_d)))\Big|+\epsilon.\end{aligned}\tag{3.15}$$

Also by using (3.12) and (3.13), we have

$$\begin{aligned}&\aleph f_1(v_s)-\aleph f_2(v_s)\\&\quad\ge C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))-\epsilon\\&\quad\ge -\Big\{\Big|C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))\Big|+\epsilon\Big\}.\end{aligned}\tag{3.16}$$

From (3.15) and (3.16), we have

$$\begin{aligned}&|\aleph f_1(v_s)-\aleph f_2(v_s)|\\&\quad< C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))+\epsilon\\&\quad\le \Big|C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))\Big|+\epsilon.\end{aligned}\tag{3.17}$$

On letting $\epsilon\to 0$ in (3.17), we obtain

$$|\aleph f_1(v_s)-\aleph f_2(v_s)|\le |C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))|.$$

By using the condition (i), we have

$$\begin{aligned}&|\aleph f_1(v_s)-\aleph f_2(v_s)|\\&\quad\le |C(v_s,v_d,f_1(\omega(v_s,v_d)))-C(v_s,v_d,f_2(\omega(v_s,v_d)))|\\&\quad\le\left(\frac{2^{2-2r}}{l}\Delta_s(g_1,g_2,f_1,f_2)\right)^{\frac{1}{r}}\\&\quad=\left(\frac{2^{2-2r}}{l}\Bigg(\,[|\aleph g_1(v_s)-\aleph g_2(v_s)|^r]^{\beta}\,[|\aleph g_1(v_s)-\aleph f_1(v_s)|^r]^{\alpha}\right.\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad [|\aleph g_2(v_s) - \aleph f_2(v_s)|^r]^\gamma \\
&\qquad\qquad \left[\frac{|\aleph g_1(v_s) - \aleph f_2(v_s)|^r + |\aleph g_2(v_s) - \aleph f_1(v_s)|^r}{2^r}\right]^{1-\alpha-\beta-\gamma}\Bigg)\Bigg)^{\frac{1}{r}} \\
&\le \left(\frac{2^{2-2r}}{l} \sup_{v_s \in \tilde{S}} \Bigg([|\aleph g_1(v_s) - \aleph g_2(v_s)|^r]^\beta \, [|\aleph g_1(v_s) - \aleph f_1(v_s)|^r]^\alpha \right. \\
&\qquad\qquad [|\aleph g_2(v_s) - \aleph f_2(v_s)|^r]^\gamma \\
&\qquad\qquad \left.\left[\frac{|\aleph g_1(v_s) - \aleph f_2(v_s)|^r + |\aleph g_2(v_s) - \aleph f_1(v_s)|^r}{2^r}\right]^{1-\alpha-\beta-\gamma}\Bigg)\right)^{\frac{1}{r}} \\
&= \left(\frac{2^{2-2r}}{l} [d(\aleph g_1, \aleph g_2)]^\beta \, [d(\aleph g_1, \aleph f_1)]^\alpha \, [d(\aleph g_2, \aleph f_2)]^\gamma \right. \\
&\qquad\qquad \left.\left[\frac{d(\aleph g_1, \aleph f_2) + d(\aleph g_2, \aleph f_1)}{2s}\right]^{1-\alpha-\beta-\gamma}\right)^{\frac{1}{r}}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|\aleph f_1(v_s) - \aleph f_2(v_s)|^r \le \frac{2^{2-2r}}{l} \Bigg(& [d(\aleph g_1, \aleph g_2)]^\beta \, [d(\aleph g_1, \aleph f_1)]^\alpha \, [d(\aleph g_2, \aleph f_2)]^\gamma \\
& \left[\frac{d(\aleph g_1, \aleph f_2) + d(\aleph g_2, \aleph f_1)}{2s}\right]^{1-\alpha-\beta-\gamma}\Bigg)
\end{aligned}
$$

Now, for all $g_1, g_2, f_1, f_2 \in \mho(\tilde{S})$, we have

$$
\begin{aligned}
s^2 d(\aleph f_1(v_s), \aleph f_2(v_s)) \le \frac{1}{l} & [d(\aleph g_1, \aleph g_2)]^\beta \, [d(\aleph g_1, \aleph f_1)]^\alpha \, [d(\aleph g_2, \aleph f_2)]^\gamma \\
& \left[\frac{d(\aleph g_1, \aleph f_2) + d(\aleph g_2, \aleph f_1)}{2s}\right]^{1-\alpha-\beta-\gamma}
\end{aligned}
$$

It is clear that Theorem 3.2 satisfies all the hypotheses of Theorem 2.2 with $\zeta(t,s) = \lambda s - t$, $\lambda = \frac{1}{l} \in (0,1)$ and $A = B = \mho(\tilde{S})$. According to Theorem 2.2, the functional equations that are defined in (3.3) have a unique bounded solution. □

4. Conclusion and Future Work

In this paper, we introduced generalized proximal $\mathcal{Z}$-contraction of the first kind and the second kind and obtained some best proximity points via simulation functions. Using similar approaches, it can be studied new best proximity points result in metric and some generalized metric spaces. The investigation of certain circumstances to exclude the identity map of X from Theorem 2.1 and Theorem 2.2 and related results is a worthwhile problem for future efforts.

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