International Journal of Analysis and Applications

New Optimum solutions of The Time-Fractional Fitzhugh-Nagumo Equations

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ABSTRACT. The purpose and objective of the present work are to show the reliability and effectiveness of the newly developed semi-numerical method, i-e, the Optimal Auxiliary Function method OAFM, by solving the fractional problems of Fitzhugh-Nagumo. We have developed OAFM mathematical formulations for nonlinear partial differential equations PDEs. The implementation of the OAFM achieves a fast serial convergence solution. The analysis shows that the proposed method has a simplified implementation and needless computational work, is extremely accurate, and converges rapidly. Tables were constructed to compare the numerical results with the problems' exact solutions to see the errors.

Received Mar. 30, 2025

2020 Mathematics Subject Classification. 35R11.

Key words and phrases. OAFM; closed-form solutions; smart grid; time-fractional; Fitzhugh-Nagumo equations.

https://doi.org/10.28924/2291-8639-23-2025-158

ISSN: 2291-8639

1. Introduction

Fraction calculation is the addition of the ordinal calculation of an integer. Fractional analysis was not applied to the existing problems of the real world. Still, after some time, the concept was validated because fractional analysis used relevant and real applications such as the propagation of sound waves in a rigid porous substance [1], Ultrasound propagation in human bones [2], viscoelastic properties in organic tissues [3], and monitoring tools [4]. In recent times, fractional computational work has been the new attraction of researchers because of its wide applications in Electromagnetism, physical problems, viscosity, and materialistic science [5-9]. Mathematical models of fractional derivatives have been preferred because they are more accurate and realistic than classical formulation models [10]. The development of fractional calculus motivates researchers, and some researchers are determined to use fractional operators and study the solution of nonlinear differential equations used in various problems. They developed many analytical and numerical methods containing fractional operators to obtain accurate approximate solutions for nonlinear differential equations. Such differential equations include various fractional operators like Caputo-Fabrizio, Hilfer, Riemann Liouville, and more. [11]. But, these operators have a command law kernel and have some restrictions in modeling problems.

Learning the exact solution to nonlinear problems plays an important role; At the same time, most fractional PDEs do not have a definitive solution; for these circumstances, we want other more reliable and effective techniques. Basic transformation methods were primarily utilized to solve such physical problems [12-15]. Such methods convert a more complex problem into an easy one. Scientists also work on perturbation techniques and other analytical and numerical methods [16-19] for nonlinear problems. This method requires few input parameters or an initial estimate; choosing these options incorrectly affects the accuracy. The homotopy definition is combined with the perturbation technique to develop the homotopy perturbation method HPM [20-22] and Homotopy Analysis methods (HAM) [23] to solve small parameter problems. This method requires an initial estimation and has good flexibility in handling the convergence area. To handle the initial assumption problem, the Optimal Homotopy Asymptotic Method (OHAM) was introduced by Marinca and Herisanu [24-28]. The method mentioned above possesses an optimal auxiliary function. It does not need an initial estimate, and for more complex problems, the given method was then extended by Ullah et al. [29-31]. Herisanu introduced the optimal Auxiliary Function Method (OAFM) to solve fractional-order nonlinear problems. The time-fractional Fitzhug-Nagumo (FN) equation is among the most significant reaction-diffusion equations used to describe neural loop communication. A mathematical model of population genetics using the FN equation is also described [32]. Khan et al. used the Homotopy Analysis method (HAM) to achieve the approximate analytical solutions to the reaction-diffusion equations. Tcher et al. offer the Power Series Method (RPSM). [33] to solve fractional reaction-diffusion equations to calculate a numerical solution. Merdan [34] used the variational iteration method (FVIM) and obtained sequential solutions of the reaction-diffusion equation. Because of the above discussion, this work is devoted to establishing the uniqueness and existence criteria of solutions of the nonlinear FN equations. Furthermore, we reveal the efficiency of the Optimal Auxilary Function method by getting the approximate solutions and 3D visual graphs of the time fractional reaction-diffusion equations.

This method is implemented with very less computational effort, and just in the first iteration, we obtain the exact solution.

The key objective of this effort is to analyze the OAFM for fractional orders of PDEs. OAFM is an efficient and reliable method for dealing with PDEs of fractional orders.

2. Some Basic Definitions

Definition1. A real valued function $\hat{g}(\eta), \eta > 0$ is in space if Ƀ ∈ R for a real number $ƛ < q$ $\hat{g}(\eta) = \eta^{q}\hat{g}_1(\eta)$, where $\hat{g}_1(\eta) \in$ Ƀ$(0,\infty)$ and is in space iff $\hat{g}^{n}(\eta) \in$ Ƀ$\eta, n \in N$

Definition: 2. The Reiman-Liouville oprator for fractional integral is

$$I^{\alpha}\hat{g}(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^u (\eta-\tau)^{n-1}\hat{g}(\tau)d\tau \tag{1}$$

$$I\hat{g}(\eta) = \hat{g}(\eta) \tag{2}$$

$$I^{\alpha}u^{\varsigma} = \frac{\Gamma(\varsigma+\alpha)}{\Gamma(\varsigma+\alpha+1)}u^{\alpha+n} \tag{3}$$

Definition: 3. In Caputo sense The fractional derivative of the function, $\hat{g}(\eta)$

$$D_u^{\alpha}\hat{g}(\eta) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\eta} (\eta-\tau)^{n-\alpha-1}\hat{g}^{n}(\tau)d\tau \tag{4}$$

Definition: 4. If $n-1 < \alpha \leq n, n \in N \; and \; \hat{g} \in$ Ƀ$_{\eta}^{n}$, $\eta \geq -1$ then

$$D_{\alpha}^{\alpha}I_{\alpha}^{\alpha}\hat{g}(\eta) = \hat{g}(\eta) = \hat{g}(\eta) - \sum_{i=0}^{n-1}\frac{(\eta-\alpha)}{I!}, \eta > 0 \tag{5}$$

3. Proposed Optimal Auxilary Function Technique

The fundamental of OAFM was highlighted in [35], where OAFM is utilized to solve different problems. To build up an OAFM use, consider the nonlinear differential Equation:

$$Ł[u(x)] + \mathcal{Y}(x) + Ŋ[u(x)] = 0 \tag{6}$$

Ł and Ŋ, respectively, represent the linear and nonlinear operators. $\mathcal{Y}$ represents the known function, while x shows the independent variable, and the unknown function at this stage is represented by $u(x)$. The initial guess is:

$$Ƀ\left(u(x), \frac{du(x)}{dx}\right) = 0 \tag{7}$$

Sometimes it takes a lot of work to get the exact solution of the highly nonlinear equations of (6) and (7) kinds [32]. If we are keen to get the approximate solution of $\underset{\sim}{u}(x)$, let us assume that it can be shown as:

$$\underset{\sim}{u}\,(x, Ki) = u_0(x) + u_1(x, Ki), i = 1, 2, \dots, r, \tag{8}$$

The first approximation and the initial guess will be obtained, as shown later. Here we substitute Equation (8) with Equation (6), and we get

$$Ł[u_0(x)] + Ł[u_1(x, k_i)] + \mathcal{Y}(x) + Ŋ[u_0(x) + u_1(x, k_i)] = 0 \tag{9}$$

Where $k_i, i = 1,2, \dots, r$ is the parameters to control the convergence, which will be carefully determined in the above Equation. The initial approximation $u_0(x)$ is the initial approximation which may be calculated from the linear Equation as given below

$$Ł[u(x)] + \mathcal{Y}(x) = 0, \qquad Ƀ\left(u_0(x), \frac{du_0(x)}{dx}\right) = 0 \tag{10}$$

while the first approximation is obtained from Equations (9) and (10)

$$Ł[u_1(x, K_i)] + [u_0(x) + u_1(x, K_i)] = 0 \;\; Ƀ\left(u_0(x), \frac{du_1(x, K_i)}{dx}\right) = 0 \tag{11}$$

The expanded form of the nonlinear term in Equation (11) is as follows

$$Ŋ[u_0(x) + u_1(x, K_i)] = Ŋ[u_0(x)] + \sum_{c=1}^{n} \frac{u_1^c(x, K_i)}{c!} Ŋ^{(c)}[u_0(x)], \dots \tag{12}$$

To keep away from the obstacles in solving Equation (11) and get the fast convergence for the solution of $\underset{\sim}{u}(x, K_i)$, another expression can be proposed instead of the last term, and this Equation can be rewritten as:

$$Ł[u_1(x, K_i)] + \hat{A}_1\big(u_0(x), K_j\big)\psi(Ŋ[u_0(x)]) + \hat{A}_2(u_0(x), K_c) = 0$$
$$Ƀ\left(u_1(x, K_i), \frac{du_1(x, K_i)}{dx}\right) = 0 \;\; , i = 1,2, \dots, r \tag{13}$$

When $\hat{A}_1$ and $\hat{A}_2$ are auxiliary functions dependent on the initial approximation $u_0(x)$and the parameters to control the convergence are K_j and K_c, j = 1,2,...,m, K = m + 1, m + 2,...,r, and $\psi(Ŋ[u_0(x)])$ are functions that depend on expressions of the nonlinear term $Ŋ[u_0(x)]$. The auxiliary functions $\hat{A}_1$and $\hat{A}_2$ and $\psi(Ŋ[u_0(x)])$ should not be only one of its kind, but such auxiliary functions have the identical form as $u_0(x)$ More precisely. , if $u_0(x)$ is a polynomial function, it is the sum of the polynomial functions $\hat{A}_1$ and $\hat{A}_2$. If $u_0(x)$ is an exponential function, then $\hat{A}_1$ and $\hat{A}_2$ should be the sum of exponential functions. The trigonometric function $u_0(x)$ is the sum of the trigonometric functions $\hat{A}_1$ and $\hat{A}_2$, and so on. If $Ŋ[u_0(x)]$ = 0, then the exact solution of the given Equation is $u_0(x)$.

Convergence of the Method: The initial parameters K_j and K_c to control the convergence can be determined accurately and efficiently by diverse methods, including the least squares method, Galerkin method, collocation method, and as well as Ritz method, but the maximum, minimum residual error is better than squared:

$$\zeta(K_1, K_2, \dots, K_r) = \int_{(\underset{\sim}{D})} \underset{\sim}{R}^2 \left(x, K_j, K_c\right) d\mathrm{Ъ}, \qquad j = 1,2, \dots, m,$$

$$K = m + 1, m + 2, \dots, r \tag{14}$$

Where

$$\underset{\sim}{R}\left(x, K_j, K_c\right) = \text{Ł}[\underset{\sim}{u}(x, K_i) + \mathcal{Y}(x) + \text{Ŋ}[\underset{\sim}{u}(x, K_i)],$$

$$j = 1,2, \dots, m, K = m + 1, m + 2, \dots, r, i = 1,2, \dots, r \tag{15}$$

in which the approximate solution $\underset{\sim}{u}(x, K_i)$ is given by Equation (8). The parameters which are unknown i-e K_1, K_2,..., K_r, can be recognized from the conditions as follows

$$\frac{\partial\zeta}{\partial K_1} = \frac{\partial\zeta}{\partial K_2} = \cdots = \frac{\partial\zeta}{\partial K_r} = 0$$

If we impose the condition mentioned below, we get the same results

$$\underset{\sim}{R}\left(x_1, K_j\right) = \underset{\sim}{R}\left(x_2, K_j\right) = \cdots = \underset{\sim}{R}\left(x_i, K_j\right) = 0 \quad x_i \in \underset{\sim}{D}, \ i = 1,2, \dots, r$$

Using the above technique, after the optimal values for the parameters K_j, $i = 1,2, \dots, r$, which control the convergence, are determined, our approximate solution is accomplished, so our method includes the auxiliary functions $\hat{A}_1$ and $\hat{A}_2$ that efficiently present an approach to fix and check the convergence of the last solution. The careful selection of functions $\hat{A}_1$ and $\hat{A}_2$ is worth mentioning according to the first-order approximation. OAFM been shown easily valid to solve nonlinear problems without large small, or parameters, including multi-degree-of-freedom systems.

4. Numerical examples

To see the fast convergence and accuracy of the abovementioned technique, we will check it on some problems in this phase.

Example 4.1 Let us apply our method to the fractional problem of the Fitzhugh Nagumo equation.

$$D_t^\alpha u - u_{xx} - u(u-q)(1-u) = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{17}$$

with corresponding initial condition as

$$u_0(x,t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{\sqrt{2}x}{4}\right) \tag{18}$$

The solution in closed form for the fractional Fitzhugh Nagumo equation with the given initial condition for $\alpha = 1$ is given as

$$u(x,t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{\sqrt{2}x + (1-2q)t}{4}\right) \tag{19}$$

Applying the Optimal Axilary Function method on Eq. (1) with a given initial condition and simplifying, we obtained

$$u_1 = \frac{1}{256a\Gamma(a)} t^a \left(4c_3 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^6 + c_4 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^8 - 2sech\left(\frac{x}{2\sqrt{2}}\right)^6 \left(cosh\left(\frac{x}{\sqrt{2}}\right) + sinh\left(\frac{x}{\sqrt{2}}\right) \left(2c_1 + (2c_1 + c_2) cosh\left(\frac{x}{\sqrt{2}}\right) + c_2 sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 - 2q + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\right)\right) \tag{20}$$

$$u(x,t) = u_0(x,t) + u_1 \tag{21}$$

Putting Eq. (18) and Eq. (20) in Eq. (21), we obtained

$$u(x,t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{256a\Gamma(a)} t^a \left(4c_3 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^6 + c_4 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^8 - 2sech\left(\frac{x}{2\sqrt{2}}\right)^6 \left(cosh\left(\frac{x}{\sqrt{2}}\right) + sinh\left(\frac{x}{\sqrt{2}}\right) \left(2c_1 + (2c_1 + c_2) cosh\left(\frac{x}{\sqrt{2}}\right) + c_2 sinh\left(\frac{x}{\sqrt{2}}\right)\right)\left(1 - 2q + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\right)\right) \tag{22}$$

To find the values of constants, i.e., $(c_1, c_2, c_3 and\ c_4)$, we will find the residual of the above Equation, which is given below

$$R = \frac{1}{\Gamma(1-a)} \int_0^t (t-r)^{-a} \left(\partial_t u(x,t)\right) dr - \partial_{xx} u_0(x,t) - u_0(x,t)(u_0(x,t) - q)\left(1 - u_0(x,t)\right) \tag{23}$$

Solving Eq.23 and using $a = 0.999999999999999, q = -1$ we obtained constants i.e $(c_1, c_2, c_3 and\ c_4)$ as follows

$$c_1 = -12.15351656708473, c_2 = 70.99180916777159,$$
$$c_3 = 79.28570017757944\ and\ c_4 = -87.07379392425483$$

For given values of $c_1, c_2, c_3\ and\ c_4$ Eq.22 becomes

$$u(x,t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{x}{2\sqrt{2}}\right) + 0.003900625 t^a \left(317.1428007103178 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^6 - 87.07379392425483 \left(1 + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)^8 - 2sech\left(\frac{x}{2\sqrt{2}}\right)^6 \left(cosh\left(\frac{x}{\sqrt{2}}\right) + \right.\right.$$

$$sinh\left(\frac{x}{\sqrt{2}}\right)\Bigg(-24.30703313416946 + 460684776033602134cosh\left(\frac{x}{\sqrt{2}}\right) +$$
$$70.99180916777159sinh\left(\frac{x}{\sqrt{2}}\right)\Bigg)\left(1 - 2q + tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\Bigg)\Bigg) \tag{24}$$

Table 1: The absolute errors for the differences between the exact solution of FN (Eq. 19) and OAFM solution (Eq.24) for various values of x when q = -1, α = 0.5, For Example, 4.1

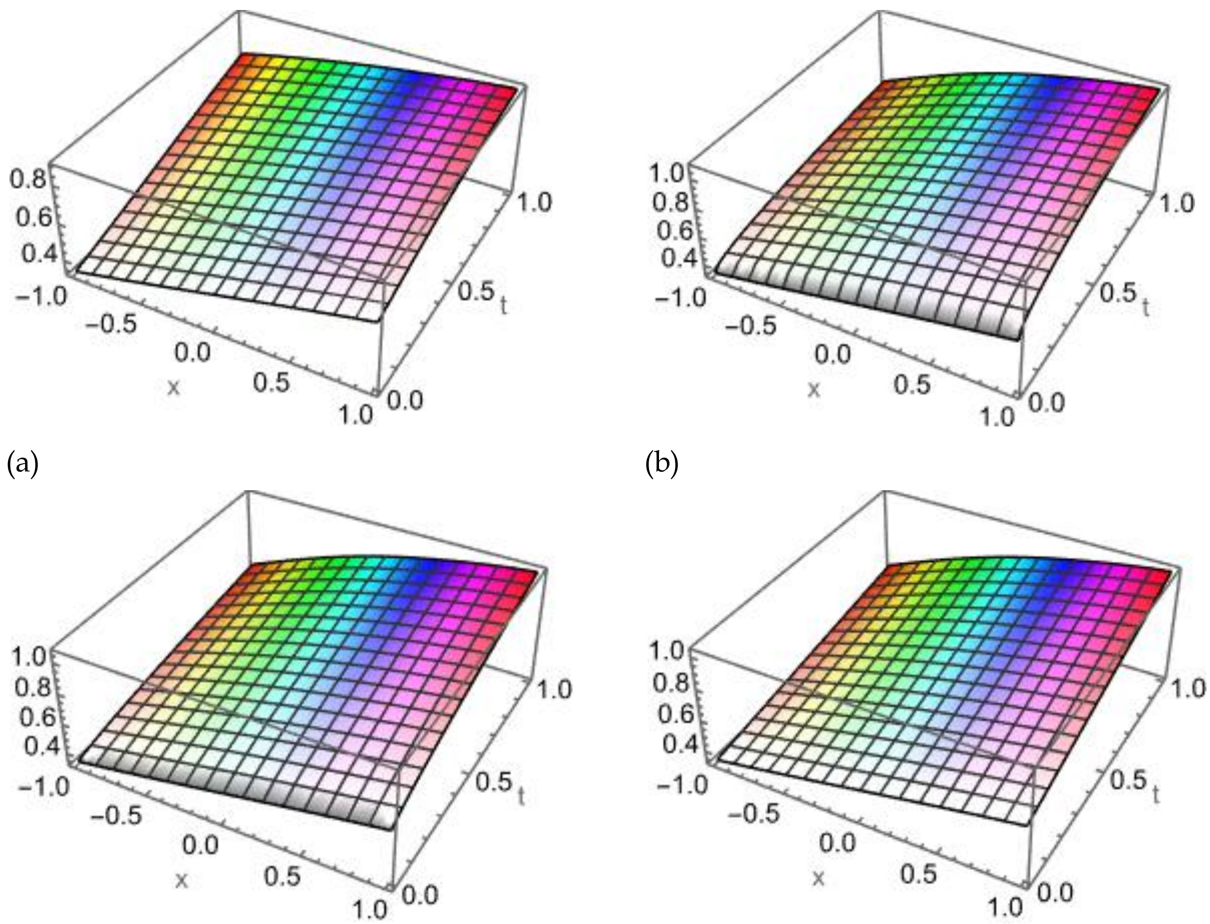
x	*OAFM Solution*	*Exact*	*Absolute Error*
-1	0.33106653448734974	0.3302417683970369	0.0008247660903128451
-0.8	0.3631982356076443	0.36223645069831173	0.0009617849093325748
-0.6	0.3965889222831253	0.39550043372307475	0.00108848856005056
-0.4	0.4309565766065897	0.42976066144224573	0.0011959151643439747
-0.2	0.46598487023482477	0.4647072002069661	0.0012776700278586683
0	0.5013353409277708	0.50000375	0.0013315909277707982
0.2	0.5366604359845695	0.5353002624176806	0.0013601735668888892
0.4	0.571615860787259	0.5702466905353214	0.0013691702519376125
0.6	0.6058711282300585	0.6045067386496589	0.0013643895803996386
0.8	0.6391179708849439	0.6377704799091787	0.0013474909757652531
1	0.6710769995136809	0.6697648670334542	0.0013121324802267686

Table 2: The absolute errors for differences between the exact solution of FN (Eq.19) and OAFM solution (Eq.24) for various values of x when q = -1, α = 0.75, For Example, 4.1

x	*OAFM Solution*	*Exact*	*Absolute Error*
-1	0.33028335355135935	0.3302417683970369	0.000041585154322454354
-0.8	0.36228532611929926	0.36223645069831173	0.0000488754209875264
-0.6	0.3955560653283021	0.39550043372307475	0.00005563160522736865
-0.4	0.4298220333281746	0.42976066144224573	0.00006137188592886256
-0.2	0.4647729529318231	0.4647072002069661	0.00006575272485698225
0	0.5000724089153483	0.50000375	0.00006865891534835189
0.2	0.5353704889039862	0.5353002624176806	0.00007022648630561346
0.4	0.5703174571926479	0.5702466905353214	0.00007076665732652643
0.6	0.60457733100941	0.6045067386496589	0.00007059235975115374
0.8	0.6378402702706287	0.6377704799091787	0.00006979036145005058
1	0.6698328796657907	0.6697648670334542	0.00006801263233657817

Table 3: The absolute errors for differences between the exact solution of FN (Eq.19) and OAFM solution (Eq.24) for various values of x when q = -1, α = 1, For Example, 4.1.

x	$OAFM\ Solution$	$Exact$	$Absolute\ Error$
-1	0.3302407713745951	0.3302417683970369	9.97022×10^{-7}
-0.8	0.36223569049690846	0.36223645069831173	7.60201×10^{-7}
-0.6	0.3954999080657829	0.39550043372307475	5.25657×10^{-7}
-0.4	0.4297603472985382	0.42976066144224573	3.14143×10^{-7}
-0.2	0.464707060014301	0.4647072002069661	1.40192×10^{-7}
0	0.5000037422870856	0.50000375	7.71291×10^{-9}
0.2	0.5353003534447612	0.5353002624176806	9.10270×10^{-8}
0.4	0.5702468619459605	0.5702466905353214	1.71410×10^{-7}
0.6	0.6045069862149762	0.6045067386496589	2.47565×10^{-7}
0.8	0.6377708006616147	0.6377704799091787	3.20752×10^{-7}
1	0.6697652358700288	0.6697648670334542	3.88365×10^{-7}



(a) (b) (c) (d)

Fig.1: (a) Exact solution when $q = -1$ For Example.4.1; (b) OAFM solution when $q = -1$ and $\alpha = 0.5$ For Example.4.1; (c) OAFM solution when $q = -1$ and $\alpha = 0.75$ For Example.4.1; (d) OAFM solution when $q = -1$ and $\alpha = 1$ For Example.4.1:

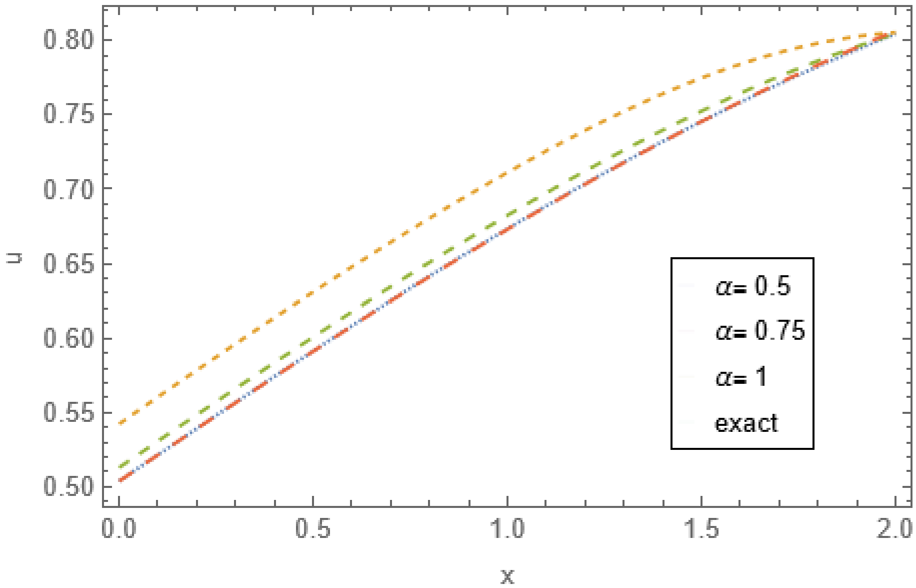


Fig. 2: visual graphs of $u(x,t)$ w.r.t x for various values of α For Example. 4.1

Example 4.2 Let us consider another fractional problem of Fitzhugh Nagumo equation having the different initial condition

$$D_t^{\alpha}u - u_{xx} - u(u-q)(1-u) = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{25}$$

with the corresponding initial condition as

$$u_0(x,t) = \frac{1}{1+e^{\frac{-x}{\sqrt{2}}}} \tag{26}$$

The solution in closed form for the fractional Fitzhugh Nagumo equation with the given initial condition for $\alpha\ = 1$ is given as

$$u(x,t) = \frac{1}{1+e^{\left(\frac{-1}{2}\right)\left(x+\left(\frac{1+2q}{\sqrt{2}}\right)t\right)}} \tag{27}$$

Applying the Optimal Axilary Function method on Eq. (25) with the given initial condition and simplifying, we obtained

$$u_1 = \frac{\left(c_4+c_3\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2 - e^{\frac{-x}{\sqrt{2}}}\left(1+e^{\frac{-x}{\sqrt{2}}}\right)\left(c_2+c_1\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2\right)\left(1-\left(1+e^{\frac{-x}{\sqrt{2}}}\right)q\right)\right)t^a}{a\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^8 \Gamma(a)} \tag{28}$$

$$u(x,t) = u_0(x,t) + u_1 \tag{29}$$

Putting Eq. (26) and Eq. (28) in Eq. (29), we obtained

$$u(x,t) = \frac{1}{1+e^{\frac{-x}{\sqrt{2}}}} + \frac{\left(c_4+c_3\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2 - e^{\frac{-x}{\sqrt{2}}}\left(1+e^{\frac{-x}{\sqrt{2}}}\right)\left(c_2+c_1\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2\right)\left(1-\left(1+e^{\frac{-x}{\sqrt{2}}}\right)q\right)\right)t^a}{a\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^8 \Gamma(a)} \tag{30}$$

To find the values of constants, i.e., $(c_1, c_2, \quad and\ c_4)$, we will find the residual of the above Equation, which is given below

$$R = \frac{1}{\Gamma(1-a)} \int_0^t (t-r)^{-a} \big(\partial_t u(x,t)\big) dr - \partial_{xx} u_0(x,t) - u_0(x,t)\big(u_0(x,t) - q\big)\big(1 - u_0(x,t)\big) \tag{31}$$

Solving Eq.31 and using $a = 0.999999999999999$, $q = -1$ we obtained constants i.e $(c_1, c_2, c_3 and\ c_4)$ as follows

$$c_1 = 12.829545830268264, c_2 = 79.89878841618389,$$
$$c_3 = 92.7267634616979\ and\ c_4 = -103.55941364237299$$

For given values of $c_1, c_2, c_3\ and\ c_4$ Eq.30 becomes

$$(UX,t) = \frac{1}{1+e^{\frac{-x}{\sqrt{2}}}} + \frac{\left(-103.55941+92.72676\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2 - e^{\frac{-x}{\sqrt{2}}}\left(1+e^{\frac{-x}{\sqrt{2}}}\right)\left(79.89878+12.82954\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^2\right)\left(1-\left(1+e^{\frac{-x}{\sqrt{2}}}\right)q\right)\right)t^a}{a\left(1+e^{\frac{-x}{\sqrt{2}}}\right)^8 \Gamma(a)} \tag{32}$$

Table 4: The absolute errors for differences between the exact solution of FN (Eq.27) and the OAFM solution (Eq.32) for various values of x when $q = -1$, $\alpha = 0.5$, For Example, 4.2:

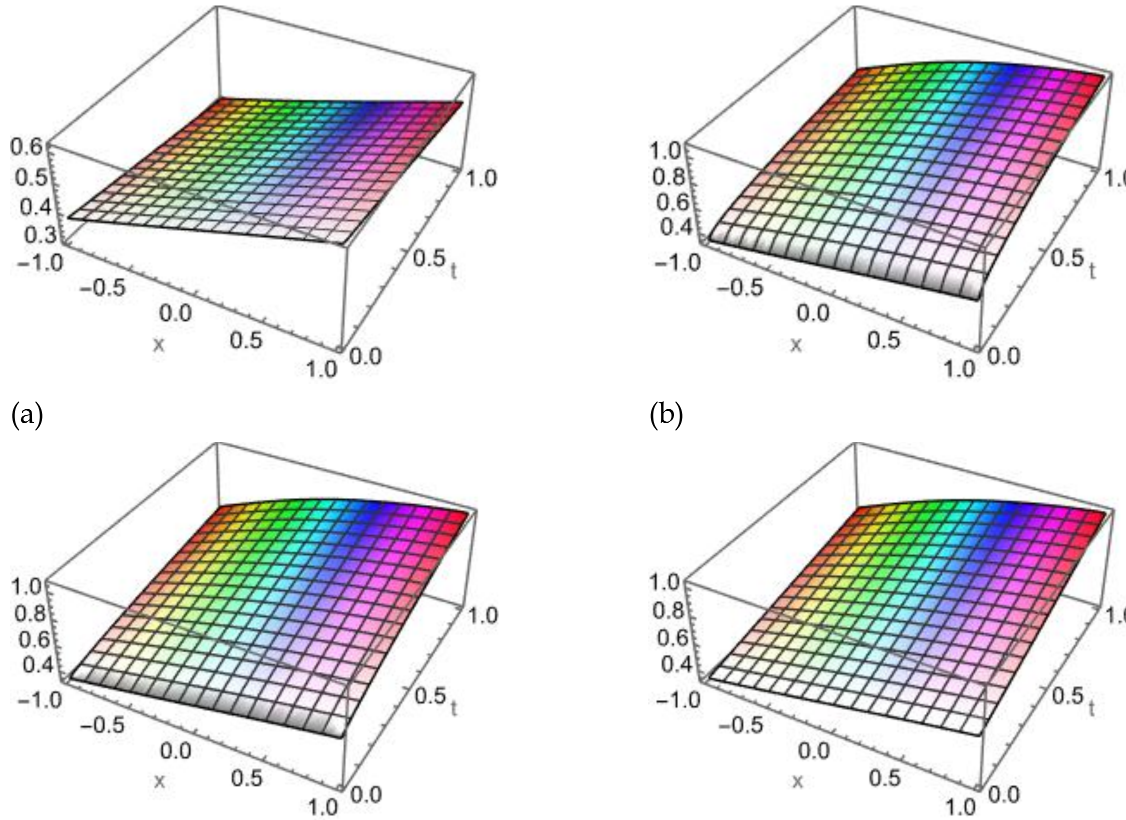
x	$OAFM\ Solution$	$Exact$	$Absolute\ Error$
-1	0.331063916240168	0.3775398379349126	0.0464759216947445
-0.8	0.363195027123445	0.4013114904378315	0.0381164633143865
-0.6	0.396585264418875	0.4255566188979189	0.0289713544790438
-0.4	0.430952780827058	0.4501651275844411	0.0192123467573827
-0.2	0.465981418743750	0.4750199308436924	0.0090385120999416
0	0.501332806699019	0.4999991161165236	0.0013336905824961
0.2	0.536659285753536	0.5249783058014168	0.0116809799521198
0.4	0.571616123026092	0.5498331222090886	0.0217830008170041
0.6	0.605871943750069	0.5744416525207819	0.0314302912292873
0.8	0.639117087538790	0.5986868106621428	0.0404302768766475
1	0.671070309245151	0.6224585003379023	0.0486118089072489

Table 5: The absolute errors for differences between the exact solution of FN (Eq.27) and the OAFM solution (Eq.32) for various values of x when q = -1, α = 0.75, For Example, 4.2:

x	$OAFM\ Solution$	$Exact$	$Absolute\ Error$
-1	0.330283211576800	0.3775398379349126	0.0472566263581116
-0.8	0.362285152139114	0.4013114904378315	0.0390263382987166
-0.6	0.395555866980466	0.4255566188979189	0.0300007519174524
-0.4	0.429821827501877	0.4501651275844411	0.0203433000825639
-0.2	0.464772765774581	0.4750199308436924	0.0102471650691112
0	0.500072271496692	0.4999991161165236	0.0000731553801691
0.2	0.535370426532662	0.5249783058014168	0.0103921207312455
0.4	0.570317471412559	0.5498331222090886	0.0204843492034705
0.6	0.604577375231015	0.5744416525207819	0.0301357227102334
0.8	0.637840222371148	0.5986868106621428	0.0391534117090054
1	0.669832516885711	0.6224585003379023	0.0473740165478091

Table 6: The absolute errors for differences between the exact solution of FN (Eq.27) and the OAFM solution (Eq.32) for various values of x when q = -1, α = 1, For Example, 4.2:

x	$OAFM\ Solution$	$Exact$	$Absolute\ Error$
-1	0.330240764036968	0.3775398379349126	0.0472990738979437
-0.8	0.362235681505145	0.4013114904378315	0.0390758089326858
-0.6	0.395499897814634	0.4255566188979189	0.0300567210832846
-0.4	0.429760336660882	0.4501651275844411	0.0204047909235584
-0.2	0.464707050341512	0.4750199308436924	0.0103128805021799
0	0.500003735184920	0.4999991161165236	0.0000046190683969
0.2	0.535300350221243	0.5249783058014168	0.0103220444198269
0.4	0.570246862680883	0.5498331222090886	0.0204137404717951
0.6	0.604506988500467	0.5744416525207819	0.0300653359796856
0.8	0.637770798186041	0.5986868106621428	0.0390839875238981
1	0.669765217120580	0.6224585003379023	0.0473067167826783



(a) (b)

(c) (d)

Fig.3: (a) Exact solution when $q = -1$ For Example.4.2; (b) OAFM solution when $q = -1$ and $\alpha = 0.5$ For Example.4.2; (c) OAFM solution when $q = -1$ and $\alpha = 0.75$ For Example.4.2; (d) OAFM solution when $q = -1$ and $\alpha = 1$ For Example.4.2:

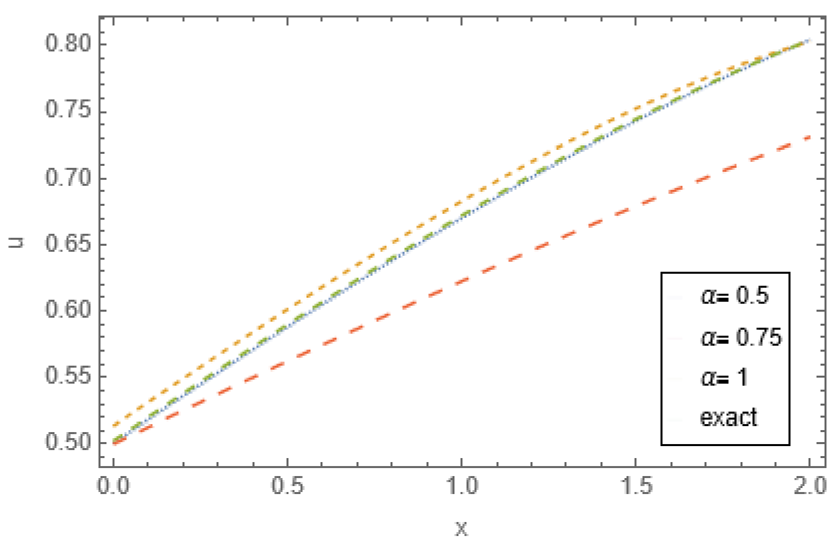


Fig. 4: visual graphs of $u(x, t)$ w.r.t x for various values of α For Example.4.2

5. Results Analysis

The OAFM has applied to the time fractional Fitzhug-Nagumo equation. We get highly accurate solutions using the proposed method, i.e., OAFM, on initial value problems. The calculations related to the examples discussed above were done by MATHEMATICA 13. The solutions obtained by our abovementioned technique are then correlated with the exact form of solution showing that the OAFM is close to exact solutions. Absolute errors of the technique for various values of "α" are then revealed in tables 1-3, and the Plots of $u(x,t)$ for various values of "α" are presented in Fig. 1 and Fig.2 for Example 4.1 and that for Example 4.2 absolute errors of the technique for various values of "α" are revealed in Tables 4-6 and Fig 3 and Fig.4. represents the plots of $u(x,t)$ for diverse values of "α" for Example 4.2. One thing is obvious when the values of "α" close to 1, then the absolute error decrease, and for $\alpha = 1$ when utilized in the OAFM, we obtain a very close outcome to the exact form solution.

6. Conclusion

This study makes use of the novel numerical approach OAFM. We solved the FNEs' equations that govern in the first-order series and did it with excellent precision. We evaluated the OAFM results with those found in the literature and the numerical results to determine the correctness and validity of the proposed method. The comparison leads to the conclusion that the suggested approach is quite accurate, and the strong correlation between our findings and the numerical outcomes supports the applicability of our approach. High nonlinear initial and boundary value problems can use OAFM, which is quite simple, even for the nonlinear initial and boundary value problems. As OAFM has auxiliary functions in which the optimal constants and the control convergence parameters exist to play a crucial part in obtaining convergent solutions that are gathered rigorously, OAFM is the optimal auxiliary constants via which we can control the convergence. Compared with other methods, OAFM requires less computational work, and even a computer with relatively small specifications can effortlessly execute the task. We may use this efficient and quick convergent method in our future work for more complicated models derived from real-world problems because it currently has no restrictions.

Acknowledgment: The author T. Alzkari extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2025-1795).

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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