Int. J. Anal. Appl. (2025), 23:117

International Journal of Analysis and Applications

Analytic Conformable Semigroups and Regularity of Solutions to Conformable Fractional Cauchy Problem

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ABSTRACT. We introduce an analytic conformable semigroup which is a solution operator of an evolution equation involving conformable fractional derivative and a sectorial linear operator. The evolution equation is called a conformable fractional Cauchy problem. We here also derive the properties of the analytic conformable semigroup by employing the properties of the analytic semigroup. The analytic conformable semigroup is then used to study the regularity of solutions to the conformable fractional Cauchy problem under Hölder continuity as a regularity condition. An example is given to show the applicability of our regularity results.

1. Introduction

Khalil et al [8] introduced a new fractional derivative T_t^α of a function $f:[0,\infty)\to\mathbb{R}$ of order $\alpha\in(0,1)$ defined by

$$T_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},\qquad t>0$$

which is called the conformable fractional derivative. It is a local operator and different from some fractional derivatives such as Riemann-Liouville and Caputo fractional derivatives which are nonlocal operator. The use of the conformable fractional derivative has some advantages compared to that of Riemann-Liouville or Caputo fractional derivatives. The first one is that the definition of the conformable fractional derivative is simpler than those of the fractional derivatives such as Riemann-Liouville and Caputo fractional derivatives ([1],[8]). The second one is that the conformable fractional derivative satisfies some properties of usual derivative

Received Apr. 1, 2025

2020 Mathematics Subject Classification. 35A01, 35A02, 35A09.

Key words and phrases. conformable fractional derivative; analytic conformable semigroup; conformable Cauchy problem; regularity.

https://doi.org/10.28924/2291-8639-23-2025-117

ISSN: 2291-8639

which are not satisfied by Riemann-Liouville and Caputo fractional derivatives such as the rule for the derivation of the product of two functions, chain rule, mean value theorem, and Rolle theorem ([1],[8]). The third one is that some real phenomena can be described by using the conformable fractional derivative as reported in ([3],[5],[6],[7],[10],[12],[19],[20],[21],[22]).

By employing the conformable fractional derivative T_t^α, we here study an analytic solution operator to the Cauchy problem in Banach space X

$$\begin{aligned} T_t^\alpha u(t) &= Au(t) + f(t), \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

with $\alpha \in (0,1)$, $A: D(A) \subseteq X \to X$ is a sectorial linear operator, $u_0 \in X$, and $f: (0,T] \to X$. A linear operator A is said to be sectorial if A satisfies the property that there exist constants $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\rho(A) \supset \Sigma_\theta = \{\lambda \in \mathbb{C}: \lambda \neq 0, |\arg(\lambda)| < \theta\},$$

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|}, \lambda \in \Sigma_\theta$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda; A) = (\lambda - A)^{-1}$ is the resolvent operator of A. We call the problem (1.1) conformable fractional Cauchy problem. The study of an analytic solution operator to the problem (1.1) is important and useful since it can be applied to investigate the existence, uniqueness, and regularity of the solutions to some real models associated with the problem 1.1, for instance, the conformable diffusion equation as studied in ([7],[21],[22]) of the form

$$T_t^\alpha u(t) = D_\alpha \Delta u(x,t) \tag{1.2}$$

with D_α is a constant associated with the equation (1.2). In [7], the multidimensional conformable diffusion equation was derived via the random walk theory. The survival function used in the diffusion process associated with the equation (1.2) is the stretched exponential function $e^{-t^\alpha/[\gamma^\alpha \Gamma(1+\alpha)]}$, $\gamma > 0$. The equation (1.2) can be used as an alternative model to describe slow diffusion phenomenon since the mean square displacement (MSD) of moving particles in the diffusion process is proportional to t^α with $0 < \alpha < 1$. This fact is similar to MSD of moving particles in slow diffusion process involving Mittag-Leffler function $E_{\alpha,1}(-t^\alpha/\gamma^\alpha)$, $\gamma > 0$ as a survival function where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \alpha > 0, z \in \mathbb{C}.$$

Such a process is modeled by the diffusion equation as derived in ([11],[13]) of the form

$${}^C D_t^\alpha u(x,t) = K_\alpha \Delta u(x,t) \tag{1.3}$$

where D_t^α is Caputo fractional derivative with $0 < \alpha < 1$ as defined in ([9],[15]) by

$${}^C D_t^\alpha u(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{d\tau} f(\tau) d\tau$$

and K_α is a constant associated with the equation (1.3).

There have been few existence, uniqueness, and regularity results of the problem (1.1). In ([2],[18]), a solution operator to the homogeneous case of the problem (1.1) called a C_0 α-semigroup was discussed. Furthermore, in ([16],[17]), the existence, uniqueness, and regularity of the mild solutions to the problem (1.1) were discussed by employing the C_0 α-semigroup. In [4], the C_0 α-semigroup is used to investigate the existence of a mild solution to the problem (1.1) with finite delay and nonlocal initial conditions. Meanwhile, as our new results, here:

1. we introduce an analytic conformable semigroup $S_\alpha(t)$ defined by

$$S_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda \frac{t^\alpha}{\alpha}} R(\lambda; A) d\lambda , \quad t > 0. \tag{1.4}$$

and derive its properties. If α =1, the operator (1.4) is an analytic semigroup. The analytic conformable semigroup (1.4) has the property

$$S_\alpha\left(t^{\frac{1}{\alpha}}\right) S_\alpha\left(s^{\frac{1}{\alpha}}\right) = S_\alpha\left((t+s)^{\frac{1}{\alpha}}\right) \tag{1.5}$$

which is called by conformable semigroup law. If α =1, the property (1.5) is the semigroup law.

2. we obtain the stronger regularity results for the problem (1.1) than those in ([16],[17]) under the Hölder continuity of f as our regularity condition by employing the analytic conformable semigroup $S_\alpha(t)$ as expressed in (1.4). The C_0 α-semigroup as used in ([16],[17]) to obtain their regularity results can not be used to obtain our regularity results under the Hölder continuity of f.

This paper consists of six sections. The research's motivation and novelty are mentioned in the first section. In second section, we briefly provide concerning conformable fractional derivative. Preliminaries results are presented in third section. Meanwhile, main results are discussed in fourth section. An example to show the applicability of the main results is given in fifth section. The last section contains conclusion.

2. Conformable Fractional Derivative

In this section, we provide briefly regarding the conformable fractional derivative and another mathematical notions associated with it.

Definition 2.1. [8] The conformable fractional derivative of $f: [0, \infty) \to \mathbb{R}$ of order $\alpha \in (0,1)$ is defined by

$$T_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for $t > 0$. If the limit exists then f is said to be α-differentiable at t. If f is α-differentiable in $(0, a)$ with $a > 0$ and $\lim_{t \to 0^+} T_t^\alpha f(t)$ exists then the conformable fractional derivative of f of order

α at $t = 0$ is defined by

$$T_t^{\alpha} f(0) = \lim_{t \to 0^+} T_t^{\alpha} f(t).$$

Theorem 2.1. [8] If $f: [0, \infty) \to \mathbb{R}$ is a differentiable function at $t > 0$ then, for $\alpha \in (0,1]$,

$$T_t^{\alpha} f(t) = t^{1-\alpha} \frac{d}{dt} f(t).$$

If $\alpha = 1$ then the conformable fractional derivative is usual derivative

Definition 2.2. [1] Given a function $f: [0, \infty) \to \mathbb{R}$ and $\alpha \in (0,1]$. The fractional Laplace transform of order α of f is defined by

$$\mathcal{L}_{\alpha}\{f(t)\}(s) = \tilde{f}_{\alpha}(s) = \int_0^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}} t^{\alpha-1} f(t)\, dt.$$

If $\alpha = 1$ then the fractional Laplace transform is usual Laplace transform, that is

$$\mathcal{L}_1\{f(t)\}(s) = \mathcal{L}\{f(t)\}(s) = \tilde{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

Theorem 2.2. [1] If $f: [0, \infty) \to \mathbb{R}$ is a differentiable function in $(0, \infty)$ and $\alpha \in (0,1]$ then

$$\mathcal{L}_{\alpha}\{T_t^{\alpha} f(s)\}(s) = s\tilde{f}_{\alpha}(s) - f(0).$$

We next define α-convolution of functions f and g as written in the following definition.

Definition 2.3. If f and g are piecewise continuous functions on $[0, \infty)$ then α-convolution $f *_{\alpha} g$ of f and g is defined by

$$(f *_{\alpha} g)(t) = \int_0^{t} f\left((t^{\alpha} - \tau^{\alpha})^{\frac{1}{\alpha}}\right) g(\tau) \tau^{\alpha-1}\, d\tau.$$

It is not difficult to show the following theorem.

Theorem 2.3. If f and g are piecewise continuous functions on $[0, \infty)$ then the fractional Laplace transform of the convolution $f *_{\alpha} g$ of f and g is given by

$$\mathcal{L}_{\alpha}\{(f *_{\alpha} g)(t)\}(s) = \tilde{f}_{\alpha}(s) \tilde{g}_{\alpha}(s).$$

Next, consider that, since

$$F_{\alpha}(s) = \mathcal{L}_{\alpha}\{f(t)\}(s) = \int_0^{\infty} e^{-s\frac{t^{\alpha}}{\alpha}} t^{\alpha-1} f(t) dt = \int_0^{\infty} e^{-s\mu} f\left((\alpha\mu)^{1/\alpha}\right) d\mu$$

then

$$g(\mu) = f\left((\alpha\mu)^{1/\alpha}\right) = \frac{1}{2\pi i} \int_{\Gamma} e^{s\mu} F_{\alpha}(s)\, ds$$

implying

$$f(\mu) = g\left(\frac{\mu^{\alpha}}{\alpha}\right) = \frac{1}{2\pi i} \int_{\Gamma} e^{s\frac{\mu^{\alpha}}{\alpha}} F_{\alpha}(s)\, ds \tag{2.1}$$

where Γ is the vertical line $\mathrm{Re}(s) = c$ such that c is greater than all real part of singularities of the integrand in the integral (2.1). Motivated by (2.1), we have a definition of the inverse of fractional Laplace transform as defined in the following definition.

Definition 2.4. Let $0 < \alpha < 1$ and $F_\alpha(s)$ be the fractional Laplace transform of order α of a function $f: [0, \infty) \to \mathbb{R}$. The inverse of the fractional Laplace transform of order α of $F_\alpha(s)$ is defined by

$$f(t) = \mathcal{L}_\alpha^{-1}\{F_\alpha(s)\}(t) = \frac{1}{2\pi i}\int_\Gamma e^{s\frac{t^\alpha}{\alpha}} F_\alpha(s)\, ds. \tag{2.2}$$

3. Preliminaries Results

3.1 Analytic Conformable Semigroup

Observe that the fractional Laplace transform of the problem (1.1) is

$$sU_\alpha(s) - u(0) = AU_\alpha(s) + F_\alpha(s).$$

It follows that

$$U_\alpha(s) = R(s; A)u_0 + R(s; A)F_\alpha(s). \tag{3.1}$$

Then, by applying the inverse of the fractional Laplace transform to the equation (3.1), we have

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{s\frac{t^\alpha}{\alpha}} R(s; A)\, u_0 ds + \frac{1}{2\pi i}\int_\Gamma e^{s\frac{t^\alpha}{\alpha}} R(s; A)\, F_\alpha(s) ds \tag{3.2}$$

is a solution to the problem (1.1). Note that

$$u(t) = \frac{1}{2\pi i}\int_\Gamma e^{s\frac{t^\alpha}{\alpha}} R(s; A)\, u_0 ds \tag{3.3}$$

is a solution to the homogeneous case of the problem (1.1). Motivated by the solution (3.3), we define an operator

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda\frac{t^\alpha}{\alpha}} R(\lambda; A)\, d\lambda, \quad t > 0 \tag{3.4}$$

where

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C}: |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C}: |\arg(\lambda)| \leq \omega, |\lambda| = r\},$$

with $r > 0$, $\omega \in (\pi/2, \theta)$, and $\Gamma_{r,\omega}$ is oriented counterclockwise. By Cauchy's theorem, the integral (3.4) is independent of $r > 0$ and $\omega \in (\pi/2, \theta)$.

We next give some properties of $S_\alpha(t)$ defined by the operator (3.4). The properties are derived by employing the properties of an analytic semigroup operator generated by the sectorial linear operator A i.e.

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{st} R(s; A)\, ds, \quad t > 0. \tag{3.5}$$

The analytic semigroup operator $S(t)$ defined by the operator (3.5) has some properties as mentioned by the following theorems. Let $B(X)$ be a set of all bounded linear operator on X and $B\big(X{:}\,D(A)\big)$ be a set all bounded linear operators from X into $D(A)$.

Theorem 3.1. [14] Let $A{:}\,D(A) \subseteq X \to X$ be a sectorial linear operator. If $S(t)$ is an analytic semigroup generated by A expressed by the operator (3.5) then

(i) $S(t) \in B(X)$ and there exists $C_1 > 0$ such that

$$\|S(t)\| \leq C_1, \qquad t > 0;$$

(ii) $S(t) \in B\big(X{:}\,D(A)\big)$ for $t > 0$ and if $x \in D(A)$ then $AS(t)x = S(t)Ax$. Moreover, there exists $M > 0$ such that

$$\|AS(t)\| \leq Mt^{-1}, \qquad t > 0;$$

(iii) the function $t \mapsto S(t)$ is infinitely continuously differentiable on $(0, \infty)$,

$$\frac{d^n}{dt^n} S(t)(t) = A^n S(t),$$

and there exists $M_n > 0$ such that

$$\left\| \frac{d^n}{dt^n} S(t) \right\| \leq M_n t^{-n}, \qquad t > 0$$

for $n = 1,2,3, \ldots.$ Moreover, the operator $S(t)$ has an analytic continuation to the sector $\Sigma_{\theta - \frac{\pi}{2}}$ and, for $z \in \Sigma_{\theta - \frac{\pi}{2}}$, $\eta \in \left(\frac{\pi}{2}, \theta\right)$,

$$S(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda; A)\ d\lambda;$$

(iv) for $t > 0$ and $x \in X$,

$$\frac{d}{dt} S(t)x = AS(t)x;$$

(v) for $s, t > 0$,

$$S(t)S(s) = S(t + s).$$

Theorem 3.2. [14] Let $A{:}\,D(A) \subseteq X \to X$ be a sectorial linear operator. If $S(t)$ is an analytic semigroup generated by A expressed by the operator (3.5) then the following statements hold.

(i) If $x \in \overline{D(A)}$ then $\lim\limits_{t \to 0^+} S(t)x = x$;

(ii) For $x \in X$ and $t \geq 0$,

$$\int_0^t S(\tau)x\, d\tau \in D(A),$$

$$A \int_0^t S(\tau)x\, d\tau = S(t)x - x.$$

Moreover, if $\tau \mapsto AS(\tau)x$ is integrable on $(0, \varepsilon)$ for some $\varepsilon > 0$ then, for $t \geq 0$,

$$S(t)x - x = \int_0^t AS(\tau)x\,d\tau;$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$ then

$$\lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax;$$

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$ then

$$\lim_{t\to 0^+} AS(t)x = Ax.$$

Theorem 3.3. [14] Let $A: D(A) \subseteq X \to X$ be a sectorial linear operator. If $S(t)$ is an analytic semigroup generated by A expressed by the operator (3.5) then, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,

$$R(\lambda: A) = \int_0^\infty e^{-\lambda t} S(t)\,dt.$$

By the relationship $S_\alpha(t) = S(t^\alpha/\alpha)$ and employing Theorem 3.1-3.3, we derive some properties of $S_\alpha(t)$ defined by the operator (3.4) as stated by the following theorems.

Theorem 3.4. Let $A: D(A) \subseteq X \to X$ be a sectorial linear operator. If $S_\alpha(t)$ is an operator defined by the operator (3.4) then

(i) $S_\alpha(t) \in B(X)$ for $t > 0$ and there exists $C_1 > 0$ such that $\|S_\alpha(t)\| \le C_1$;

(ii) $S_\alpha(t) \in B\big(X: D(A)\big)$ for $t > 0$ and if $x \in D(A)$ then $AS_\alpha(t)x = S_\alpha(t)Ax$. Moreover, there exists $M = M(\alpha) > 0$ such that

$$\|AS_\alpha(t)x\| \le \frac{M}{t^\alpha},\ t > 0;$$

(iii) The function $t \mapsto S_\alpha(t)$ is differentiable in $(0, \infty)$ and

$$D_t S_\alpha(t) = t^{\alpha-1} AS_\alpha(t)$$

and there exists $N = N(\alpha) > 0$ such that

$$\|D_t S_\alpha(t)\| \le \frac{N}{t},\ t > 0;$$

Moreover, the operator $S_\alpha(t)$ has an analytic continuation $S_\alpha(z)$ to the sector $\Sigma_{\theta-\frac{\pi}{2}}$ and for $z \in \Sigma_{\theta-\frac{\pi}{2}}$, $\eta \in (\pi/2, \theta)$,

$$S_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda\frac{z^\alpha}{\alpha}} R(\lambda; A)\ d\lambda;$$

(iv) for $t > 0$ and $x \in X$,

$$T_t^\alpha S_\alpha(t)x = AS_\alpha(t)x;$$

(v) for $t, s > 0$,

$$S_\alpha\big(t^{1/\alpha}\big) S_\alpha\big(s^{1/\alpha}\big) = S_\alpha\big((t+s)^{1/\alpha}\big).$$

Proof.

(i) By Theorem 3.1(i), for $t > 0$,

$$\|S_\alpha(t)\| = \left\|S\left(\frac{t^\alpha}{\alpha}\right)\right\| \le C_1;$$

(ii) By Theorem 3.1(ii), if $x \in D(A)$ then, for $t > 0$,

$$AS_\alpha(t)x = AS\left(\frac{t^\alpha}{\alpha}\right)x = S\left(\frac{t^\alpha}{\alpha}\right)Ax = S_\alpha(t)Ax$$

and

$$\|AS_\alpha(t)\| = \left\|AS\left(\frac{t^\alpha}{\alpha}\right)\right\| \le M\left(\frac{t^\alpha}{\alpha}\right)^{-1} = \frac{\alpha M}{t^\alpha}.$$

(iii) By theorem 3.1(iii), for $t > 0$,

$$S'_\alpha(t) = \frac{d}{dt}S\left(\frac{t^\alpha}{\alpha}\right) = t^{\alpha-1}S'\left(\frac{t^\alpha}{\alpha}\right) = t^{\alpha-1}AS\left(\frac{t^\alpha}{\alpha}\right) = t^{\alpha-1}AS_\alpha(t)$$

and

$$\|S'_\alpha(t)\| = \|t^{\alpha-1}AS_\alpha(t)\| = t^{\alpha-1}\|AS_\alpha(t)\| \le t^{\alpha-1}Mt^{-\alpha} = \frac{M}{t}.$$

Now, we suppose $0 < \delta < \theta - \pi/2$ and $\eta = \theta - \delta$. If $z \in \Sigma_{\eta-\pi/2}$ with $\lambda = |\lambda|e^{\pm\eta i}$ and $|\lambda| \ge r$ then $\lambda z^\alpha = |\lambda||z|^\alpha e^{i(\alpha\arg(z)\pm\eta)}$ with $\pi/2 < \alpha\arg(z) + \eta < 3\pi/2$ and $-3\pi/2 < \alpha\arg(z) - \eta < -\pi/2$. It means that $\mathrm{Re}(\lambda z^\alpha) < 0$ implying

$$\begin{aligned}\|S_\alpha(z)\| &= \left\|\frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda\frac{z^\alpha}{\alpha}}R(\lambda;A)\,d\lambda\right\| \\ &= \left\|\frac{1}{2\pi i}\int_{\Gamma_{|z|^{-\alpha},\eta}} e^{\lambda\frac{z^\alpha}{\alpha}}R(\lambda;A)\,d\lambda\right\| \\ &\le \frac{1}{2\pi}\int_{\Gamma_{|z|^{-\alpha},\eta}} \frac{e^{|\lambda|\frac{|z|^\alpha}{\alpha}\cos(\alpha\arg(z)\pm\eta)}}{|\lambda|}\,d|\lambda| \\ &\le \frac{1}{2\pi}\int_{\Gamma_{1,\eta}} e^{\frac{\rho}{\alpha}\cos(\alpha\arg(z)\pm\eta)}\rho^{-1}\,d\rho\end{aligned}$$

which is bounded. Therefore, for $z \in \Sigma_{\eta-\pi/2}$, the function $z \mapsto S_\alpha(z)$ is bounded. Now, observe that the value of z^α is not a unique with

$$z^\alpha = e^{\alpha\log z} = e^{\alpha\left(\ln|z|+i(\mathrm{Arg}(z)+2k\pi)\right)} = |z|^\alpha e^{i\alpha(\mathrm{Arg}(z)+2k\pi)}, \qquad k = 0, \pm1, \pm2, \ldots.$$

If $k = 0$, we have the principle value of z^α, that is $z^\alpha = |z|^\alpha e^{i\alpha\mathrm{Arg}(z)}$. It is a unique. Thus, for $z \in \Sigma_{\theta-\frac{\pi}{2}}$, $\eta \in (\pi/2, \theta)$,

$$z \mapsto S_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda\frac{z^\alpha}{\alpha}}R(\lambda;A)\,d\lambda,$$

with the principle value of z^α, can be considered as an analytic continuation of $S_\alpha(t)$ to the sector $\Sigma_{\theta-\frac{\pi}{2}}$. Since union of the sector $\Sigma_{\eta-\pi/2}$ is $\Sigma_{\theta-\pi/2}$, the function ia also analytic on $\Sigma_{\theta-\pi/2}$;

(iv) In proof of (iii), we proved that for $t > 0$ and $x \in X$,

$$S'_\alpha(t)x = t^{\alpha-1}AS_\alpha(t)x$$

implying

$$T_t^\alpha S_\alpha(t)x = t^{1-\alpha}S'_\alpha(t)x = AS_\alpha(t)x;$$

(v) By Theorem 3.1(v), for $t, s > 0$,

$$S_\alpha\big(t^{1/\alpha}\big)S_\alpha\big(s^{1/\alpha}\big)x = S\left(\frac{t}{\alpha}\right)S\left(\frac{s}{\alpha}\right)x = S\left(\frac{t+s}{\alpha}\right) = S_\alpha\big((t+s)^{1/\alpha}\big). \qquad \blacksquare$$

Theorem 3.5. Let $A: D(A) \subseteq X \to X$ be a sectorial linear operator. If $S_\alpha(t)$ is an operator defined by the operator (3.4) then the following statements hold.

(i) If $x \in \overline{D(A)}$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$;

(ii) For $x \in X$ and $t \geq 0$,

$$\int_0^t \tau^{\alpha-1}S_\alpha(\tau)x\,d\tau \in D(A), \quad A\int_0^t \tau^{\alpha-1}S_\alpha(\tau)x\,d\tau = S_\alpha(t)x - x;$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$ then

$$\lim_{t\to 0^+}\frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\alpha}Ax.$$

Proof.

(i) By Theorem 3.2(i), if $x \in \overline{D(A)}$ then

$$\lim_{t\to 0^+} S_\alpha(t)x = \lim_{t\to 0^+} S\left(\frac{t^\alpha}{\alpha}\right)x = \lim_{t\to 0^+} S(t)x = x;$$

(ii) By Theorem 3.2(ii), for $t \geq 0$ and $x \in X$,

$$\int_0^t \tau^{\alpha-1}S_\alpha(\tau)x\,d\tau = \int_0^t \tau^{\alpha-1}S\left(\frac{\tau^\alpha}{\alpha}\right)x\,d\tau = \int_0^{\frac{t^\alpha}{\alpha}} S(r)x\,dr \in D(A),$$

$$A\int_0^t \tau^{\alpha-1}S_\alpha(\tau)x\,d\tau = A\int_0^{\frac{t^\alpha}{\alpha}} S(r)x\,dr = S\left(\frac{t^\alpha}{\alpha}\right)x - x = S_\alpha(t)x - x.$$

Furthermore, if $\tau \mapsto \tau^{\alpha-1}AS_\alpha(\tau)x$ is integrable on $(0,\varepsilon)$ for some $\varepsilon > 0$ then, for $t \geq 0$,

$$S_\alpha(t)x - x = \int_0^t \tau^{\alpha-1}AS_\alpha(\tau)x\,d\tau;$$

(iii) By Theorem 3.2(iii), if $x \in D(A)$ and $Ax \in \overline{D(A)}$ then

$$\lim_{t\to 0^+}\frac{S_\alpha(t)x-x}{t^\alpha}=\frac{1}{\alpha}\lim_{t\to 0^+}\frac{S\left(\frac{t^\alpha}{\alpha}\right)x-x}{\frac{t^\alpha}{\alpha}}=\frac{1}{\alpha}\lim_{t\to 0^+}\frac{S(t)x-x}{t}=\frac{1}{\alpha}Ax. \qquad \blacksquare$$

Theorem 3.6. Let $A: D(A) \subseteq X \to X$ be a sectorial linear operator. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,

$$R(\lambda : A)=\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}}t^{\alpha-1}S_\alpha(t)\,dt.$$

Proof. By Theorem 3.3, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,

$$\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}}t^{\alpha-1}S_\alpha(t)\,dt=\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}}t^{\alpha-1}S\left(\frac{t^\alpha}{\alpha}\right)dt=\int_0^\infty e^{-\lambda\tau}S(\tau)\,d\tau=R(\lambda : A). \qquad \blacksquare$$

We call the operator $S_\alpha(t)$ defined by (3.4) the analytic conformable semigroup generated by the sectorial linear operator $A: D(A) \subseteq X \to X$. The analytic conformable semigroup is a solution operator to the homogeneous case of the problem (1.1) i.e.

$$\begin{aligned} T_t^\alpha u(t) &= Au(t), \;\; t>0 \\ u(0) &= u_0. \end{aligned}$$

3.2 A Solution to Conformable Fractional Cauchy Problem

We define Banach space $L^{\alpha,p}\big((0,T];X\big)$ by

$$L^{\alpha,p}\big((0,T];X\big)=\left\{f:(0,T]\to X:\int_0^T \|f(t)\|^p t^{\alpha-1}dt<+\infty\right\}, 0<\alpha<1, p\geq 1$$

with its norm

$$\|f\|_{\alpha,p}=\int_0^T \|f(t)\|^p t^{\alpha-1}dt.$$

Motivated by the solution (3.2), we get a solution to the problem (1.1) as stated by the following theorem.

Theorem 3.9. Let $u_0 \in X$ and $f \in L^{\alpha,1}\big((0,T];X\big)$. If $u: [0,T] \to X$ is a solution to the problem (1.1) then

$$u(t)=S_\alpha(t)u_0+\int_0^t S_\alpha\big((t^\alpha-\tau^\alpha)^{1/\alpha}\big)f(\tau)\tau^{\alpha-1}\,d\tau,\; 0<t\leq T.$$

Proof. Since $S_\alpha(t)$ is the analytic conformable semigroup generated by A and u is a solution to the problem (1.1), then

$$v(\tau)=S_\alpha\big((t^\alpha-\tau^\alpha)^{1/\alpha}\big)u(\tau)=S\left(\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)u(\tau)$$

is differentiable for $0 < \tau < t$ and by, Theorem 3.1(ii-iv),

$$v'(\tau)=-\tau^{\alpha-1}S'\left(\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)u(\tau)+S\left(\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)u'(\tau)$$

$$= -\tau^{\alpha-1} A S\left(\frac{t^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right) u(\tau) + S\left(\frac{t^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right)\left(\tau^{\alpha-1} A u(\tau) + \tau^{\alpha-1} f(\tau)\right)$$

$$= S\left(\frac{t^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right) f(\tau)\tau^{\alpha-1}. \quad (3.6)$$

If $f \in L^{\alpha,1}\big((0,T];X\big)$ then $S(t^\alpha/\alpha - \tau^\alpha/\alpha) f(\tau)\tau^{\alpha-1}$ is integrable. By integrating both side of the equation (3.6) from 0 to t, we have

$$v(t) - v(0) = \int_0^t S\left(\frac{t^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right) f(\tau)\tau^{\alpha-1}\, d\tau$$

or

$$u(t) = S\left(\frac{t^\alpha}{\alpha}\right) u(0) + \int_0^t S\left(\frac{t^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right) f(\tau)\tau^{\alpha-1}\, d\tau.$$

It means that

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha\left((t^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right) f(\tau)\tau^{\alpha-1}\, d\tau,\ 0 < t \le T. \quad \blacksquare$$

Remark 3.10. If $f = 0$ then $t \mapsto u(t) = S_\alpha(t)u_0$ is the unique solution to the homogeneous case of the problem (1.1). Theorem 3.9 also implies the uniqueness of the solution to the problem (1.1).

4. Main Results

In this section, we show the regularity of mild solution to the problem (1.1). We first define two types of solutions to the problem (1.1)

Definition 4.1. Let $u_0 \in X$ and $f \in L^{\alpha,1}\big((0,T];X\big)$. A function $u: [0,T] \to X$ is a mild solution to the problem (1.1) if it satisfies

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha\big((t^\alpha - \tau^\alpha)^{1/\alpha}\big) f(\tau)\tau^{\alpha-1}\, d\tau,\ 0 < t \le T.$$

Definition 4.2. Let $u_0 \in X$. A function $u: [0,T] \to X$ is a classical solution to the problem (1.1) if $u \in C\big((0,T];D(A)\big) \cap C([0,T];X)$, $T_t^\alpha u \in C\big((0,T];X\,\big)$, and it satisfies the the problem (1.1).

The following theorem shows us regarding the Hölder continuity of a mild solution to the problem (1.1).

Theorem 4.1. Let A be a sectorial linear operator, $S_\alpha(t)$ be an analytic conformable semigroup generated by A, and $f \in L^{\alpha,p}\big((0,T];X\big)$ with $p > 1$.

(i) If u is a mild solution to the problem (1.1) then u is Hölder continuous with exponent $\alpha(p-1)/p$ on $[\varepsilon,T]$ for every $\varepsilon > 0$.

(ii) If $u_0 \in D(A)$ then u is Hölder continuous with the same exponent on $[0,T]$.

Proof. Consider the mild solution u to to the problem (1.1) i.e.

$$u(t)=S_\alpha(t)u_0+\int_0^t S_\alpha\left((t^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)f(\tau)\tau^{\alpha-1}\,d\tau=S_\alpha(t)u_0+v(t),\qquad t>0.$$

Note that based on Theorem 3.5(ii), for $t\in[\varepsilon,T]$ and $h>0$,

$$\begin{aligned}\|S_\alpha(t+h)u_0-S_\alpha(t)u_0\| &\le \int_t^{t+h}\|AS_\alpha(\tau)u_0\|\tau^{\alpha-1}d\tau\\ &\le M(\alpha)\|u_0\|\int_t^{t+h}\tau^{-1}d\tau\\ &=\frac{M(\alpha)\|u_0\|}{t}h\\ &\le\frac{M(\alpha)\|u_0\|}{\varepsilon}h.\end{aligned}$$

It means that $S_\alpha(t)u_0$ is Lipschitz continuous on $[\varepsilon,T]$. We now consider, for $t\in[0,T]$ and $h>0$,

$$\begin{aligned}v(t+h)-v(t)&=\int_0^{t+h}S_\alpha\left(((t+h)^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)f(\tau)\tau^{\alpha-1}\,d\tau-\int_0^t S_\alpha\left((t^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)f(\tau)\tau^{\alpha-1}\,d\tau\\ &=\int_0^t\left[S_\alpha\left(((t+h)^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)-S_\alpha\left((t^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)\right]f(\tau)\tau^{\alpha-1}\,d\tau\\ &\quad+\int_t^{t+h}S_\alpha\left(((t+h)^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)f(\tau)\tau^{\alpha-1}\,d\tau\\ &=I_1+I_2.\end{aligned}$$

Therefore, by using Hölder's inequality, we have, for $t\in[0,T]$ and $h>0$,

$$\begin{aligned}\|I_2\|&\le\int_t^{t+h}\left\|S_\alpha\left(((t+h)^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}\right)\right\|\|f(\tau)\tau^{\alpha-1}\|\,\tau^{\alpha-1}d\tau\\ &=\frac{1}{\alpha}\int_{t^\alpha}^{(t+h)^\alpha}\left\|S_\alpha\left(((t+h)^\alpha-r)^{\frac{1}{\alpha}}\right)\right\|\left\|f\left(r^{1/\alpha}\right)\right\|dr\\ &\le\frac{1}{\alpha}\left(\int_{t^\alpha}^{(t+h)^\alpha}C_1^{p/(p-1)}dr\right)^{(p-1)/p}\left(\int_{t^\alpha}^{(t+h)^\alpha}\left\|f\left(r^{1/\alpha}\right)\right\|^p dr\right)^{1/p}\\ &=\frac{C_1}{\alpha}((t+h)^\alpha-t^\alpha)^{(p-1)/p}\left(\int_t^{t+h}\|f(s)\|^p s^{\alpha-1}dr\right)^{\frac{1}{p}}\\ &\le\frac{C_1}{\alpha}h^{\alpha(p-1)/p}\|f\|_{\alpha,p}.\end{aligned}$$

We next estimate I_1. Based on Theorem 3.4(i), we observe that for $h > 0$, $t \in [0,T]$, and $0 < \tau < t$,

$$\left\| S_\alpha\left(((t+h)^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right) - S_\alpha\left((t^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right)\right\| \le 2C_1.$$

By Theorem 3.2(ii) and Theorem 3.5(ii), for $h > 0$, $t \in (0,T]$, and $0 < \tau < t$,

$$\begin{aligned}\left\| S_\alpha\left(((t+h)^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right) - S_\alpha\left((t^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right)\right\| &\le \left\| \int_{(t^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}}^{((t+h)^\alpha-\tau^\alpha)^{\frac{1}{\alpha}}} r^{\alpha-1} A S_\alpha(r)\, dr \right\| \\ &= \left\| \int_{\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}} AS(r)\, dr \right\| \\ &\le M \int_{\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}} \|AS(r)\|\, dr \\ &\le \frac{M}{\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}}\left(\frac{(t+h)^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}\right) \\ &\le \frac{M}{t^\alpha - \tau^\alpha} h^\alpha.\end{aligned}$$

Consequently, for $t \in [0,T]$ and $h > 0$,

$$\left\| S_\alpha\left(((t+h)^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right) - S_\alpha\left((t^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right)\right\| \le C\min\left\{1, \frac{h^\alpha}{t^\alpha - \tau^\alpha}\right\}$$

where $C = \max\{2C_1, M\}$. Therefore, for $t \in [0,T]$ and $h > 0$,

$$\begin{aligned}\|I_1\| &\le C\int_0^t \min\left\{1, \frac{h^\alpha}{t^\alpha - \tau^\alpha}\right\} \|f(\tau)\| \tau^{\alpha-1} d\tau \\ &= C\int_0^{t^\alpha} \min\left\{1, \frac{h^\alpha}{r}\right\} \left\|f\left((t^\alpha - r)^{1/\alpha}\right)\right\| dr \\ &\le C\left(\int_0^{t^\alpha} \left(\min\left\{1, \frac{h^\alpha}{r}\right\}\right)^{\frac{p}{(p-1)}} dr\right)^{\frac{(p-1)}{p}} \left(\int_0^{t^\alpha} \left\|f\left((t^\alpha - r)^{\frac{1}{\alpha}}\right)\right\|^p dr\right)^{\frac{1}{p}} \\ &= \alpha C\left(\int_0^{t^\alpha} \left(\min\left\{1, \frac{h^\alpha}{r}\right\}\right)^{\frac{p}{(p-1)}} dr\right)^{\frac{(p-1)}{p}} \left(\int_0^{t} \|f(s)\|^p s^{\alpha-1} ds\right)^{\frac{1}{p}} \\ &= \alpha C\|f\|_{\alpha,p}\left(\int_0^{t^\alpha} \left(\min\left\{1, \frac{h^\alpha}{r}\right\}\right)^{p/(p-1)} dr\right)^{(p-1)/p}\end{aligned}$$

$$= \alpha C \|f\|_{\alpha,p} \left(\int_0^{h^\alpha} dr + \int_{h^\alpha}^{\infty} \left(\frac{h^\alpha}{r} \right)^{p/(p-1)} dr \right)^{(p-1)/p}$$
$$= \alpha C \|f\|_{\alpha,p} \left(h^\alpha + h^\alpha \int_{h^\alpha}^{\infty} \left(\frac{1}{r} \right)^{p/(p-1)} dr \right)^{(p-1)/p}$$
$$\leq \alpha C \|f\|_{\alpha,p} \left(h^\alpha + h^{\alpha p/(p-1)} \int_{h^\alpha}^{\infty} \left(\frac{1}{r} \right)^{p/(p-1)} dr \right)^{(p-1)/p}$$
$$= \alpha C \|f\|_{\alpha,p} \left(h^\alpha + h^{\alpha p/(p-1)} (1-p) \left[\left(\frac{1}{r} \right)^{1/(p-1)} \right]_{h^\alpha}^{\infty} \right)^{(p-1)/p}$$
$$= \alpha C \|f\|_{\alpha,p} \left(h^\alpha + h^{\alpha p/(p-1)} (1-p) \left(-h^{-\alpha/(p-1)} \right) \right)^{(p-1)/p}$$
$$= \alpha C \|f\|_{\alpha,p} \left(h^\alpha - h^\alpha (1-p) \right)^{(p-1)/p}$$
$$= \alpha C \|f\|_{\alpha,p} p^{(p-1)/p} h^{\alpha(p-1)/p} .$$

Thus, the part (i) is proven.

Next, by Theorem 3.4(i) and Theorem 3.5(ii), if $u_0 \in D(A)$ then, for $t \in [0,T]$ and $h > 0$,

$$\|S_\alpha(t+h)u_0 - S_\alpha(t)u_0\| \leq \int_t^{t+h} \|S_\alpha(\tau)Au_0\| \tau^{\alpha-1} d\tau$$
$$\leq C_1 \|Au_0\| \int_t^{t+h} \tau^{\alpha-1} d\tau$$
$$= \frac{C_1 \|Au_0\|}{\alpha} ((t+h)^\alpha - t^\alpha)$$
$$\leq \frac{C_1 \|Au_0\|}{\alpha} h^\alpha.$$

Therefore, $S_\alpha(t)u_0$ is Hölder continuous with exponent α on $[0,T]$. This proves the part (ii). ■

The following lemma is needed to prove the regularity of the solution to the problem (1.1) under Hölder continuity condition on f.

Lemma 4.2. Let A be a sectorial linear operator and $S_\alpha(t)$ be an analytic conformable semigroup generated by A with $\alpha = 1/n$, $n = 1,2,3,4, \dots$. Let $f: [0,T] \to X$ and there exist a constant $K > 0$ and $0 < \vartheta < 1$ such that

$$\|f(t) - f(s)\| \leq K|t-s|^\vartheta. \tag{4.1}$$

If

$$v_1(t) = \int_0^t S_\alpha \left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}} \right) \left(f(s) - f(t) \right) s^{\alpha-1} ds$$

then $v_1(t) \in D(A)$ for $t \in (0,T]$ and, for every $\varepsilon > 0$, $Av_1(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\varepsilon, T]$.

Proof. Consider that, by Theorem 3.4(i) and the condition (4.1), for $t \in (0,T]$ and $0 < s < t \le T$,

$$\left\| S_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\left(f(s) - f(t)\right)s^{\alpha-1}\right\| \le C_1 K (t-s)^\vartheta s^{\alpha-1} \in L^1\left((0,T):X\right).$$

Observe now that, by Theorem 3.4(ii) and the condition (4.1), for $t \in [0,T]$,

$$\begin{aligned}\left\| AS_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\left(f(s) - f(t)\right)s^{\alpha-1}\right\| &\le MK\frac{(t-s)^\vartheta}{t^\alpha - s^\alpha}s^{\alpha-1} \\ &= MK\frac{(t^{\alpha n} - s^{\alpha n})^\vartheta}{t^\alpha - s^\alpha}s^{\alpha-1} \\ &= MK\frac{(t^\alpha - s^\alpha)^\vartheta\left(t^{\alpha(n-1)} + t^{\alpha(n-2)}s^\alpha + \cdots + s^{\alpha(n-1)}\right)^\vartheta}{t^\alpha - s^\alpha}s^{\alpha-1} \\ &\le nMKT^{\alpha\vartheta(n-1)}\frac{(t^\alpha - s^\alpha)^\vartheta}{t^\alpha - s^\alpha}s^{\alpha-1} \\ &= nMKT^{\vartheta-\alpha\vartheta}(t^\alpha - s^\alpha)^{\vartheta-1}s^{\alpha-1} \in L^1\left((0,T):X\right).\end{aligned}$$

Since A is closed, we have, for $t \in (0,T]$,

$$Av_1(t) = \int_0^t AS_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\left(f(s) - f(t)\right)s^{\alpha-1}\,ds.$$

Then, $v_1(t) \in D(A)$, for $t \in (0,T]$.

Given $\varepsilon > 0$. We next show the Hölder continuity of $Av_1(t)$ on $[\varepsilon, T]$. Consider that, for $0 < t < t+h < T$ with $0 < h < 1$,

$$\begin{aligned}Av_1(t+h) - Av_1(t) &= A\int_0^t \left(S_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right) - S_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\right)\left(f(s) - f(t)\right)s^{\alpha-1}\,ds \\ &\quad + A\int_0^t S_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\left(f(t) - f(t+h)\right)s^{\alpha-1}\,ds \\ &\quad + A\int_t^{t+h} S_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\left(f(s) - f(t+h)\right)s^{\alpha-1}\,ds \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Note that, by Theorem 3.1(iii), for $0 < s < t < t+h \le T$,

$$\begin{aligned}\left\| AS_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right) - AS_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\right\| &= \left\| AS\left(\frac{(t+h)^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right) - AS\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\right\| \\ &= \left\| \int_{\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}} D_\tau^2 S(\tau)\,d\tau \right\|\end{aligned}$$

$$\leq \int_{\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}} \|D_\tau^2 S(\tau)\|\, d\tau$$

$$\leq M_2 \int_{\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}} \tau^{-2}\, d\tau$$

$$= \alpha M_2 \left(\frac{1}{t^\alpha - s^\alpha} - \frac{1}{(t+h)^\alpha - s^\alpha}\right)$$

$$= \alpha M_2 \frac{(t+h)^\alpha - t^\alpha}{(t^\alpha - s^\alpha)((t+h)^\alpha - s^\alpha)}$$

$$\leq \alpha M_2 h^\alpha \frac{1}{(t^\alpha - s^\alpha)((t+h)^\alpha - s^\alpha)}.$$

Then, by the condition (4.1), for $\varepsilon \leq s < t < t+h \leq T$,

$$\|I_1\| \leq \alpha M_2 h^\alpha \int_0^t \left\| AS_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right) - AS_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\right\| \|f(s) - f(t)\| s^{\alpha-1} ds$$

$$\leq \alpha M_2 K h^\alpha \int_0^t \frac{(t-s)^\vartheta}{(t^\alpha - s^\alpha)((t+h)^\alpha - s^\alpha)} s^{\alpha-1}\, ds$$

$$\leq \alpha n M_2 K T^{\alpha\vartheta(n-1)} h^\alpha \int_0^t \frac{(t^\alpha - s^\alpha)^{\vartheta-1}}{(t+h)^\alpha - s^\alpha} s^{\alpha-1}\, ds$$

$$= M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha \int_0^t \frac{(t^\alpha - s^\alpha)^{\vartheta-1}}{(t+h)^\alpha - s^\alpha} s^{\alpha-1} ds$$

$$= M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha \left(\int_0^h \frac{(t^\alpha - s^\alpha)^{\vartheta-1}}{(t+h)^\alpha - s^\alpha} s^{\alpha-1}\, ds + \int_h^t \frac{(t^\alpha - s^\alpha)^{\vartheta-1}}{(t+h)^\alpha - s^\alpha} s^{\alpha-1}\, ds\right)$$

$$= \frac{M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha}{\alpha} \left(\int_0^{h^\alpha} \frac{r^{\vartheta-1}}{(t+h)^\alpha - t^\alpha + r} dr + \int_{h^\alpha}^{t^\alpha} \frac{r^{\vartheta-1}}{(t+h)^\alpha - t^\alpha + r} dr\right)$$

$$\leq \frac{M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha}{\alpha} \left(\frac{1}{(t+h)^\alpha - h^\alpha} \int_0^{h^\alpha} r^{\vartheta-1}\, ds + \int_{h^\alpha}^{\infty} r^{\vartheta-2}\, dr\right)$$

$$\leq \frac{M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha}{\alpha} \left(\frac{h^{\alpha\vartheta}}{((\varepsilon+h)^\alpha - h^\alpha)\vartheta} + \frac{h^{\alpha(\vartheta-1)}}{1-\vartheta}\right)$$

$$= \frac{M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha}{\alpha} \left(\frac{h^{\alpha\vartheta}}{\varepsilon\vartheta}\left((\varepsilon+h)^{\alpha(n-1)} + (\varepsilon+h)^{\alpha(n-2)} h^\alpha + \cdots + h^{\alpha(n-1)}\right) + \frac{h^{\alpha(\vartheta-1)}}{1-\vartheta}\right)$$

$$\leq \frac{M_2 K T^{\vartheta - \alpha\vartheta} h^\alpha}{\alpha} \left(\frac{h^{\alpha\vartheta}}{\varepsilon\vartheta} (\varepsilon+h)^{1-\alpha} + \frac{h^{\alpha(\vartheta-1)}}{1-\vartheta}\right)$$

$$\leq \frac{M_2KT^{\vartheta-\alpha\vartheta}}{\alpha}\left(\frac{h^{\alpha\vartheta}}{\varepsilon\vartheta}h^{\alpha}(\varepsilon+h)^{1-\alpha}+\frac{h^{\alpha\vartheta}}{1-\vartheta}\right)$$
$$\leq \frac{M_2KT^{\vartheta-\alpha\vartheta}}{\alpha}\left(\frac{\varepsilon+h}{\varepsilon\vartheta}+\frac{1}{1-\vartheta}\right)h^{\alpha\vartheta}$$
$$\leq \frac{M_2KT^{\vartheta-\alpha\vartheta}}{\alpha}\left(\frac{T}{\varepsilon\vartheta}+\frac{1}{1-\vartheta}\right)h^{\alpha\vartheta}.$$

Next, consider that, by Theorem 3.4(i) and the condition (4.1), for $0 < s < t < t+h < T$,

$$\begin{aligned}
\|I_2\| &\leq \left\| A\int_0^t S_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\big(f(t)-f(t+h)\big)s^{\alpha-1}\,ds \right\| \\
&= \left\| A\int_0^t S\left(\frac{(t+h)^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)s^{\alpha-1}\,ds\big(f(t)-f(t+h)\big)\right\| \\
&= \left\| A\int_{((t+h)^\alpha - t^\alpha)/\alpha}^{(t+h)^\alpha/\alpha} S(r)\,dr\big(f(t)-f(t+h)\big)\right\| \\
&= \left\| S\left(\frac{(t+h)^\alpha}{\alpha}\right) - S\left(\frac{(t+h)^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)\right\| \left\|\big(f(t)-f(t+h)\big)\right\| \\
&= \left\|S_\alpha(t+h) - S_\alpha\big(((t+h)^\alpha - t^\alpha)^{1/\alpha}\big)\right\|\left\|\big(f(t)-f(t+h)\big)\right\| \\
&\leq 2C_1h^\vartheta \leq 2C_1h^{\alpha\vartheta}.
\end{aligned}$$

and

$$\begin{aligned}
\|I_3\| &\leq \int_t^{t+h} \left\|AS_\alpha\left(((t+h)^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\big(f(s)-f(t+h)\big)\right\| s^{\alpha-1}\,ds \\
&\leq KM\int_t^{t+h}\frac{(t+h-s)^\vartheta}{(t+h)^\alpha - s^\alpha}s^{\alpha-1}\,ds \\
&\leq nKM(t+h)^{\alpha\vartheta(n-1)}\int_t^{t+h}((t+h)^\alpha - s^\alpha)^{\vartheta-1}s^{\alpha-1}\,ds \\
&\leq nKMT^{\alpha\vartheta(n-1)}\int_{t^\alpha}^{(t+h)^\alpha}((t+h)^\alpha - r)^{\vartheta-1}\,dr \\
&= \frac{KMT^{\vartheta-\alpha\vartheta}}{\alpha\vartheta}((t+h)^\alpha - t^\alpha)^\vartheta \\
&\leq \frac{KMT^{\vartheta-\alpha\vartheta}}{\alpha\vartheta}h^{\alpha\vartheta}.
\end{aligned}$$

■

The following theorem provides the regularity of the solution to the problem (1.1) under Hölder continuity condition on f.

Theorem 4.3. Let A be a sectorial linear operator and $S_\alpha(t)$ be an analytic conformable semigroup generated by A with $\alpha = 1/n$, $n = 1,2,3,4,\ldots.$ Let $f: [0,T] \to X$ and there exist a constant $K > 0$ and $0 < \vartheta < 1$ such that, for $s,t \in [0,T]$,

$$\|f(t) - f(s)\| \le K|t - s|^\vartheta.$$

If u is a solution to the problem (1.1) on $[0,T]$ then, for every $\delta > 0$, Au and $T_t^\alpha u$ are Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$.

Proof. By Theorem 3.9, for $0 < t \le T$,

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha\left((t^\alpha - \tau^\alpha)^{\frac{1}{\alpha}}\right) f(\tau)\tau^{\alpha-1}\, d\tau = S_\alpha(t)u_0 + v(t).$$

We first show that $AS_\alpha(t)u_0$ is Hölder continuous with exponent α on $[\delta, T]$ for every $\delta > 0$. Consider that, for $t, h > 0$,

$$\begin{aligned}
\|AS_\alpha(t+h) - AS_\alpha(t)\| &= \left\| AS\left(\frac{(t+h)^\alpha}{\alpha}\right) - AS\left(\frac{t^\alpha}{\alpha}\right)\right\| \\
&= \left\| \int_{\frac{t^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}} D_\tau^2 S(\tau)\, d\tau \right\| \\
&\le \int_{\frac{t^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}} \|D_\tau^2 S(\tau)\|\, d\tau \\
&\le M_2 \int_{\frac{t^\alpha}{\alpha}}^{\frac{(t+h)^\alpha}{\alpha}} \tau^{-2}\, d\tau \\
&= \alpha M_2 \left(\frac{1}{t^\alpha} - \frac{1}{(t+h)^\alpha}\right) \\
&= \alpha M_2 \frac{(t+h)^\alpha - t^\alpha}{t^\alpha (t+h)^\alpha} \\
&\le \alpha M_2 h^\alpha t^{-\alpha}(t+h)^{-\alpha}.
\end{aligned}$$

We then show that $Av(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. In order to do this, we decompose v as

$$v(t) = \int_0^t S_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right)\big(f(s) - f(t)\big)s^{\alpha-1}ds + \int_0^t S_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right) f(t)s^{\alpha-1}ds = v_1(t) + v_2(t).$$

From Lemma 4.2, we proved that $Av_1(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$ for every $\delta > 0$. We next show $Av_2(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. Note that, by Theorem 3.5(ii), for $t \ge 0$,

$$Av_2(t) = A\int_0^t S_\alpha\left((t^\alpha - s^\alpha)^{\frac{1}{\alpha}}\right) f(t)s^{\alpha-1}ds = A\int_0^t S_\alpha(r)r^{\alpha-1}f(t)\,dr = S_\alpha(t)f(t) - f(t).$$

We now consider that, by Theorem 3.4(i), Theorem 3.5(ii), and the condition (4.1), for $t \geq \delta$ and $h > 0$,

$$\begin{aligned}
&\|S_\alpha(t+h)f(t+h) - S_\alpha(t)f(t)\| \\
&\quad = \|S_\alpha(t+h)f(t+h) - S_\alpha(t+h)f(t) + S_\alpha(t+h)f(t) - S_\alpha(t)f(t)\| \\
&\quad \leq \|S_\alpha(t+h)f(t+h) - S_\alpha(t+h)f(t)\| + \|S_\alpha(t+h)f(t) - S_\alpha(t)f(t)\| \\
&\quad \leq \|S_\alpha(t+h)\|\|f(t+h) - f(t)\| + \|S_\alpha(t+h) - S_\alpha(t)\|\|f(t)\| \\
&\quad \leq C_1Kh^\vartheta + \int_t^{t+h} \|\tau^{\alpha-1}AS_\alpha(\tau)x\|\,d\tau\|f(t)\| \\
&\quad \leq C_1Kh^\vartheta + \int_t^{t+h} \tau^{-1}\,d\tau\|f\|_\infty \\
&\quad \leq C_1Kh^\vartheta + \frac{Mh}{\delta}\|f\|_\infty \\
&\quad \leq C'h^{\alpha\vartheta}
\end{aligned}$$

with $C' = \max\{C_1K, M\|f\|_\infty/\delta\}$. It means that $Av_2(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. Thus, $Av(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. Since $AS_\alpha(t)u_0$ is Hölder continuous with exponent α and $Av(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$, $Au(t)$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. Consequently, since u is the solution to the problem (1.1) and f is Hölder continuous with exponent ϑ on $[0, T]$, $T_t^\alpha u$ is Hölder continuous with exponent $\alpha\vartheta$ on $[\delta, T]$. This completes the proof. ■

5. Examples

Consider the nonlinear conformable diffusion equation

$$\begin{aligned}
T_t^\alpha u(x,t) &= \Delta u(x,t) + k\int_0^t (t-\tau)^{-\gamma}u(x,\tau)\,d\tau, \ \ (x,t) \in \Omega \times (0,T) \\
u(x,0) &= u_0(x) \text{ in } \Omega
\end{aligned} \tag{5.1}$$

where $0 < \gamma < 1$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary, and k is a positive constant. We next investigate the existence and uniqueness of a solution to the problem (5.1) with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T). \tag{5.2}$$

The abstract formulation of the problem (5.1) with Neumann boundary condition (5.2) is

$$\begin{aligned}
&T_t^\alpha u = Au + f(t,u), \ \ 0 < t < T \\
&u(0) = u_0 \text{ in } \Omega
\end{aligned}$$

in $X = \{u: u \in L^2(\Omega)\}$ where $0 < \alpha < 1$, $A = \Delta$, and

$$f(t,u) = k\int_0^t (t-\tau)^{-\gamma} u(x,\tau)\, d\tau$$

with $0 < \gamma < 1$. Note that $u \geq 0$ in $\Omega \times (0,T)$. We next set

$$D(A) = \left\{u \in H^2(\Omega): \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}.$$

The operator A is dissipative and self adjoint. It means that A is sectorial in X.

Next, observe that for $0 < s < t < T$ and $u \in Y = BC\big((0,T);X\big)$,

$$\begin{aligned} f(t,u) - f(s,u) &= k\int_0^t (t-\tau)^{-\gamma} u(\cdot,\tau)\, d\tau - k\int_0^s (s-\tau)^{-\gamma} u(\cdot,\tau)\, d\tau \\ &\leq k\int_0^t (t-\tau)^{-\gamma} u(\cdot,\tau)\, d\tau - k\int_0^s (t-\tau)^{-\gamma} u(\cdot,\tau)\, d\tau \\ &= k\int_s^t (t-\tau)^{-\gamma} u(\cdot,\tau)\, d\tau \\ &\leq k\|u\|_Y \int_s^t (t-\tau)^{-\gamma}\, d\tau \\ &= k\|u\|_Y \int_0^{t-s} \tau^{-\gamma}\, d\tau \\ &= \frac{k\|u\|_Y}{1-\gamma}(t-s)^{1-\gamma}. \end{aligned}$$

It follows that f is Hölder continuous with exponent $1-\gamma$. According to Theorem 4.3, if u solve the problem (5.1) then u is a classical solution.

6. Conclusion

By employing the analytic conformal semigroup generated by the sectorial linear operator A and imposing the condition on f i.e. the Hölder continuity of f as our regularity condition, we obtain stronger results regarding the regularity of solutions to the Cauchy problem (1.1) than those in ([16],[17]). To obtain our regularity results under the Hölder continuity of f, we can not use C_0 α-semigroup as used in ([16],[17]) to obtain their regularity results. For further work, a global solution to the Cauchy problem (1.1) under the regularity conditions is an interesting topic to study.

Acknowledgment: This work was supported by Jenderal Soedirman University through the Research Grant of Fundamental Research Scheme (Riset Dasar UNSOED) with grant number: 27.199/UN23.37/PT.01.03/II/2023.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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