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Solving Damped Harmonic Oscillator as a Second-Order Differential Equations and Caputo Fractional Differential Equation in Bipolar Menger Probabilistic b-Metric Spaces

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Abstract. In this study, we explore the newly proposed bipolar Menger probabilistic b-metric spaces and present several novel fixed-point theorems within this framework. We also provide a range of complex examples and apply our main results to the analysis of the damped harmonic oscillator, modeled by second-order differential equations. Furthermore, we demonstrate the applicability of our theoretical results to significant problem: Caputo fractional differential equations with integral boundary conditions. The proposed methods and results contribute to the broader understanding of probabilistic metric spaces and their utility in advanced mathematical modeling and analysis.

1. Introduction

The foundational theory of metric spaces was established by Fréchet in 1906 [3]. Since then, the field has seen various adaptations and expansions, including alterations to the metric function and relaxations of traditional axioms, thereby broadening the concept's applicability. This research introduces an innovative structure that expands upon probabilistic metric spaces by allowing metrics to encompass a product of two distinct nonempty sets. The concept of a bipolar metric, essential to this study, was first introduced by Mutlu and Gurdal in 2016 [34], inspired by practical scenarios where "distance" involves disparate set elements. Examples include measuring distances between points and lines in Euclidean space, between sets and their elements, or between a group of celestial bodies and the inverse luminosities of stars.

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Definition 1.1. *[34] If Ξ and Θ are nonempty sets, then a mapping $\mu : \Xi \times \Theta \to [0;+\infty)$ satisfying:*

(μ1) *$\mu(\zeta,\zeta) = 0$ for any $\zeta \in \Xi \cap \Theta$;*
(μ2) *if $\mu(\zeta,\varsigma) = \mu(\varsigma,\zeta) = 0$ for some $\zeta \in \Xi$ and $\varsigma \in \Theta$, then $\zeta = \varsigma$;*
(μ3) *$\mu(\zeta,\varsigma) = \mu(\varsigma,\zeta)$ for all $\zeta,\varsigma \in \Xi \cap \Theta$;*
(μ4) *$\mu(\zeta_1,\varsigma_1) \leq \mu(\zeta_1,\varsigma_2) + \mu(\zeta_2,\varsigma_1) + \mu(\zeta_2,\varsigma_2)$ for all $\zeta_1,\zeta_2 \in \Xi$ and $\varsigma_1,\varsigma_2 \in \Theta$.*

is a bipolar metric and a triple (Ξ,Θ,μ) is a bipolar metric space.

The notion of a b-metric space was originally put forward by Bakhtin [1] and later expanded by Czerwik [29]. Subsequent research has delved into its properties, defining convergence, Cauchy sequences, and establishing numerous fixed-point results with implications for nonlinear functional analysis. For a comprehensive overview, readers are directed to extensive literature on the subject [4–12]. Conversely, probabilistic metric spaces were initially conceptualized by Menger [15] in 1942. Fixed-point theorems within these spaces were further explored by Sehgal and Bharucha-Reid [16]. Schweizer and Sklar [18] have also extensively studied the characteristics of these spaces, known as Menger probabilistic metric spaces (*MPM*-spaces), focusing on both single-valued and multi-valued mappings [19–28]. In 2015, Hasanvand and Khanehgir [17] introduced the Menger probabilistic b-metric space (*MPbM*-space), contributing a fixed-point theorem for single-valued operators. The development of this area was significantly influenced by Branciari's 2000 introduction of the quadrilateral inequality, which was pivotal in generalizing the Banach contraction theorem and developing the concept of rectangular metric spaces [13]. This paper presents the integration of probabilistic b-metric spaces with bipolar metric concepts into what we term bipolar Menger probabilistic b-metric spaces (*BIMPM*-spaces). These spaces are crucial in non-Hausdorff topological contexts, particularly within the Tarskian framework for semantics in programming languages in computer science.

For foundational concepts concerning b-metric spaces, bipolar metric spaces, *MPM*-spaces, distribution functions, t-norms, and H-type (Hadzić type) t-norms among others, references [1,13, 14,22,25,29] and their citations are recommended.

We will proceed by introducing the necessary definitions.

Definition 1.2. *[17] Suppose $\Xi \neq \emptyset$, $\mathcal{T}$ is a continuous t-norm, $F : \Xi \times \Xi \to D^+$ (The set of all Menger distance distribution functions is denoted by D^+) is a mapping, and $\varrho \in (0,1]$. Then, $(\Xi,F,\mathcal{T})$ is a MPbM-space when for every $\zeta,\varsigma,z \in \Xi$ and $\iota,\kappa > 0$,*

(PM1) $F_{\zeta,\varsigma}(\iota) = 1$ *iff* $\zeta = \varsigma$,
(PM2) $F_{\zeta,\varsigma}(\iota) = F_{\varsigma,\zeta}(\iota)$,
(PM3) $F_{\zeta,\varsigma}(\iota+\kappa) \geq \mathcal{T}(F_{\zeta,z}(\varrho\iota),F_{\varsigma,z}(\varrho\kappa))$.

A *MPM*-space with $\varrho = 1$ is called a *MPbM*-space, meaning the class of *MPbM*-spaces encompasses a broader range than that of *MPM*-spaces. For more details on *MPM*-spaces, see [17].

Example 1.1. *[17] Let $\Xi = \mathbb{R}^+$ and define $\mathcal{T}(u,v) = \min\{u,v\}$. Define $F : \Xi \times \Xi \to D^+$ for $\zeta, \varsigma \in Xi$ by,*

$$F_{\zeta,\varsigma}(\iota) = \begin{cases} \frac{\iota}{\iota+|\zeta-\varsigma|^2}, & \text{if } \iota > 0 \\ 0 & \text{elswhere.} \end{cases}$$

Then, $(\Xi, F, \mathcal{T})$ constitutes a complete MPbM-space with $\varrho = \frac{1}{2}$.

Next, we explore (Φ)-functions, defined as functions $\phi : [0,\infty) \to [0,\infty)$ satisfying:
(ϕ_1) $\phi(r) < r$ for all $r > 0$;
(ϕ_2) $lim_{n\to\infty}\phi^n(r) = 0$ for all $r > 0$.
These functions are called (Φ)-functions, and their collection is denoted by Φ.

The structure of this document includes four sections. Section 2 introduces a novel *FP* theorem applicable to single-valued mappings within the *BIMPbM*-space. Section 3 demonstrates the application of our findings by establishing the existence of solutions for initial value problems related to the damped harmonic oscillator, framed within second-order differential equations. Section 4 offers a summary of the conclusions

2. New *FP* Theorem in Bipolar *MPbM*-Spaces

This section presents essential definitions in bipolar *MPbM*-spaces.

Definition 2.1. *Let $\Xi, \Theta \neq \emptyset$, $\mathcal{T}$ is a continuous t-norm, $\varrho \in (0,1]$ and $F : \Xi \times \Theta \to D^+$, then $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is defined as a bipolar Menger probabilistic b-metric $(BIMPbM)$ space. For all $\zeta \in \Xi, \varsigma \in \Theta$ the following properties hold:*

- $(F1)$ *$F_{\zeta,\zeta}(\iota) = 1$ for any $\zeta \in \Xi \cap \Theta$;*
- $(F2)$ *if $F_{\zeta,\varsigma}(\iota) = F_{\varsigma,\zeta}(\iota) = 1$ for some $\zeta \in \Xi$ and $\varsigma \in \Theta$, then $\zeta = \varsigma$;*
- $(F3)$ *$F_{\zeta,\varsigma}(\iota) = F_{\varsigma,\zeta}(\iota)$ for all $\zeta, \varsigma \in \Xi \cap \Theta$;*
- $(F4)$ *$F_{\zeta_1,\varsigma_1}(\iota_1 + \iota_2 + \iota_3) \geq \mathcal{T}(\mathcal{T}(F_{\zeta_1,\varsigma_2}(\varrho\iota_1), F_{\zeta_2,\varsigma_1}(\varrho\iota_2)), F_{\zeta_2,\varsigma_2}(\varrho\iota_3))$ for all $\zeta_1, \zeta_2 \in \Xi$ and $\varsigma_1, \varsigma_2 \in \Theta$.*

Example 2.1. *Let $\Xi = \{0, \frac{1}{2}, 1, 2, 3, 4, 5\}$ and $\Theta = \{0, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, 6\}$ be equipped with $F_{\zeta,\varsigma}(\iota) = \frac{\iota}{\iota+|\zeta-\varsigma|^2}$, $\mathcal{T}(a,b) = min\{a,b\}$ and $\varrho = \frac{1}{2}$. Then, $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is a complete $(BIMPbM)$ space.*

Definition 2.2. *Let $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is a $(BIMPbM)$-space. Then the functions $F^{\Xi} : \Xi \times \Xi \to D^+$ and $F^{\Theta} : \Theta \times \Theta \to D^+$ which are defined as $F^{\Xi}_{\zeta_1,\zeta_2}(\iota) = min_{\varsigma\in\Theta}\{F_{\zeta_1,\varsigma}(\iota), F_{\zeta_2,\varsigma}(\iota)\}$ for all $\zeta_1, \zeta_2 \in \Xi$ and $F^{\Theta}_{\varsigma_1,\varsigma_2}(\iota) = min_{\zeta\in\Xi}\{F_{\varsigma_1,\zeta}(\iota), F_{\varsigma_2,\zeta}(\iota)\}$ for all $\varsigma_1, \varsigma_2 \in \Theta$, are called inner probabilistic metrics generated by (Ξ, Θ, F)*

Definition 2.3. *Let $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is a $(BIMPbM)$-space. If the inner probabilistic metric F^{Ξ} is a probabilistic metric on Ξ, then we say that Θ characterizes Ξ, and if F^{Θ} is a probabilistic metric, we say that Ξ characterizes Θ . If Ξ and Θ characterize each other, then the space $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is called bicharacterized.*

In the following sections, we will discuss bipolar convergence, bipolar Cauchy sequences, and bipolar completeness in the *BIMPbM*-space.

Let $(\Xi,\Theta,F,\mathcal{T},\varrho)$ be a *BIMPbM*-space. A left sequence (ζ_n) converges to a right point ς if and only if for every $\Im > 0$ there exists an $n_0 \in \mathbb{N}$ such that $F_{\zeta_n,\varsigma}(\iota) > 1-\Im$ for all $n \geq n_0$. Similarly a right sequence (ς_n) converges to a left point ζ if and only if, for every $\Im > 0$ there exists an $n_0 \in \mathbb{N}$ such that, whenever $n \geq n_0, F_{\zeta,\varsigma_n}(\iota) > 1-\Im$.

Lemma 2.1. *Let $(\Xi,\Theta,F,\mathcal{T},\varrho)$ be a BIMPbM-space. If $(\Xi,\Theta,F,\mathcal{T},\varrho)$ is bicharacterized then every convergent sequence has a unique limit.*

Proof. Let $\{\zeta_n\}$ be a left sequence such that both $\zeta_n \to \varsigma_1 \in \Theta$ and $\zeta_n \to \varsigma_2 \in \Theta$. Then for each $\zeta \in \Xi$ we have

$$F_{\zeta,\varsigma_2}(\iota_1+\iota_2+\iota_3) \geq \mathcal{T}(\mathcal{T}(F_{\zeta,\varsigma_1}(\iota_1),F_{\zeta_n,\varsigma_1}(\iota_2)),F_{\zeta_n,\varsigma_2}(\iota_3))$$

and

$$F_{\zeta,\varsigma_1}(\iota_1+\iota_2+\iota_3) \geq \mathcal{T}(\mathcal{T}(F_{\zeta,\varsigma_2}(\iota_1),F_{\zeta_n,\varsigma_2}(\iota_2)),F_{\zeta_n,\varsigma_1}(\iota_3))$$

Since

$$lim_{n\to\infty}F_{\zeta_n,\varsigma_1}(\iota_2) = lim_{n\to\infty}F_{\zeta_n,\varsigma_1}(\iota_3) = 1$$

and

$$lim_{n\to\infty}F_{\zeta_n,\varsigma_2}(\iota_3) = lim_{n\to\infty}F_{\zeta_n,\varsigma_2}(\iota_2) = 1$$

so, $F_{\zeta,\varsigma_1}(\iota_1) = F_{\zeta,\varsigma_2}(\iota_1)$ for all $\zeta \in \Xi$. Hence $F^{\Theta}_{\varsigma_1,\varsigma_2}(\iota_1) = min_{\zeta\in\Xi}\{F_{\zeta,\varsigma_1}(\iota_1,F_{\zeta,\varsigma_2}(\iota_1\} = 1$ and since Ξ characterizes Θ , F^{Θ} is a probabilistic metric so that $\varsigma_1 = \varsigma_2$. □

Now we define the continuity of maps.

Definition 2.4. *Let $(\Xi,\Theta,F,\mathcal{T},\varrho)$ be a BIMPbM-space.*

- *1. A sequence (ζ_n,ς_n) on the set $\Xi\times\Theta$ is called a bisequence on $(\Xi,\Theta,F,\mathcal{T},\varrho)$.*
- *2. if both sequences (ζ_n) and (ς_n) converge, then the bisequence (ζ_n,ς_n) is considered convergent. Should (ζ_n) and (ς_n) converge to the same point u in $\Xi\cap\Theta$, the bisequence is deemed biconvergent.*
- *3. a bisequence (ζ_n,ς_n) in $(\Xi,\Theta,F,\mathcal{T},\varrho)$ qualifies as a Cauchy bisequence if, for any $\Im > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that for all integers $n,m \geq n_0$, $F_{\zeta_n,\varsigma_n}(\iota) \geq 1-\Im$.*

Definition 2.5. *states that a BIMPbM-space is complete if every Cauchy bisequence within it converges.*

We now introduce a new common *FP* theorem for single-valued mapping in a complete *BIMPbM*-space as per

Theorem 2.1. *Let $(\Xi,\Theta,F,\mathcal{T},\varrho)$ be a complete BIMPbM-space with $\varrho \in (0,1]$, satisfying $\mathcal{T}(a,a) \geq a$ for a in $[0,1]$. Assume further that a continuous operator $\varpi : \Xi\cup\Theta \to \Xi\cup\Theta$ exists, with $\varpi(\Xi) \subset \Xi$ and $\varpi(\Theta) \subset \Theta$, such that:*

$$F_{\varpi\zeta,\varpi\varsigma}((\varrho)^k\varphi(\iota)) \geq \lambda \min\{F_{\zeta,\varsigma}(\varrho^{k-1}\varphi(\frac{\iota}{c}), \\ F_{\zeta,\varpi\varsigma}(\varrho^{k-1}\varphi(\frac{\iota}{c}),F_{\varpi\zeta,\varsigma}(\varrho^{k-1}\varphi(\frac{\iota}{c})\}$$

where $\lambda \geq 1$ and c in $(0,1)$. Then, this operator has a fixed point.

Proof. With the initial points $\zeta_0 \in \Xi$ and $\varsigma_0 \in \Theta$ we start by constructing an iterative bisequence $\{(\zeta_n, \varsigma_n)\}$ such that:

$$\zeta_1 = \varpi\zeta_0,\ \zeta_2 = \varpi^2\zeta_0,\ \zeta_3 = \varpi^3\zeta_0,\ ...,\ \zeta_n = \varpi^n\zeta_0, ...$$

and

$$\varsigma_1 = \varpi\varsigma_0,\ \varsigma_2 = \varpi^2\varsigma_0,\ \varsigma_3 = \varpi^3\varsigma_0,\ ...,\ \varsigma_n = \varpi^n\varsigma_0, ...$$

Since φ is continuous at 0, we can find a $r > 0$ so that $r > \varphi(r)$. So, it yields

$$\begin{aligned} F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) &= F_{\varpi\zeta_{n-1},\varpi\varsigma_{n-1}}(\varrho^k\varphi(r)) \\ &\geq \lambda \min\{F_{\varpi\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varpi\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c})\} \\ &\geq \lambda \min\{F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\} \\ &\geq \min\{F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\} \end{aligned}$$

At this stage, we need to show that

$$F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \geq F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}). \tag{2.1}$$

Now there are two situations that we will examine.

Case 1:

Let us assume that $F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum.

So $F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \geq F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c})$, then

$$\begin{aligned} F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) &\geq F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq F_{\zeta_n,\varsigma_{n-2}}(\varrho^{k-2}\varphi(\frac{r}{c^2})) \\ \geq ... &\geq F_{\zeta_n,\varsigma_0}(\varrho^{k-n}\varphi(\frac{r}{c^n})) \end{aligned}$$

for every $n \in \mathbb{N}$. By letting $n \to \infty$ we obtain $F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \geq 1$. Therefore, we get $\zeta_n = \varsigma_n$, which contradicts the condition $\zeta_n \neq \varsigma_n$.

Case 2:

Let us assume that $F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum.

So $F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \geq F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})$, then

$$\begin{aligned} F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) &\geq F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq F_{\zeta_{n-2},\varsigma_n}(\varrho^{k-2}\varphi(\frac{r}{c^2})) \\ \geq ... &\geq F_{\zeta_0,\varsigma_n}(\varrho^{k-n}\varphi(\frac{r}{c^n})) \end{aligned}$$

for every $n \in \mathbb{N}$. By letting $n \to \infty$ we obtain $F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \geq 1$. Therefore, we get $\zeta_n = \varsigma_n$, which contradicts the condition $\zeta_n \neq \varsigma_n$.

So, we obtain that $F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum, and (2.1) is true.

From (2.1) we have:

$$\begin{aligned} F_{\zeta_n,\varsigma_n}(\varrho^k\iota) &\geq F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) \\ &\geq F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq \cdots \geq F_{\zeta_0,\varsigma_0}(\varrho^{k-n}\varphi(\frac{r}{c^n}), \end{aligned}$$

that is, $F_{\zeta_n,\varsigma_n}(\varrho^k\iota) \geq F_{\zeta_0,\varsigma_0}(\varrho^{k-n}\varphi(\frac{r}{c^n})$ for any $n \in \mathbb{N}$. Also

$$\begin{aligned} F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) &= F_{\varpi\zeta_{n-1},\varpi\varsigma_n}(\varrho^k\varphi(r)) \\ &\geq \lambda \min\{F_{\varpi\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varpi\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\} \\ &\geq \min\{F_{\varpi\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varpi\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\} \\ &\geq \min\{F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), \\ &\quad F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}), F_{\zeta_{n-1},\varsigma_{n+1}}(\varrho^{k-1}\varphi(\frac{r}{c})\}. \end{aligned}$$

Now we need to show that

$$F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \geq F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}). \tag{2.2}$$

Now there are two situations that we will examine.

Case1:

Let us assume that $F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum.

So $F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \geq F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})$, then

$$\begin{aligned} F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) &\geq F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2})) \\ &\geq \ldots \geq F_{\zeta_n,\varsigma_1}(\varrho^{k-n}\varphi(\frac{r}{c^n})) \end{aligned}$$

for every $n \in \mathbb{N}$. By letting $n \to \infty$ we obtain $F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \geq 1$. Therefore, we get $\zeta_n = \varsigma_{n+1}$, which contradicts the condition $\zeta_n \neq \varsigma_{n+1}$.

Case2:

Let us assume that $F_{\zeta_{n-1},\varsigma_{n+1}}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum.

So $F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \geq F_{\zeta_{n-1},\varsigma_{n+1}}(\varrho^{k-1}\varphi(\frac{r}{c})$, then

$$\begin{aligned} F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) &\geq F_{\zeta_{n-1},\varsigma_{n+1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq F_{\zeta_{n-2},\varsigma_{n+1}}(\varrho^{k-2}\varphi(\frac{r}{c^2})) \\ \geq \ldots &\geq F_{\zeta_0,\varsigma_{n+1}}(\varrho^{k-n}\varphi(\frac{r}{c^n})) \end{aligned}$$

for every $n \in \mathbb{N}$. By letting $n \to \infty$ we obtain $F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \geq 1$. Therefore, we get $\zeta_n = \varsigma_{n+1}$, which contradicts the condition $\zeta_n \neq \varsigma_{n+1}$.

So, we obtain that $F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})$ is a minimum, and (2.2) is true.

From (2.2) we have:

$$\begin{aligned} F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\iota) &\geq F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\varphi(r)) \\ &\geq F_{\zeta_{n-1},\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c}) \\ &\geq \cdots \geq F_{\zeta_0,\varsigma_1}(\varrho^{k-n}\varphi(\frac{r}{c^n}), \end{aligned}$$

that is, $F_{\zeta_n,\varsigma_{n+1}}(\varrho^k\iota) \geq F_{\zeta_0,\varsigma_1}(\varrho^{k-n}\varphi(\frac{r}{c^n})$ for any $n \in \mathbb{N}$.

We consider $m, n \in \mathbb{N}$, with $m > n$. Then, by $(F4)$ and the strictly non-decreasing feature of φ, it yields

$$\begin{aligned} F_{\zeta_m,\varsigma_n}((m-n)\iota) &\geq \min\{F_{\zeta_m,\varsigma_{n+1}}((m-n-2)\iota), \\ &\quad F_{\zeta_n,\varsigma_{n+1}}(\iota), F_{\zeta_n,\varsigma_n}(\iota)\} \\ &\geq \min\{F_{\zeta_m,\varsigma_{n+1}}((m-n-2)\iota), \\ &\quad F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n})\} \\ &\geq \min\{F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}), \\ &\quad \cdots, F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^{m-2}}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-2}}), \\ &\quad F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^{m-1}}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-1}})\} \\ &= \min\{F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n})\}. \end{aligned}$$

Since $\varrho^{1-n}\varphi(\frac{r}{c^n}) \to \infty$ as $n \to \infty$, there exists a $n_0 \in \mathbb{N}$ so that $F_{\zeta_0,\varsigma_1}(\varrho^{1-n}\varphi(\frac{r}{c^n}) > 1 - \upsilon$ and $F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}) > 1 - \upsilon$ for a fixed $\upsilon \in (0,1)$, whenever $n \geq n_0$. Thus, $F_{\zeta_m,\varsigma_n}((m-n)\iota) > 1 - \upsilon$ for every $m > n \geq n_0$. Since $\iota > 0$ and $0 < \upsilon < 1$ are arbitrary, we conclude that $\{(\zeta_n, \varsigma_n)\}$ is a Cauchy bisequence in the complete *BIMPbM*-space $(\Xi, \Theta, F, \mathcal{T}, \varrho)$. So, there exists a $u \in \Xi \cap \Theta$ and $\varpi(\varsigma_n) = \varsigma_{n+1} \to u \in \Xi \cap \Theta$ guarantees that $(\varpi(\varsigma_n))$ has unique limit. Since ϖ is continuous $(\varpi(\varsigma_n)) \to \varpi(u)$, so $\varpi(u) = u$. Hence u is a fixed point of ϖ. This complete the proof. □

Corollary 2.1. *Let $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is a complete BIMPbM-space with $\varrho \in (0,1]$, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Additionally, let us assume that $\varpi : \Xi \cup \Theta \to \Xi \cup \Theta$ is a continuous operator that satisfies:*

$$F_{\varpi\zeta,\varpi\varsigma}(\varphi(\iota)) \geq \lambda \min\{F_{\zeta,\varsigma}(\iota),$$
$$F_{\varpi\zeta,\varsigma}(\iota), F_{\zeta,\varpi\varsigma}(\iota)\}$$

where $\lambda \geq 1$. Then, ϖ has a FP.

Definition 2.6. *Let $(\Xi_1, \Theta_1, F_1, \mathcal{T}, \varrho)$ and $(\Xi_2, \Theta_2, F_2, \mathcal{T}, \varrho)$ be two BIMPbM-space and $\varpi : \Xi_1 \cup \Theta_1 \to \Xi_2 \cup \Theta_2$ be a function. If $\varpi(\Xi_1) \subset \Xi_2$ and $\varpi(\Theta_1) \subset \Theta_2$, then ϖ is called a covariant map, or a map from $(\Xi_1, \Theta_1, F_1, \mathcal{T}, \varrho)$ to $(\Xi_2, \Theta_2, F_2, \mathcal{T}, \varrho)$. If $\varpi : (\Xi_1, \Theta_1, F_1, \mathcal{T}, \varrho) \to (\Theta_2, \Xi_2, F_2, \mathcal{T}, \varrho)$ is a map, then ϖ is called a contravariant map from $(\Xi_1, \Theta_1, F_1, \mathcal{T}, \varrho)$ to $(\Theta_2, \Xi_2, F_2, \mathcal{T}, \varrho)$ and this is denoted as*

$$\varpi : (\Xi_1, \Theta_1, F_1, \mathcal{T}, \varrho) \rightleftharpoons (\Theta_2, \Xi_2, F_2, \mathcal{T}, \varrho)$$

Theorem 2.2. *Let $(\Xi, \Theta, F, \mathcal{T}, \varrho)$ is a complete BIMPbM-space with $\varrho \in (0,1]$, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Additionally, let us assume that $\varpi : \Xi \cup \Theta \to \Xi \cup \Theta$ is a contravariant continuous operator that satisfies:*

$$F_{\varpi\zeta,\varpi\varsigma}((\varrho)^k \varphi(\iota)) \geq \lambda F_{\varsigma,\zeta}(\varrho^{k-1}\varphi(\frac{\iota}{c})).$$

where $\lambda \geq 1$. Then, ϖ has a FP.

Proof. Let $\zeta_0 \in \Xi$, $\varpi\zeta_0 = \varsigma_0 \in \Theta$ and $\varpi\varsigma_0 = \zeta_1$. For each $n \in \mathbb{N}$ define $\varpi\zeta_n = \varsigma_n$ and $\varpi\varsigma_n = \zeta_{n+1}$. Then (ζ_n, ς_n) is a bisequence on $(\Xi, \Theta, F, \mathcal{T}, \varrho)$.

Since φ is continuous at 0, we can find a $r > 0$ so that $r > \varphi(r)$. So, it yields

$$\begin{aligned}
F_{\zeta_n,\varsigma_n}(\varrho^k\varphi(r)) &= F_{\varpi\varsigma_{n-1},\varpi\zeta_n}(\varrho^k\varphi(r)) \\
&\geq \lambda F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\
&\geq F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\
&= F_{\varpi\varsigma_{n-1},\varpi\zeta_{n-1}}(\varrho^{k-1}\varphi(\frac{r}{c}) \\
&\geq \lambda F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2}) \\
&\geq F_{\zeta_{n-1},\varsigma_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2}) \\
&= F_{\varpi\varsigma_{n-2},\varpi\zeta_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2})) \\
&\geq \\
&\vdots \\
&\geq F_{\zeta_0,\varsigma_0}(\varrho^{k-n}\varphi(\frac{r}{c^n}).
\end{aligned}$$

Also

$$\begin{aligned}
F_{\zeta_{n+1},\varsigma_n}(\varrho^k\varphi(r)) &= F_{\varpi\varsigma_n,\varpi\zeta_n}(\varrho^k\varphi(r))\\
&\geq \lambda F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\\
&\geq F_{\zeta_n,\varsigma_n}(\varrho^{k-1}\varphi(\frac{r}{c})\\
&= F_{\varpi\varsigma_{n-1},\varpi\zeta_n}(\varrho^{k-1}\varphi(\frac{r}{c})\\
&\geq \lambda F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2})\\
&\geq F_{\zeta_n,\varsigma_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2})\\
&= F_{\varpi\varsigma_{n-1},\varpi\zeta_{n-1}}(\varrho^{k-2}\varphi(\frac{r}{c^2}))\\
&\geq\\
&\vdots\\
&\geq F_{\zeta_0,\varsigma_0}(\varrho^{k-n}\varphi(\frac{r}{c^n}).
\end{aligned}$$

We consider $m, n \in \mathbb{N}$, with $m > n$. Then, by $(F4)$ and the strictly non-decreasing feature of φ, it yields

$$\begin{aligned}
F_{\zeta_m,\varsigma_n}((m-n)\iota) &\geq \min\{F_{\zeta_m,\varsigma_{n+1}}((m-n-2)\iota),\\
&\quad F_{\zeta_{n+1},\varsigma_n}(\iota), F_{\zeta_{n+1},\varsigma_{n+1}}(\iota)\}\\
&\geq \min\{F_{\zeta_m,\varsigma_{n+1}}((m-n-2)\iota),\\
&\quad F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n})\}\\
&\geq \min\{F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}),\\
&\quad \cdots, F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-2}}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-2}}),\\
&\quad F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-1}}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^{m-1}})\}\\
&= \min\{F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}), F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n})\}.
\end{aligned}$$

Since $\varrho^{1-n}\varphi(\frac{r}{c^n}) \to \infty$ as $n \to \infty$, there exists a $n_0 \in \mathbb{N}$ so that $F_{\zeta_0,\varsigma_0}(\varrho^{1-n}\varphi(\frac{r}{c^n}) > 1 - \upsilon$ for a fixed $\upsilon \in (0,1)$, whenever $n \geq n_0$. Thus, $F_{\zeta_m,\varsigma_n}((m-n)\iota) > 1 - \upsilon$ for every $m > n \geq n_0$. Since $\iota > 0$ and $0 < \upsilon < 1$ are arbitrary, we conclude that $\{(\zeta_n, \varsigma_n)\}$ is a Cauchy bisequence in the complete *BIMPbM*-space $(\Xi, \Theta, F, \mathcal{T}, \varrho)$. So, there exists a $u \in \Xi \cap \Theta$ and $\varpi(\varsigma_n) = \zeta_n \to u \in \Xi \cap \Theta$ guarantees that $(\varpi(\varsigma_n))$ has unique limit. Since ϖ is continuous $(\varpi(\varsigma_n)) \to \varpi(u)$, so $\varpi(u) = u$. Hence u is a fixed point of ϖ. This complete the proof.

□

Example 2.2. *Let $\Xi = [0,1]$ and $\Theta = [1,2]$ be equipped with $F_{\zeta,\varsigma}(\iota) = \frac{\iota}{\iota+|\zeta-\varsigma|^2}$ for each $\zeta \in \Xi$, $\varsigma \in \Theta$, $\mathcal{T}(a,b) = min\{a,b\}$ and $\varrho = \frac{1}{2}$. Then, $(\Xi,\Theta,F,\mathcal{T},\varrho)$ is a complete BIMPbM-space. Define $\phi(t) = \frac{t}{4}$ for each $t \in [0,\infty)$, $\lambda = 1$, $c = \frac{1}{2}$ and $\varpi : \Xi \cup \Theta \rightrightarrows \Xi \cup \Theta$ by*

$$\varpi(\iota) = \begin{cases} \frac{\iota}{4}, & \iota \in (0,1) \\ 1, & \iota \in [1,2]. \end{cases}$$

For each $\zeta \in \Xi$ and $\varsigma \in \Theta$ we have

$$\begin{aligned} F_{\varpi\zeta,\varpi\varsigma}(\varrho^k\phi(\iota)) &= \frac{\frac{\iota}{2^k}}{\frac{\iota}{2^k} + |\frac{\zeta}{4} - 1|^2} \\ &= \frac{\iota}{\iota + 2^k|\frac{\zeta}{4} - 1|^2} \\ &= \frac{\iota}{\iota + \frac{2^k}{16}|\zeta - \frac{1}{4}|^2} \\ &= \frac{\iota}{\iota + 2^{k-4}|\zeta - \frac{1}{4}|^2} \\ &\geq \frac{\iota}{\iota + 2^{k-2}|\zeta - \frac{1}{4}|^2} \\ &\geq \frac{\iota}{\iota + 2^{k-2}|\zeta - \varsigma|^2} \\ &= F_{\zeta,\varsigma}(\varrho^{k-1}\phi(\frac{\iota}{c})) \\ &\geq \{F_{\zeta,\varsigma}(\varrho^{k-1}\phi(\frac{\iota}{c})), F_{\varpi\zeta,\varsigma}(\varrho^{k-1}\phi(\frac{\iota}{c})), \\ &\quad F_{\zeta,\varpi\varsigma}(\varrho^{k-1}\phi(\frac{\iota}{c}))\}. \end{aligned}$$

Thus, all the criteria of Theorem 2.1 are satisfied and ϖ has a FP $\iota = 1$.

3. Two Illustrative Applications

3.1. **Damped Harmonic Oscillators.**

In classical mechanics, the harmonic oscillator is defined as a system which, when displaced from its equilibrium position, experiences a restoring force F directly proportional to the displacement x, represented as

$$\vec{F} = -k\vec{x}$$

where k is a positive constant. If damping is present, characterized by a frictional force proportional to velocity, the system is described as a damped oscillator. In such systems, friction or damping reduces the motion's speed proportionally to the frictional force applied. Unlike the simple undriven harmonic oscillator where only the restoring force acts on the mass, the damped harmonic oscillator also experiences a frictional force opposing the motion. Hence, the balance offorces for damped harmonic oscillators is:

$$F = F_{ext} - k\omega - c\frac{d\omega}{dt} = m\frac{d^2\omega}{dt^2}$$

When there is no external force (i.e., $F_{ext} = 0$),

$$\frac{d^2\omega}{dt^2} + 2v\bar{\omega}\frac{d\omega}{dt} + \bar{\omega}2\omega = 0 \tag{3.1}$$

where $\bar{\omega} = \sqrt{\frac{k}{m}}$ is the undamped angular frequency of the oscillator and $v = \frac{c}{2\sqrt{km}}$ is the damping ratio. A damped harmonic oscillator is critically damped for $v = 1$, i.e., the system comes back to the steady state as quickly as time permits without oscillation (however, overshoot can happen as in the case of doors).

Green's function associated to equation 3.1 in case of critically damped motion under conditions $\omega(0) = 0, \acute{\omega}(0) = a$ is given by

$$G(\ell, \jmath) = \begin{cases} -\jmath e^{\rho(\jmath-\ell)}, & 0 \leq \jmath \leq \ell \leq 1 \\ -\ell e^{\rho(\jmath-\ell)}, & 0 \leq \ell \leq \jmath \leq 1. \end{cases} \tag{3.2}$$

where $\rho > 0$ is a constant, calculated in terms of v and $\bar{\omega}$.

Let $\Xi = C([0,1], \mathbb{R}^+)$ be the set of all continuous functions on $[0,1]$ and $\Theta = C([0,1], \mathbb{R}^+)$ be the set of all continuous functions on $[0,1]$. Consider $\mu : \Xi \times \Theta \to [0, \infty)$ to be defined by

$$\mu(\zeta, \varsigma) = max_{\ell \in [0,1]} |\zeta(\ell) - \varsigma(\ell)|^2 e^{-2\ell\iota}, \qquad \zeta \in \Xi, \varsigma \in \Theta, L > 0.$$

One can see that μ is a complete bipolar b-metrics with $s = 2$. Next, we define the mapping $F : \Xi \times \Theta \to D^+$ as

$$F_{\zeta,\varsigma}(\iota) = \chi(\iota - \mu(\zeta, \varsigma))$$

for $\iota > 0$, where

$$\chi(\iota) = \begin{cases} 0 & \text{if } \iota \leq 0 \\ 1 & \text{if } \iota > 0. \end{cases}$$

We know that (Ξ, Θ, F, min) is a complete *BIMPbM*-space with coefficient $\varrho = \frac{1}{2}$.

In the next theorem, we consider the equation of critically damped harmonic oscillators in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

Theorem 3.1. *Let (Ξ, Θ, F, min) be a complete BIMPbM-space whit $\varrho = \frac{1}{2}$ and $\varpi : \Xi \cup \Theta \to \Xi \cup \Theta$ be a self mapping such that:*

$$\varpi\omega(\ell) = \int_0^\ell G(\ell, \jmath)\Omega(\jmath, \omega(\jmath))d\jmath \tag{3.3}$$

where $\Omega : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an increasing function which satisfies

$$|\Omega(\ell, \varpi\omega(\ell)) - \Omega(\ell, \varpi\vartheta(\ell))| \leq \frac{1}{2} max\{|\omega(\ell)) - \vartheta(\ell)|, |\varpi\omega(\ell)) - \vartheta(\ell)|, |\omega(\ell)) - \varpi\vartheta(\ell)|\}$$

Then the differential equation 3.1 has a unique solution

Proof. Finding solution of equation 3.1 is equivalent to solving the integral equation

$$\omega(\ell) = \int_0^{\ell} G(\ell, \jmath)\Omega(\jmath, \omega(\jmath))d\jmath.$$

Therefore, z is a solution of equation 3.1 iff z is a fixed point of ϖ. Now, for all $\omega \in \Xi$ and $\vartheta \in \Theta$ we have

$$\begin{aligned}
\mu(\varpi\omega, \varpi\vartheta) &= max_{\ell\in[0,1]}(|\varpi\omega(\ell) - \varpi\vartheta(\ell)|^2 e^{-2L\ell}) \\
&= max_{\ell\in[0,1]}(|\int_0^{\ell} G(\ell, \jmath)\Omega(\jmath, \omega(\jmath))d\jmath - \int_0^{\ell} G(\ell, \jmath)\Omega(\jmath, \vartheta(\jmath))d\jmath|^2 e^{-2L\ell}) \\
&\leq max_{\ell\in[0,1]}((\int_0^{\ell} G(\ell, \jmath)|\Omega(\jmath, \omega(\jmath)) - \Omega(\jmath, \vartheta(\jmath))|)^2 e^{-2L\ell} d\jmath) \\
&\leq max_{\ell\in[0,1]}((\int_0^{\ell} G(\ell, \jmath)|\frac{1}{2}max\{|\omega(\ell) - \vartheta(\ell)|, \\
&\qquad |\varpi\omega(\ell) - \vartheta(\ell)|, |\omega(\ell) - \varpi\vartheta(\ell)|\}|)^2 e^{-2L\ell} d\jmath) \\
&\leq \frac{1}{4}max\{\mu(\omega, \vartheta), \mu(\varpi\omega, \vartheta), \mu(\omega, \varpi\vartheta)\}max_{\ell\in[0,1]}(\int_0^{\ell} G(\ell, \jmath)d\jmath)^2
\end{aligned}$$

on the other hand $max_{\ell\in[0,1]} \int_0^{\ell} G(\ell, \jmath)d\jmath < 1$ so we have

$$\mu(\varpi\omega, \varpi\vartheta) \leq \frac{1}{4}max\{\mu(\omega, \vartheta), \mu(\varpi\omega, \vartheta), \mu(\omega, \varpi\vartheta)\}$$

Putting $c = \frac{1}{2}$, for any $r > 0$ and $k \in \mathbb{N}$ we derive

$$\begin{aligned}
F_{\varpi\omega,\varpi\vartheta}(\frac{r}{2^k}) &= \chi(\frac{r}{2^k} - \mu(\varpi\omega, \varpi\vartheta)) \\
&\geq \chi(\frac{r}{2^k} - \frac{c}{2}max\{\mu(\omega, \vartheta), \mu(\varpi\omega, \vartheta), \mu(\omega, \varpi\vartheta)\}) \\
&= \chi(\frac{r}{2^{k-1}c} - max\{\mu(\omega, \vartheta), \mu(\varpi\omega, \vartheta), \mu(\omega, \varpi\vartheta)\}) \\
&= min\{F_{\omega,\vartheta}(\frac{r}{2^{k-1}c}), F_{\varpi\omega,\vartheta}(\frac{r}{2^{k-1}c}).F_{\omega,\varpi\vartheta}(\frac{r}{2^{k-1}c})\}
\end{aligned}$$

Therefore, by Theorem 2.1, ϖ whit $\phi(r) = r$ has a fixed point, which is a solution to the equation 3.1. □

3.2. **Fractional differential equations.** Suppose the Caputo fractional differential equation defined as follows, where D^c_{ϱ} denotes the Caputo fractional derivative of order ϱ:

$$D^c_{\varrho}(\zeta(\iota)) = h(\iota, \zeta(\iota)), \qquad (0 < \iota < 1, 0 < \varrho \leq 2), \tag{3.4}$$

where $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$\zeta(0) = 0, \qquad \zeta(1) = \int_0^{\wp} \zeta(\kappa)d\kappa \quad (0 < \wp < 1).$$

The Caputo derivative of order ϱ for a continuous function $\varpi : \mathbb{R}^+ \to \mathbb{R}$ is defined as follows:

$$D^c_{\varrho}\varpi(\iota) = \frac{1}{\Gamma(n - \varrho)} \frac{d^n}{d\iota^n} \int_0^{\iota} \frac{\varpi(\kappa)}{(\iota - \kappa)^{\varrho-n+1}} d\kappa \qquad (n = [\varrho] + 1),$$

where this expression is valid pointwise on $(0,+\infty)$. Here's a captivating theorem related to the topic:

Theorem 3.2. *Let (Ξ,Θ,F,min) be a complete BIMPbM-space whit $\varrho=\frac{1}{2}$ and $G:\Xi\cup\Theta\to\Xi\cup\Theta$ be a self mapping defined by $G(\zeta)=\frac{\varpi\zeta(\iota)}{4}$. Suppose the following conditions are met:*

(i) *For each $\zeta,\varsigma\in\Xi$ and $\iota\in I(I=[0,1])$, there exists $L>0$ such that*

$$\|h(\iota,\varpi\zeta(\iota))-h(\iota,\varpi\varsigma(\iota))\|\leq\frac{\Gamma(\varrho+1)}{5}e^{-Ls}\frac{1}{4}max[|\zeta(\kappa)-\varsigma(\kappa)|,|\varpi\zeta(\kappa)-\varsigma(\kappa)|,|\zeta(\kappa)-\varpi\varsigma(\kappa)|];$$

(ii) *Exist $\varpi:\mathbb{R}^+\to\mathbb{R}$ such that*

$$\begin{aligned}\varpi\zeta(\iota) &= \frac{1}{\Gamma(\varrho)}\int_0^\iota(\iota-\kappa)^{\varrho-1}h(\kappa,\zeta(\kappa))d\kappa\\ &-\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^1(1-\kappa)^{\varrho-1}h(\kappa,\zeta(\kappa))d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^\wp(\int_0^\kappa(\kappa-k)^{\varrho-1}h(k,\zeta(k))dk)d\kappa.\end{aligned}$$

Thus, (3.4) *has a solution in Ξ.*

Proof. If the following is satisfied:

$$\begin{aligned}\zeta(\iota) &= \frac{1}{\Gamma(\varrho)}\int_0^\iota(\iota-\kappa)^{\varrho-1}h(\kappa,\zeta(\kappa))d\kappa\\ &-\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^1(1-\kappa)^{\varrho-1}h(\kappa,\zeta(\kappa))d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^\wp(\int_0^\kappa(\kappa-k)^{\varrho-1}h(k,\zeta(k))dk)d\kappa,\end{aligned}$$

Then, $\zeta(\kappa)$ is a solution of (3.4). Define $\mu(\zeta,\varsigma)=\max_{\iota\in I}(|\zeta(\iota)-\varsigma(\iota)|^2e^{-Lt})$ for $\zeta,\varsigma\in\Xi$, where L meets condition (i). Now, for all $\zeta\in\Xi$ and $\varsigma\in\Theta$ we have

$$\begin{aligned}\mu(\varpi\zeta,\varpi\varsigma)\leq&\max_{\iota\in I}\frac{1}{\Gamma(\varrho)}\int_0^\iota|\iota-\kappa|^{\varrho-1}\|h(\kappa,\zeta(\kappa))-h(\kappa,\varsigma(\kappa))\|d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^1|1-\kappa|^{\varrho-1}\|h(\kappa,\zeta(\kappa))-h(\kappa,\varsigma(\kappa))\|d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^\wp\|\int_0^\kappa(\kappa-k)^{\varrho-1}(h(k,\zeta(k))-h(k,\varsigma(k)))dk\|d\kappa\\ &\leq\max_{\iota\in I}\frac{1}{\Gamma(\varrho)}\int_0^\iota|\iota-\kappa|^{\varrho-1}\frac{\Gamma(\varrho+1)}{5}e^{-Ls}\frac{1}{4}max[|\zeta(\kappa)-\varsigma(\kappa)|,|\varpi\zeta(\kappa)-\varsigma(\kappa)|,|\zeta(\kappa)-\varpi\varsigma(\kappa)|]d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^1|1-\kappa|^{\varrho-1}\frac{\Gamma(\varrho+1)}{5}e^{-Ls}\frac{1}{4}max[|\zeta(\kappa)-\varsigma(\kappa)|,|\varpi\zeta(\kappa)-\varsigma(\kappa)|,\\ &|\zeta(\kappa)-\varpi\varsigma(\kappa)|]d\kappa\\ &+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^\wp(\int_0^\kappa|\kappa-k|^{\varrho-1}\frac{\Gamma(\varrho+1)}{5}e^{-Ls}\frac{1}{4}max[|\zeta(\kappa)-\varsigma(\kappa)|,|\varpi\zeta(\kappa)-\varsigma(\kappa)|,\end{aligned}$$

$$|\zeta(\kappa) - \varpi\varsigma(\kappa)|]dk)d\kappa$$
$$\leq \frac{\Gamma(\varrho+1)}{5}\frac{1}{4}max[\mu(\zeta,\varsigma),\mu(\varpi\zeta,\varsigma),\mu(\zeta,\varpi\varsigma)](\max_{\iota\in I}\frac{1}{\Gamma(\varrho)}\int_0^{\iota}|\iota-\kappa|^{\varrho-1}d\kappa$$
$$+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^1|1-\kappa|^{\varrho-1}d\kappa$$
$$+\frac{2\iota}{(2-\wp^2)\Gamma(\varrho)}\int_0^{\wp}(\int_0^{\kappa}|\kappa-k|^{\varrho-1}dk)d\kappa)$$
$$\leq \frac{1}{4}max[\mu(\zeta,\varsigma),\mu(\varpi\zeta,\varsigma),\mu(\zeta,\varpi\varsigma)],$$

for any $\zeta \in \Xi, \varsigma \in \Theta$. Putting $c = \frac{1}{2}$, for any $r > 0$ and $k \in \mathbb{N}$ we derive

$$\begin{aligned} F_{\varpi\zeta,\varpi\varsigma}(\frac{r}{2^k}) &= \chi(\frac{r}{2^k} - \mu(\varpi\zeta,\varpi\varsigma)) \\ &\geq \chi(\frac{r}{2^k} - \frac{c}{2}max\{\mu(\zeta,\varsigma),\mu(\varpi\zeta,\varsigma),\mu(\zeta,\varpi\varsigma)\}) \\ &= \chi(\frac{r}{2^{k-1}c} - max\{\mu(\zeta,\varsigma),\mu(\varpi\zeta,\varsigma),\mu(\zeta,\varpi\varsigma)\}) \\ &= min\{F_{\zeta,\varsigma}(\frac{r}{2^{k-1}c}),F_{\varpi\zeta,\varsigma}(\frac{r}{2^{k-1}c}).F_{\zeta,\varpi\varsigma}(\frac{r}{2^{k-1}c})\} \end{aligned}$$

Theorem 2.1 ensures a *FP* of *G*, which solves the Caputo equation (3.4). □

Example 3.1. *Let us consider the following fractional differential equation :*

$$D^c_{\varrho}w(\iota) + w(\iota) = \frac{2}{\Gamma(3-\varrho)}\iota^{2-\varrho} + \iota^3, \tag{3.5}$$

with initial condition: $w(0) = 0$, $w^{'}(0) = 0$.
Equation 3.5 has the exact solution with $\varrho = 1.9$*:*

$$w(\iota) = \iota^2$$

By Equation 3.4, we can express Equation 3.5 in the homotopy form;

$$D^c_{\varrho}w(\iota) + uw(\iota) - \frac{2}{\Gamma(3-\varrho)}\iota^{2-\varrho} - \iota^3 = 0, \tag{3.6}$$

the solution of Equation 3.5 is:

$$w(\iota) = w_0(\iota) + uw_1(\iota) + u^2w_2(\iota) + ... \tag{3.7}$$

Substituting Equation 3.7 in 3.6 and collecting terms with the power of u, we get

$$\begin{cases} u^0 : D^c_{\varrho}w_0(\iota) = 0 \\ u^1 : D^c_{\varrho}w_1(\iota) = -w_0(\iota) + h(\iota) \\ u^2 : D^c_{\varrho}w_2(\iota) = -w_1(\iota) \\ u^3 : D^c_{\varrho}w_3(\iota) = -w_2(\iota). \end{cases}$$

Applying U^{ϱ} and the inverse operation of D^{c}_{ϱ}, on both sides of Equation 3.1 and fractional integral operation U^{ϱ} of order $\varrho > 0$, we have

$$w_0(\iota) = \sum_{i=0}^{1} w^{i}(0)\frac{\iota^{i}}{i!} = w(0)\frac{\iota^{0}}{0!} + w^{'}(0)\frac{\iota^{1}}{1!}$$

$$w_1(\iota) = -U^{\varrho}[w_0(\iota) + U^{\varrho}h(\iota)] = \iota^2 + \frac{\Gamma(4)}{\Gamma(4+\varrho)}\iota^{3+\varrho}$$

$$w_2(\iota) = -U^{\varrho}[w_1(\iota)] = \frac{2}{\Gamma(3+\varrho)}\iota^{2+\varrho} - \frac{6}{\Gamma(3+2\varrho)}\iota^{3+2\varrho}$$

$$w_3(\iota) = -U^{\varrho}[w_2(\iota)] = \frac{2}{\Gamma(3+2\varrho)}\iota^{2+2\varrho} - \frac{6}{\Gamma(3+3\varrho)}\iota^{3+3\varrho}$$

Hence the solution of Equation 3.5 is

$$w(\iota) = w_0(\iota) + w_1(\iota) + w_2(\iota) + ... \tag{3.8}$$

$$w(\iota) = \iota^2 + \frac{\Gamma(4)}{\Gamma(4+\varrho)}\iota^{3+\varrho} - \frac{2}{\Gamma(3+\varrho)}\iota^{2+\varrho} - \frac{6}{\Gamma(3+2\varrho)}\iota^{3+2\varrho} + ... \tag{3.9}$$

when $\varrho = 1.9$

$$\begin{aligned} w(\iota) &= \iota^2 + \frac{6}{\Gamma(5.9)}\iota^{4.9} - \frac{2}{\Gamma(4.9)}\iota^{3.9} - \frac{6}{\Gamma(7.8)}\iota^{6.8} + ... \\ &= \iota^2 - \textit{ small terms} \\ &\approx \iota^2 \end{aligned}$$

For $\varrho = 1.9$ and $\sim= 51$, the results (both numerical and exact) using the matrix approach are presented in Table 1 and the maximum error observed was $\sim= 51$ is 0.039016195358901

ι	$w(\iota)$	$w_{\zeta}(\iota)$	$\lvert w(\iota) - w_{\zeta}(\iota)\rvert$
0.10000	0.01000	0.00862	0.00138
0.20000	0.04000	0.03769	0.00231
0.30000	0.09000	0.08654	0.00346
0.40000	0.16000	0.15474	0.00526
0.50000	0.25000	0.24193	0.00807
0.60000	0.36000	0.34786	0.01214
0.70000	0.49000	0.47244	0.01756
0.80000	0.64000	0.61581	0.02419
0.90000	0.81000	0.77841	0.03159
1.00000	1.00000	0.96098	0.03902

TABLE 1. The numerical and exact solution using the matrix approach method where $\sim= 51$

4. Conclusion

As the *BIMPbM*-spaces is relatively new addition to the existing literature, therefore, in this note, we endeavor to further enrich this notion by introducing the idea of *BIMPbM*-space wherein we combined concept of the *MPbM*-space with concept of the *BIM*-space. Then, by using these concepts, new fixed point were proven. Section 2 introduces new *FP* theorems for single-valued operators in *BIMPbM*-spaces, along with application for solving damped harmonic oscillator, as second-order differential equations and we were able to investigate the existence of a solution for the Caputo fractional differential equations and also give numerical example for equation.

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