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BOUNDS OF CERTAIN DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract. In this paper we study explicit bounds of certain dynamic integral inequalities on time scales. These estimates give the bounds on unknown functions which can be used in studying the qualitative aspects of certain dynamic equations. Using these inequalities we prove the uniqueness of some partial integro-differential equations on time scales.

1. Introduction

In 1989 German Mathematician Stefan Hilger [4] initiated the study of time scale in his Ph.D dissertation. Dynamic inequalities on time scales has applications in various fields. During past few years many authors have studied various types of dynamic equations and inequalities on time scales, its properties and applications [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Motivated by need for the diverse applications and to widen the scope of such inequalities in this paper we obtain some explicit bounds of certain dynamic inequalities on time scales.

2. Main Results

Let $\mathbb{R}$ denotes the real number $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes the arbitrary time scales and $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty)$. Let C_{rd} be the set of all rd continuous function. We assume here basic understaing of time scale calculus. The research monograph [2, 3] gives the basic information on time scales calculus.

Now we give here our main results.

Theorem 2.1 Let $p(t,s), q(t,s) \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}}, \mathbb{R}_+)$ and be nondecreasing for $t \in I_{\mathbb{T}}$ for each $s \in I_{\mathbb{T}}$ and

$$u(t) \le c + \int_{t_0}^{t} p(t,\tau) u(\tau)\,\Delta\tau + \int_{t_0}^{\alpha} q(t,\tau) u(\tau)\,\Delta\tau, \tag{2.1}$$

for $t \in I_{\mathbb{T}}$ where $c \ge 0$ is a constant. If

$$k(t) = \int_{t_0}^{\alpha} q(t,\tau)\, e_p(\tau, s)\Delta\tau < 1, \tag{2.2}$$

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then

$$u(t) \leq \frac{c}{1-k(t)} e_p(t,\tau). \tag{2.3}$$

Proof. For fixed X, $t_0 \leq X \leq \alpha$ then for $t_0 \leq t \leq X$ we have

$$u(t) \leq c + \int_{t_0}^{t} p(X,\tau) u(\tau) \Delta\tau + \int_{t_0}^{\alpha} q(X,\tau) u(\tau) \Delta\tau. \tag{2.4}$$

Define a function $w(t,X)$, $t_0 \leq t \leq X$ by right hand side of (2.4) we get

$$u(t) \leq w(t,X), t_0 \leq t \leq X, \tag{2.5}$$

$$w(t_0,X) = c + \int_{t_0}^{\alpha} q(X,\tau) u(\tau) \Delta\tau, \tag{2.6}$$

and

$$w^{\Delta}(t,X) = p(X,t) u(t) \leq p(X,t) w(t), \tag{2.7}$$

for $t_0 \leq \mathbb{T}$.

By taking $t = s$ and integrating it with respect to s from t_0 to X we have

$$w(X,X) \leq w(t_0,X) e_p(t_0,X). \tag{2.8}$$

Since X is arbitrary in (2.5) and (2.8) X replaced by t and $u(t) \leq w(t,t)$ we get

$$u(t) \leq w(t_0,t) e_p(t_0,s), \tag{2.9}$$

where

$$w(t_0,t) = c + \int_{t_0}^{\alpha} q(t,\tau) u(\tau) \Delta\tau. \tag{2.10}$$

Using (2.9) on the right hand side of (2.10) and by (2.2) we have

$$w(t_0,t) \leq \frac{c}{1-k(t)}. \tag{2.11}$$

Using (2.11) in (2.9) we get the result

Theorem 2.2 Let $f(t,\tau), g(t,\tau), h(t,\tau) \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}}, \mathbb{R}_+)$, $f(t,\tau), g(t,\tau)$ are nondecreasing in t for each $\tau \in I_{\mathbb{T}}$ and if

$$u(t) \leq c + \int_{t_0}^{t} f(t,\tau) \left[u(\tau) + \int_{t_0}^{\tau} h(\tau,s) u(s) \Delta s \right] \Delta\tau + \int_{t_0}^{\alpha} g(t,\tau) u(\tau) \Delta\tau. \tag{2.12}$$

for $t \in I_{\mathbb{T}}$. Then

$$u(t) \leq \frac{c}{1-k_1(t)} e_f(t_0,\tau), \tag{2.13}$$

where

$$k_1(t) = \int_{t_0}^{\tau} g(t,\tau) e_F(t_0,\tau) \Delta\tau < 1, \tag{2.14}$$

$$F(t,\tau)=f(t,\tau)\left[1+\int_{t_0}^{\tau}h(\tau,\xi)\,\Delta\xi\right]. \tag{2.15}$$

Proof. Let $c>0$ and for any fix $X\in I_{\mathbb{T}}$ then for $t_0\le t\le X$. From (2.13) we have

$$u(t)\le c+\int_{t_0}^{t}f(X,\tau)\left[u(\tau)+\int_{t_0}^{\tau}h(\tau,s)\,u(s)\,\Delta s\right]\Delta\tau+\int_{t_0}^{\alpha}g(X,\tau)u(\tau)\,\Delta\tau. \tag{2.16}$$

Define a function $w(t,X)$, $t\in[t_0,X]$, $u(t)\le w(t,X), w(t,x)>0$

$$w(t_0,X)=c+\int_{t_0}^{\alpha}g(X,\tau)u(\tau)\,\Delta\tau, \tag{2.17}$$

and

$$\begin{aligned} w^{\Delta}(t,X)&=f(X,t)\left[u(\tau)+\int_{t_0}^{t}h(t,s)\,u(s)\,\Delta s\right]\\ &\le f(X,t)\left[w(t)+\int_{t_0}^{t}h(t,s)\,w(s,X)\,\Delta s\right]. \end{aligned} \tag{2.18}$$

From (2.17) and since $w(t,X)$ is nondecreasing in t we have

$$\frac{w^{\Delta}(t,X)}{w(t,X)}\le f(X,t)\left[1+\int_{t_0}^{t}h(t,s)\,\Delta s\right], \tag{2.19}$$

for $t\in[t_0,X]$.

Now taking $t=\xi$ and integrating with respect to ξ from t_0 to X we get

$$w(X,X)\le w(t_0,X)\,e_F(t_0,\tau) \tag{2.20}$$

Since X is arbitrary with X replaced by t we have for $t\in I_{\mathbb{T}}$,

$$w(t,t)\le w(t_0,t)\,e_F(t_0,\tau), \tag{2.21}$$

and

$$w(t_0,t)=c+\int_{t_0}^{\alpha}g(t,\tau)u(\tau)\,\Delta\tau, \tag{2.22}$$

for $t\in I_{\mathbb{T}}$.

Since $u(t)\le w(t,t)$ we get from (2.21)

$$u(t)\le w(t_0,t)\,e_F(t_0,\tau). \tag{2.23}$$

Now from (2.23), (2.22) and from (2.14) we have

$$w(t_0,t)\le\frac{c}{1-k_1(t)}. \tag{2.24}$$

Using (2.24) in (2.25) we get

$$u(t)\le\frac{c}{1-k_1(t)}e_F(t_0,\tau). \tag{2.25}$$

Theorem 2.3 Let $a, b, c \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$ and

$$u(t) \leq c + \int_{t_0}^{t} a(\tau) \left[u(\tau) + \int_{t_0}^{\tau} b(s) u(s) \Delta s + \int_{t_0}^{\alpha} d(s) u(s) \Delta s \right] \Delta\tau, \tag{2.26}$$

$$z = \int_{t_0}^{\alpha} d(s) e_{a+b}(t_0, s) \Delta s < 1, \tag{2.27}$$

then

$$u(t) \leq \frac{c}{1-z} e_{a+b}(t_0, \tau), \tag{2.28}$$

for $t \in I_{\mathbb{T}}$.

Proof. Now define a function $w(t)$ by right hand side of (2.26) then $w(t_0) = c, u(t) \leq w(t)$ and

$$\begin{aligned} w^{\Delta}(t) &= a(t) \left[u(t) + \int_{t_0}^{t} b(s) u(s) \Delta s + \int_{t_0}^{\alpha} d(s) u(s) \Delta s \right] \\ &\leq a(t) \left[w(t) + \int_{t_0}^{t} b(s) w(s) \Delta s + \int_{t_0}^{\alpha} d(s) w(s) \Delta s \right], \end{aligned} \tag{2.29}$$

for $t \in I_T$. Now define a function $v(t)$ by

$$v(t) = w(t) + \int_{t_0}^{t} b(s) w(s) \Delta s + \int_{t_0}^{\alpha} d(s) w(s) \Delta s, \tag{2.30}$$

then $w(t) \leq v(t)$, $w^{\Delta}(t) \leq a(t) v(t)$

$$v(t_0) = c + \int_{t_0}^{\alpha} d(s) w(s) \Delta s, \tag{2.31}$$

and

$$\begin{aligned} v^{\Delta}(t) &= w^{\Delta}(t) + b(t) w(t) \\ &\leq a(t) v(t) + b(t) w(t) \\ &\leq [a(t) + b(t)] w(t). \end{aligned} \tag{2.32}$$

We get

$$v(t) \leq v(t_0) e_{a+b}((t_0, t)), \tag{2.33}$$

for $t \in I_{\mathbb{T}}$. Using (2.33) in $w(t) \leq v(t)$ we have

$$w(t) \leq v(t_0) e_{a+b}(t_0, t). \tag{2.34}$$

Now from (2.33), (2.31) and from (2.27) we have

$$v(t_0) \leq \frac{c}{1-z}, \tag{2.35}$$

Using (2.34) in (2.33) and we have $u(t) \leq w(t)$ we get (2.28).

3. Applications

Now we consider the following dynamic equation

$$y(t) = a(t) + \int_{t_0}^{t} G\left(t, \tau, y(\tau), \int_{t_0}^{\tau} b(\tau, s, y(\tau))\,\Delta s\right)\Delta\tau$$

$$+\int_{t_0}^{\alpha} d(t, \tau, x(\tau))\Delta\tau \tag{3.1}$$

for $t \in I_{\mathbb{T}}$ where $y(t)$ is unknown function and $\alpha \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$, $b, d \in C_{rd}(I_{\mathbb{T}} \times \mathbb{R}_n, \mathbb{R}_n)$, $G \in C_{rd}(I_{\mathbb{T}} \times \mathbb{R}_n \times \mathbb{R}_n, \mathbb{R}_n)$.

Now we give the application of Theorem 2.2 for studying certain properties of solution of equation (3.1).

Theorem 3.1 Suppose that the function a, b, d, G as in (3.1) satisfy the conditions

$$|a(t)| \le c, \tag{3.2}$$

$$|b(t, \tau, y)| \le h(t, \tau)\,|y|, \tag{3.3}$$

$$|d(t, \tau, y)| \le g(t, \tau)\,|y|, \tag{3.4}$$

$$|G(t, \tau, y, x)| \le f(t, \tau)\,(|y| + |x|), \tag{3.5}$$

where $f(t,\tau), g(t,\tau), h(t,\tau)$ and c are as given in Theorem 2.2. Let $k_1(t)$ be as in (2.14). If $y(t), t \in I_{\mathbb{T}}$ is a solution of (3.1) then

$$|y(t)| \le \frac{c}{1 - k_1(t)} e_f(\tau, t_0), \tag{3.6}$$

for $t \in I_{\mathbb{T}}$ where F is defined by (2.15).

Proof. Since $y(t)$ is solution of equation (3.1) and using $(3.2) - (3.5)$ we get

$$|y(t)| \le c + \int_{t_0}^{t} f(t, \tau)\left(|y(x)| + \int_{t_0}^{\tau} h(\tau, s)\,|y(\tau)|\Delta s\right)\Delta\tau$$

$$+\int_{t_0}^{\alpha} g(t, \tau)\,|y(\tau)|\,\Delta\tau \tag{3.7}$$

Now an application of Theorem 2.2 to (3.7) we get (3.6).

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