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On the Negative Spectrum of One-Dimensional Schrödinger Operators on Quantum Trees with Point Interactions

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Abstract. We present an explicit algorithm to determine the number of negative eigenvalues of Schrödinger operators on rooted quantum trees equipped with delta or delta-prime vertex interactions. We employ the methods of [Behrndt and Luger [5]], and the structure of trees to generate a sequence which has the same number of negative elements as the original Laplace operator. We show that the number of negative eigenvalues of the Schrödinger operators with delta interactions equals the number of negative terms in this sequence, while for delta-prime interactions, it reduces to the number of negative interaction strengths.

1. Introduction

Schrödinger operators with potentials localized on a finite or discrete set of points are commonly referred to as solvable models in quantum mechanics. These models are termed "solvable" because their resolvents can be explicitly computed in terms of interaction strengths and source locations. Consequently, their spectral properties, including the spectrum and eigenfunctions, can be determined in a closed form. Such models have been widely studied in the physics literature, particularly in the contexts of atomic, nuclear, and solid-state physics. In many cases one is particularly interested in the spectral properties of its self-adjoint operator on the graph. The aim of this paper is to derive a formula for the number of negative eigenvalues of the Schrödinger operator over quantum tree graphs with point iterations.

Let a metric graph Γ and consider an arbitrary edge of finite length $e_i \in \mathcal{I}$ of Γ. We identify it with an interval $[0, d_i]$ and denote the space of all square integrable function defined on $[0, d_i]$

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by $L^2((0,d_i))$. Similarly for an infinite length edge $e_j \in \mathcal{E}$, identified with interval $(0,\infty)$, we can consider space of all square integrable functions $L^2((0,\infty))$. Now, the space of all square integrable functions on the graph can be defined as:

$$L^2(\Gamma) := \bigoplus_{e_i \in \mathcal{I}} L^2((0,d_i)) + \bigoplus_{e_j \in \mathcal{E}} L^2((0,\infty)).$$

For a vertex v_k with degree greater than two the delta vertex conditions can be compactly written as:

$$\begin{cases} f \text{ is continuous at } v_k \\ \sum_{x_i \in v_k} \partial f(x_i) = \alpha_k f(v_k) \end{cases} \tag{1.1}$$

where $\alpha_k \in \mathbb{R}$. The symbol ∂ denotes the normal derivative which is defined as

$$\partial f(x_i) = \begin{cases} \lim\limits_{x \to x_i} \dfrac{df}{dx}(x), \quad x_i \quad \text{is the left end point of the interval} \\ -\lim\limits_{x \to x_i} \dfrac{df}{dx}(x), \quad x_i \quad \text{is the right end point of the interval.} \end{cases}$$

Similarly for vertex v_k the delta-prime vertex conditions are given by

$$\begin{cases} \partial f \text{ is continuous at } v_k \\ \sum\limits_{x_i \in v_k} f(x_i) = \beta_k \, \partial f(v_k). \end{cases} \tag{1.2}$$

where the potentials $\beta_k \in \mathbb{R}$. For detailed study on the most general form of self-adjoint Vertex conditions we refer to [3, 10–12].

The objective of this paper is to investigate some spectral properties of Schrödinger operator with delta point and deltra-prime point interations over quantum trees. In the space $L^2(\mathbb{R})$, they are given by

$$L_{V,\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k \in I} \alpha_k \delta_k(v), \;\; L_{V,\beta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{k \in I} \beta_k \langle \cdot, \delta_k^{'} \rangle \delta_k^{'}(v)$$

The following result can be used to calculate the $\kappa_-(.)$ of the Schrödinger operator with a general form $A\underline{f} + B\underline{f}' = 0$ of self-adjoint vertex conditions at vertices.

Theorem 1.1 (Behrndt and Luger [5])**.** *Consider a connected finite graph* Γ*, and take a self-adjoint realization of the Laplacian L in* $L^2(\Gamma)$*, that is,*

$$L = -\frac{d^2}{dx^2}, \quad \text{domain}(L) = \left\{ f \in W_2^2(\Gamma) : A\underline{f} + B\underline{f}' = 0 \right\}$$

where A and B are matrices, and M is the Titchmarsh Weyl M-function. Then L satisfies

$$\kappa_-(L) = \kappa_-\left(-AB^* - BMB^*\right). \tag{1.3}$$

In [6], the authors give an algorithm for the negative eigenvalues of Laplacian with finitely many delta and delta prime interactions on the real line. We present their results below.

Consider the Laplace operator $L_{V,\alpha}$ defined on the real line with finitely many delta interactions all separated by a finite distance d_i. The set of points where the delta potential is placed is denoted by V.

We consider that each vertex v_i is equipped with delta conditions having strength α_i. That is

$$v_1 : \left\{ \begin{array}{c} f_0(0) = f_1(0) \\ f_0'(0) + f_1'(0) = \alpha_1 f_0(0) \end{array} \right\},$$

$$v_2 : \left\{ \begin{array}{c} f_1(d_1) = f_2(0) \\ -f_1'(d_1) + f_2'(0) = \alpha_2 f_1(d_1) \end{array} \right\},$$

$$v_3 : \left\{ \begin{array}{c} f_2(d_2) = f_3(0) \\ -f_2'(d_2) + f_3'(0) = \alpha_3 f_2(d_2) \end{array} \right\},$$

$$\vdots$$

$$v_n : \left\{ \begin{array}{c} f_{n-1}(d_{n-1}) = f_n(0) \\ -f_{n-1}'(d_{n-1}) + f_n'(0) = \alpha_n f_{n-1}(d_{n-1}) \end{array} \right\}.$$

Define the sequence $\gamma = \{\gamma_k\}_{k=1}^n$ by

$$\begin{aligned} &\gamma_1 = \alpha_1 + {d_1}^{-1}, \\ &\gamma_k = \alpha_k + {d_k}^{-1} + {d_{k-1}}^{-1} - {d_{k-1}}^{-2}{\gamma_{k-1}}^{-1}, \quad \text{for} \quad k = \{2, 3, \ldots, n-1\}, \\ &\gamma_n = \alpha_n + {d_{n-1}}^{-1} - d_{n-1}^{-2}{\gamma_{n-1}}^{-1}, \\ &\text{if} \quad \gamma_k = 0, \quad \text{then} \quad \gamma_{k+1} := -\infty. \end{aligned}$$

The result which relates the number of negative terms in the sequence $\gamma = \{\gamma_k\}_{k=1}^n$ to the $\kappa_-(L_{V,\alpha})$ of the Laplace operator on the real line is the following [6]

$$\kappa_-(L_{V,\alpha}) = \kappa_-(\gamma) + N_\infty(\gamma).$$

Here N_∞ is the the number of negative infinite $\gamma = \{\gamma_k\}_{k=1}^n$ within the sequence.

we take the following delta-prime conditions on each vertex v_i.

$$v_1 : \left\{ \begin{array}{c} f_0'(0) = f_1'(0) \\ f_0(0) + f_1(0) = \beta_1 f_0'(0) \end{array} \right\},$$

$$v_2 : \left\{ \begin{array}{c} -f_1'(d_1) = f_2'(0) \\ f_1(d_1) + f_2(0) = -\beta_2 f_1'(d_1) \end{array} \right\},$$

$$v_3 : \left\{ \begin{array}{c} -f_2'(d_2) = f_3'(0) \\ f_2(d_2) + f_3(0) = -\beta_3 f_2'(d_2) \end{array} \right\},$$

$$\vdots$$

$$v_n : \left\{ \begin{array}{c} -f_{n-1}'(d_{n-1}) = f'(0) \\ f_{n-1}(d_{n-1}) + f(0) = -\beta_n f_{n-1}'(d_{n-1}) \end{array} \right\}.$$

We define the sequence γ_i as follows

$$\gamma_1 = {d_1}^{-1},$$
$$\gamma_2 = \beta_1 \ ,$$
$$\gamma_3 = {d_2}^{-1},$$
$$\gamma_4 = \beta_2 \ ,$$
$$\vdots$$
$$\gamma_{2n-3} = {d_{n-1}}^{-1},$$
$$\gamma_{2n-2} = \beta_{n-1},$$
$$\gamma_{2n-1} = \beta_n.$$

The number of negative eigenvalues $\kappa_-(L_{V,\beta})$ is given by [6].

$$\kappa_-(L_{V,\beta}) = \kappa_-(\beta_1) + \kappa_-(\beta_2) + \ldots + \kappa_-(\beta_n).$$

2. Main Results

In this section we present our results on the negative eigenvalues of Schrödinger operator on the quantum tree graphs. Our main results are the following

Theorem 2.1. *The number of negative eigenvalues κ_- of the Schrödinger operator on Γ with delta interactions is*

$$\kappa_- = \#\{\gamma_{i,j} < 0\}.$$

Theorem 2.2. *The number of negative eigenvalues κ_- of the Schrödinger operator on Γ with delta-prime interactions is*

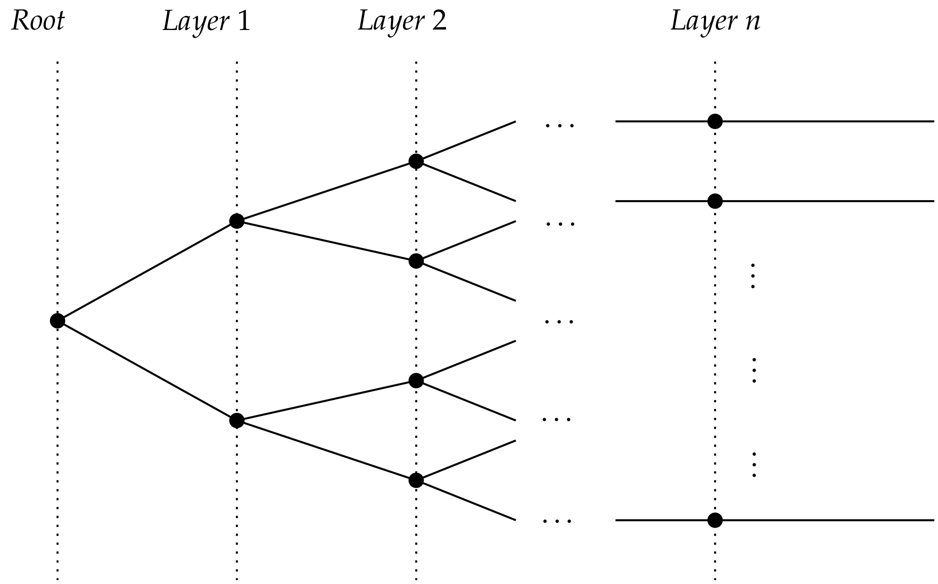
$$\kappa_- = \#\{\beta_{i,j} < 0\}.$$

3. Proofs

3.1. **The General Case.**

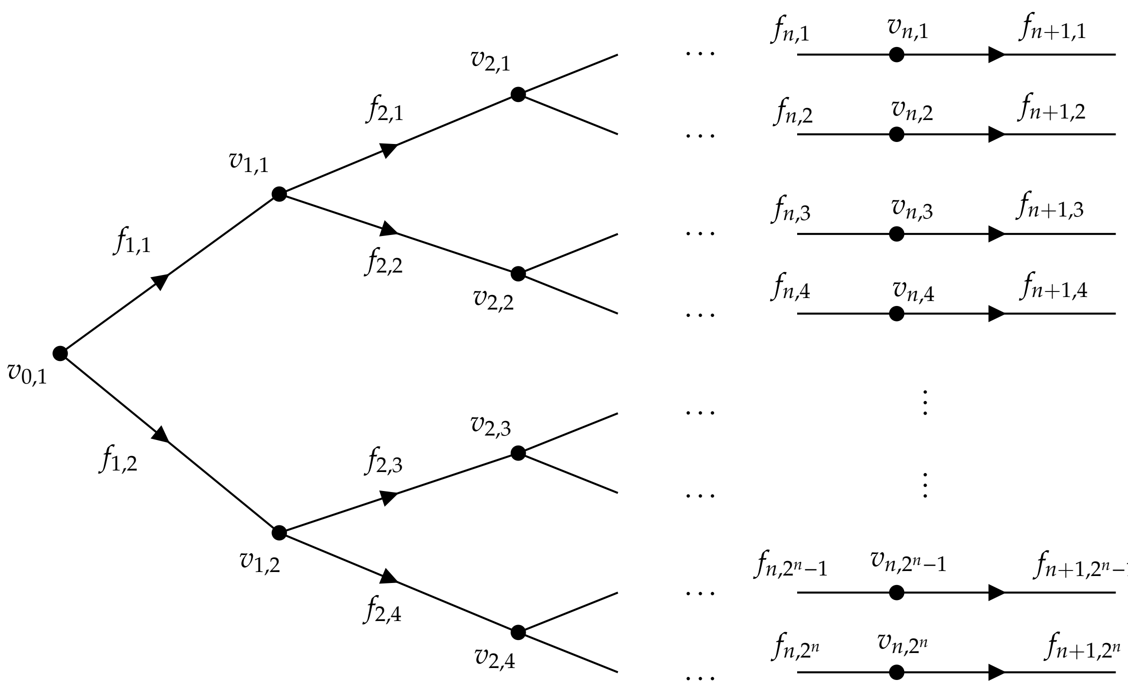
In this section we look at the result of the matrix $(-AB^* - BMB^*)$ for any vertex conditions that satisfy $A\underline{f} + B\underline{f}' = 0$. We set up the notation for the tree graph and highlight some properties of the tree structure that we take advantage of.

We collect vertices into sets called layers $l = \{l_i\}_{i=0}^n$ where n is the total number of layers. And $|l_i|$ is the number of vertices in l_i. They are numbered starting with the root as zero and successively increasing the index with each connected vertex.



Each layer l_i contains the vertices $\{v_{i,j}\}_{j=1}^{|l_i|}$ where $j \in \{1, \ldots, |l_i|\}$ is the index for vertices and $i \in \{0, \ldots, n\}$ is the index for layers. This indexing used for functions $f_{i,j}$ and distances $d_{i,j}$ such that it matches the vertex at its right end-point.

For example, a regular tree graph that branches out to two vertices in each successive layer.



We also group the connected vertices of any vertex $v_{i,j}$ into two sets: $l^+(v_{i,j})$ for all connected vertices in the next layer l_{i+1} and $l^-(v_{i,j})$ for all connected vertices in the previous layer l_{i-1}.

Before converting the vertex conditions to the form $A\underline{f} + B\underline{f}' = 0$ we arrange them according to their layer in the order $\{n, n-1, \ldots, 1, 0\}$, this causes the matrix $(-AB^* - BMB^*)$ to have the following structure

$$(-AB^* - BMB^*) = \begin{bmatrix} [P_n] & [Q_n] & 0 & 0 & \dots & 0 & 0 \\ [Q_n]^T & [P_{n-1}] & [Q_{n-1}] & 0 & \dots & 0 & 0 \\ 0 & [Q_{n-1}]^T & [P_{n-2}] & [Q_{n-2}] & \dots & 0 & 0 \\ 0 & 0 & [Q_{n-2}]^T & [P_{n-3}] & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & [P_1] & [Q_1] \\ 0 & 0 & 0 & 0 & \dots & [Q_1] & [P_0] \end{bmatrix}, \tag{3.1}$$

where $[P_i]$ contains the vertex information on the layer l_i , $[Q_i]$ contains information on the connections of vertices between layers l_i and l_{i-1}. The 0 represent zero matrices. For each $[P_i]$ the internal structure is given by

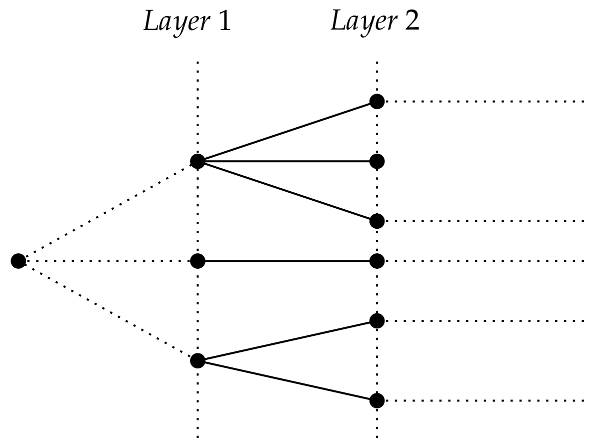
$$[P_i] = \begin{bmatrix} [p_{i,1}] & 0 & \dots & 0 \\ 0 & [p_{i,2}] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & [p_{i,|l_i|}] \end{bmatrix},$$

where $[p_{i,j}]$ are decided by the vertex conditions. We have arranged the vertex conditions in a decreasing order of layers l_i to take advantage of the tree structure as each vertex $v_{i,j}$ will only be connected to one vertex in the previous layer $\left|l^-(v_{i,j})\right| = 1$. We define $[Q_i] := Q_i^v\ Q_i^c$ where $[Q_i^v] = \left[[q_{i,1}] \quad [q_{i,2}] \quad \cdots \quad [q_{i,|l_i|}],\right]$ where these $[q_{i,j}]$ are decided by the vertex conditions. Q_i^c is a matrix that contains information on how each vertex between the layers l_i and l_{i-1} are connected. In Q_i^c the rows represent the vertices of layer l_i and the columns represent the vertices of layer l_{i-1}. The ones at the intersection of these rows and columns mean that their respective vertices are connected while the zeros represent no connection.

$$\begin{array}{c} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \vdots \\ v_{i,|l_i|-1} \\ v_{i,|l_i|} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ \begin{matrix} v_{i-1,1} & v_{i-1,2} & \cdots & v_{i-1,|l_{i-1}|} \end{matrix} \end{array}.$$

We have arranged the vertex conditions in a decreasing order of layers l_i to take advantage of the tree structure as each vertex $v_{i,j}$ will only be connected to one vertex in the previous layer $\left|l^-(v_{i,j})\right| = 1$.

Consider the relation between the two layers shown in the following figure.



The Q_2^c for the connections between these two layers is given by the following matrix

$$\begin{array}{c} v_{2,1} \\ v_{2,2} \\ v_{2,3} \\ v_{2,4} \\ v_{2,5} \\ v_{2,6} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \\ \qquad\; v_{1,1} \quad v_{1,2} \quad v_{1,3}$$

So the $[Q_2]$ in this case will be

$$[Q_2] = \begin{bmatrix} [q_{2,1}] & 0 & 0 \\ [q_{2,2}] & 0 & 0 \\ [q_{2,3}] & 0 & 0 \\ 0 & [q_{2,4}] & 0 \\ 0 & 0 & [q_{2,5}] \\ 0 & 0 & [q_{2,6}] \end{bmatrix}.$$

In the following section we use this machinery specifically for the case of tree graphs where each vertex is equipped with delta conditions.

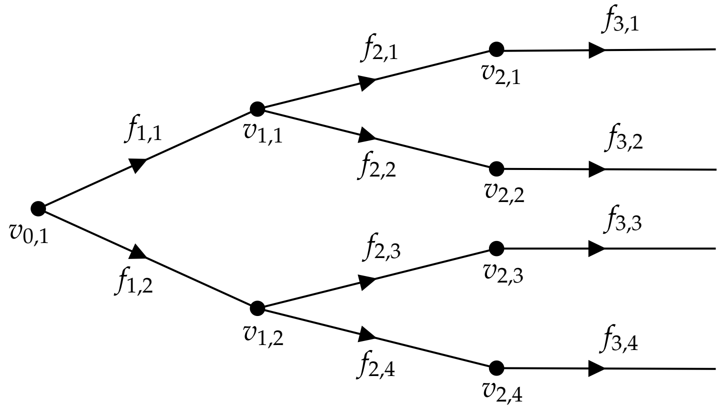
3.2. **The Delta Interaction Case.** We are considering the Laplace operator $L_{V,\alpha}$ with delta interactions (1.1) on its vertices. We start with the resultant matrix $(-AB^* - BMB^*)$ from (1.3). Its follows the form in (3.1). The vertex information $p_{i,j}$ with delta vertex interactions consists of the potentials and all the attached distances

$$p_{i,j} := \alpha_{i,j} + {d_{i,j}}^{-1} + \sum_{r \in J(i)} {d_{i+1,r}}^{-1},$$

where $J(i) := \{r \text{ such that } v_{i+1,r} \in l^+(v_{i,j})\}$. The $q_{i,j}$ are given below

$$q_{i,j} := -d_{i,j}.$$

We first demonstrate our result by applying Lemma 4 in [18] on $(-AB^* - BMB^*)$ for a simple example of tree graph with three layers as shown in the figure below



The indexing of vertices matches with their preceding distance and function. The resultant form (3.1) requires that the vertex interactions by arranged in the order $\{n, n-1, \ldots, 2, 1, 0\}$. The arrangement of the vertices within the layer does not affect the resulting sequence just rearranges their positions.

$$Layer2: \left\{ \begin{array}{l} v_{2,1}: \left\{ \begin{array}{c} f_{2,1}(d_{2,1}) = f_{3,1}(0) \\ -f'_{2,1}(d_{2,1}) + f'_{3,1}(0) = \alpha_{2,1} f_{2,1}(d_{2,1}) \end{array} \right\} \\ v_{2,2}: \left\{ \begin{array}{c} f_{2,2}(d_{2,2}) = f_{3,2}(0) \\ -f'_{2,2}(d_{2,2}) + f'_{3,2}(0) = \alpha_{2,2} f_{2,2}(d_{2,2}) \end{array} \right\} \\ v_{2,3}: \left\{ \begin{array}{c} f_{2,3}(d_{2,3}) = f_{3,3}(0) \\ -f'_{2,3}(d_{2,3}) + f'_{3,3}(0) = \alpha_{2,3} f_{2,3}(d_{2,3}) \end{array} \right\} \\ v_{2,4}: \left\{ \begin{array}{c} f_{2,4}(d_{2,4}) = f_{3,4}(0) \\ -f'_{2,4}(d_{2,4}) + f^{2,4}_{3,4}(0) = \alpha_{2,4} f_{2,4}(d_{2,4}) \end{array} \right\} \end{array} \right. ,$$

$$Layer1: \left\{ \begin{array}{l} v_{1,1}: \left\{ \begin{array}{c} f_{1,1}(d_{1,1}) = f_{2,1}(0) \\ f_{2,1}(0) = f_{2,2}(0) \\ -f'_{1,1}(d_{1,1}) + f'_{2,1}(0) + f'_{2,2}(0) = \alpha_{1,1} f_{1,1}(d_{1,1}) \end{array} \right\} \\ v_{1,2}: \left\{ \begin{array}{c} f_{1,2}(d_{1,2}) = f_{2,3}(0) \\ f_{2,3}(0) = f_{2,4}(0) \\ -f'_{1,2}(d_{1,2}) + f'_{2,3}(0) + f'_{2,4}(0) = \alpha_{1,2} f_{1,2}(d_{1,2}) \end{array} \right\} \end{array} \right. ,$$

$$Root: \left\{ v_{0,1}: \left\{ \begin{array}{c} f_{1,1}(0) = f_{1,2}(0) \\ f'_{1,1}(0) + f'_{1,2}(0) = \alpha_{0,1} f_{1,1}(0) \end{array} \right\} \right. .$$

Now we compute the resultant matrix $-AB^* - BMB^*$ from (1.3) for these vertex conditions after converting them to the form $A\underline{f} + b\underline{f'} = 0$ where M is the M-function for this case. The zero rows and columns have already been removed.

$$\begin{bmatrix} \alpha_{2,1}+{d_{2,1}}^{-1} & 0 & 0 & 0 & -{d_{2,1}}^{-1} & 0 & 0 \\ 0 & \alpha_{2,2}+{d_{2,2}}^{-1} & 0 & 0 & -{d_{2,2}}^{-1} & 0 & 0 \\ 0 & 0 & \alpha_{2,3}+{d_{2,3}}^{-1} & 0 & 0 & -{d_{2,3}}^{-1} & 0 \\ 0 & 0 & 0 & \alpha_{2,4}+{d_{2,4}}^{-1} & 0 & -{d_{2,4}}^{-1} & 0 \\ -{d_{2,1}}^{-1} & -{d_{2,2}}^{-1} & 0 & 0 & \alpha_{1,1}+{d_{1,1}}^{-1}+{d_{2,1}}^{-1}+{d_{2,2}}^{-1} & 0 & -{d_{1,1}}^{-1} \\ 0 & 0 & -{d_{2,3}}^{-1} & -{d_{2,4}}^{-1} & 0 & \alpha_{1,2}+{d_{1,2}}^{-1}+{d_{2,3}}^{-1}+{d_{2,4}}^{-1} & -{d_{1,2}}^{-1} \\ 0 & 0 & 0 & 0 & -{d_{1,1}}^{-1} & -{d_{1,2}}^{-1} & \alpha_{0,1}+{d_{1,1}}^{-1}+{d_{1,2}}^{-1} \end{bmatrix}$$

Now we apply Lemma 4 in [18], to decompose this matrix to obtain the following sequence of equations

$$\gamma_{2,1} = \alpha_{2,1} + {d_{2,1}}^{-1},$$

$$\gamma_{2,2} = \alpha_{2,2} + {d_{2,2}}^{-1},$$

$$\gamma_{2,3} = \alpha_{2,3} + {d_{2,3}}^{-1},$$

$$\gamma_{2,4} = \alpha_{2,4} + {d_{2,4}}^{-1},$$

$$\gamma_{1,1} = \alpha_{1,1} + {d_{1,1}}^{-1} + {d_{2,1}}^{-1} + {d_{2,2}}^{-1} - \gamma_{2,1}^{-1}{d_{2,1}}^{-2} - \gamma_{2,2}^{-1}{d_{2,2}}^{-2},$$

$$\gamma_{1,2} = \alpha_{1,2} + {d_{1,2}}^{-1} + {d_{2,3}}^{-1} + {d_{2,4}}^{-1} - \gamma_{2,3}^{-1}{d_{2,3}}^{-2} - \gamma_{2,4}^{-1}{d_{2,4}}^{-2},$$

$$\gamma_{0,1} = \alpha_{0,1} + {d_{1,1}}^{-1} + {d_{1,2}}^{-1} - \gamma_{1,1}^{-1}{d_{1,1}}^{-2} - \gamma_{1,2}^{-1}{d_{1,2}}^{-2}.$$

For a regular tree graph where each vertex has 2 branches in the next layer, the κ_- of $L_{V,\alpha}$ depends on the number of negative $\gamma_{i,j}$ which are given by

$$\gamma_{n,j} = \alpha_{n,j} + {d_{n,j}}^{-1},$$

$$\gamma_{i,j} = \alpha_{i,j} + {d_{i,j}}^{-1} + \sum_{r=2j-1}^{2j} {d_{i+1,r}}^{-1} - \sum_{r=2j-1}^{2j} {\gamma_{i+1,r}}^{-1}{d_{i+1,r}}^{-2},$$

$$\gamma_{0,1} = \alpha_{0,1} + \sum_{r=1}^{2} {d_{1,r}}^{-1} - \sum_{r=1}^{2} {\gamma_{1,r}}^{-1}{d_{1,r}}^{-2}.$$

For a regular tree graph with c number of branches emanating from each vertex, we have the following result

$$\gamma_{n,j} = \alpha_{n,j} + {d_{n,j}}^{-1},$$

$$\gamma_{i,j} = \alpha_{i,j} + {d_{i,j}}^{-1} + \sum_{r=cj-(c-1)}^{cj} {d_{i+1,r}}^{-1} - \sum_{r=cj-(c-1)}^{cj} {\gamma_{i+1,r}}^{-1}{d_{i+1,r}}^{-2},$$

$$\gamma_{0,1} = \alpha_{0,1} + \sum_{r=1}^{c} {d_{1,r}}^{-1} - \sum_{r=1}^{c} {\gamma_{1,r}}^{-1}{d_{1,r}}^{-2}.$$

The following is a general result for any metric tree

$$\gamma_{n,j} = \alpha_{n,j} + {d_{n,j}}^{-1},$$
$$\gamma_{i,j} = \alpha_{i,j} + {d_{i,j}}^{-1} + \sum_{r\in J(i,j)} {d_{i+1,r}}^{-1} - \sum_{r\in J(i,j)} {\gamma_{i+1,r}}^{-1}{d_{i+1,r}}^{-2},$$
$$\gamma_{0,1} = \alpha_{0,1} + \sum_{r\in J(0,1)} {d_{1,r}}^{-1} - \sum_{r\in J(0,1)} {\gamma_{1,r}}^{-1}{d_{1,r}}^{-2}.$$

where $J(i,j) := \{r \text{ such that } v_{i+1,r} \in l^+(v_{i,j})\}$ which returns the vertex index of the vertices attached to $v_{i,j}$ in the next layer. There also occur negative infinities

$$\text{if} \quad \gamma_{i,j} = 0 \quad \text{then} \quad \gamma_{i-1,\overline{J(i)}} = -\infty,$$

where $\overline{J(i,j)} := \{t \text{ such that } v_{i-1,t} \in l^-(v_{i,j})\}$ which returns the vertex index of the vertices attached to $v_{i,j}$ in the next layer.. The $\kappa_-(L_{V,\alpha})$ is equivalent to the number of negative $\gamma_{i,j}$.

$$\kappa_-(L_{V,\alpha}) = \kappa_-(\gamma)$$

where $\gamma = \gamma_{i,j}$ such that $i \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{N}$. Counting the number of negative entries in the sequence will give you the $\kappa_-(L_{V,\alpha})$.

3.3. **The Delta Prime Interaction Case.**

We now consider the Laplace operator $L_{V,\beta}$ with delta prime interactions (1.2). The same considerations form the general case apply. The sequence of the vertex conditions starts at the largest layer and goes backwards.

The resultant matrix $(-AB^* - BMB^*)$ (1.3) in the delta prime case has the structure shown in (3.1) along with the internal structure of P_i and Q_i. The general vertex information occupying is

$$[p_{i,j}] = \begin{bmatrix} d_{i,j}^{-1} + d_{i+1,r_1}^{-1} & d_{i,j}^{-1} & \dots & d_{i,j}^{-1} & -\beta_{i,j}d_{i,j}^{-1} \\ d_{i,j}^{-1} & d_{i,j}^{-1} + d_{i+1,r_2}^{-1} & \dots & d_{i,j}^{-1} & -\beta_{i,j}d_{i,j}^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{i,j}^{-1} & d_{i,j}^{-1} & \dots & d_{i,j}^{-1} + d_{i+1,r_m}^{-1} & -\beta_{i,j}d_{i,j}^{-1} \\ -\beta_{i,j}d_{i,j}^{-1} & -\beta_{i,j}d_{i,j}^{-1} & \dots & -\beta_{i,j}d_{i,j}^{-1} & \beta_{i,j} - \beta_{i,j}^2 d_{i,j}^{-1} \end{bmatrix},$$

with $\{r_1, r_2, \dots, r_m\} \in J(i,j)$ where m is the number vertices in $l^+(v_{i,j})$. And the connectivity information in Q_i is a matrix with the same number if rows as $p_{i,j}$ and the same number of columns as $p_{i-1,\overline{J(i)}}$. It is given in the form

$$[q_{i,j}] = \begin{bmatrix} 0 & \dots & -d_{i,j}^{-1} & \dots & 0 \\ 0 & \dots & -d_{i,j}^{-1} & \dots & 0 \\ 0 & \dots & \vdots & \dots & 0 \\ 0 & \dots & -d_{i,j}^{-1} & \dots & 0 \\ 0 & \dots & \beta_{i,j}d_{i,j}^{-1} & \dots & 0 \end{bmatrix}$$

where the all columns are zero other than one and this column aligns with the diagonal entry of $p_{i-1,\overline{J(i)}}$ that contains $d_{i,j}^{-1}$.

For the nth layer the diagonal entries of P_n are

$$[p_{n,j}] = \begin{bmatrix} d_{n,j}^{-1} & -\beta_{n,j} d_{n,j}^{-1} \\ -\beta_{n,j} d_{n,j}^{-1} & \beta_{n,j} - \beta_{n,j}^2 d_{n,j}^{-1} \end{bmatrix}$$

and their connection information Q_n

$$[q_{n,j}] = \begin{bmatrix} 0 & \cdots & -d_{n,j}^{-1} & \cdots & 0 \\ 0 & \cdots & \beta_{n,j} d_{n,j}^{-1} & \cdots & 0 \end{bmatrix}.$$

We now apply Lemma 4 in [18] on the matrix $(-AB^* - BMB^*)$. Considering first the decomposition of the entries of $[P_n]$

$$\begin{bmatrix} \begin{bmatrix} d_{n,j}^{-1} & -\beta_{n,j} d_{n,j}^{-1} \\ -\beta_{n,j} d_{n,j}^{-1} & \beta_{n,j} - \beta_{n,j}^2 d_{n,j}^{-1} \end{bmatrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{bmatrix} -d_{n,j}^{-1} & 0 & 0 \\ \beta_{n,j} d_{n,j}^{-1} & 0 & 0 \end{bmatrix} \\ \vdots \quad\quad \vdots & \ddots & \vdots \quad\quad \vdots \quad\quad \vdots \\ \begin{bmatrix} -d_{n,j}^{-1} & \beta_{n,j} d_{n,j}^{-1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{bmatrix} d_{n-1,t}^{-1} + d_{n,j}^{-1} & \cdots & -\beta_{n-1,t} d_{n-1,t}^{-1} \\ \vdots & \ddots & \vdots \\ -\beta_{n-1,t} d_{n-1,t}^{-1} & \cdots & \beta_{n-1,t} - \beta_{n-1,t}^2 d_{n-1,t}^{-1} \end{bmatrix} \end{bmatrix}.$$

where $t \in \overline{J(i,j)}$. The result of this decomposition is $\gamma_{n,1}^1 = -d_{n,1}^{-1}$ and the matrix

$$\begin{bmatrix} \begin{bmatrix} \beta_{n,j} \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots \quad\quad \vdots \quad\quad \vdots \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{bmatrix} d_{n-1,t}^{-1} & \cdots & -\beta_{n-1,t} d_{n-1,t}^{-1} \\ \vdots & \ddots & \vdots \\ -\beta_{n-1,t} d_{n-1,t}^{-1} & \cdots & \beta_{n-1,t} - \beta_{n-1,t}^2 d_{n-1,t}^{-1} \end{bmatrix} \end{bmatrix}.$$

Applying the Lemma 4 in [18] on this matrix yields $\gamma_{n,1}^2 = \beta_{n,1}$ and the $d_{n,1}^{-1}$ in the diagonal entry $p_{n-1,j}$ is removed. When all the elements of the layer n have been converted to sequences then the first element $p_{i,1}$ in the next layer will have the following form

$$\begin{bmatrix} \begin{bmatrix} d_{i,j}^{-1} & d_{i,j}^{-1} & \dots & -\beta_{i,j}d_{i,j}^{-1} \\ d_{i,j}^{-1} & d_{i,j}^{-1} & \dots & -\beta_{i,j}d_{i,j}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{i,j}d_{i,j}^{-1} & -\beta_{i,j}d_{i,j}^{-1} & \dots & \beta_{i,j}-\beta_{i,j}^2 d_{i,j}^{-1} \end{bmatrix} & \begin{matrix} \dots \\ \dots \\ \dots \\ \dots \end{matrix} & \begin{bmatrix} -d_{i,j}^{-1} & \dots & 0 \\ -d_{i,j}^{-1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \beta_{i,j}d_{i,j}^{-1} & \dots & 0 \end{bmatrix} \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \ddots & \vdots \quad \vdots \quad \vdots \\ \begin{bmatrix} -d_{i,j}^{-1} & -d_{i,j}^{-1} & \dots & \beta_{i,j}d_{i,j}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{bmatrix} d_{i-1,t}^{-1}+d_{i,j}^{-1} & \dots & -\beta_{i-1,t}d_{i-1,t}^{-1} \\ \vdots & \ddots & \vdots \\ -\beta_{i-1,t}d_{i-1,t}^{-1} & \dots & \beta_{i-1,t}-\beta_{i-1,t}^2 d_{i-1,t}^{-1} \end{bmatrix} \end{bmatrix}.$$

Now we will apply Lemma 4 in [18] to this matrix. As the first element is decomposed the rest of the vertex information is simplified to zeros. Giving us $\gamma_{i,1}^1 = -d_{i,1}^{-1}$ and the matrix

$$\begin{bmatrix} \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \beta_{i,j} \end{bmatrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \\ \vdots \quad \vdots \quad \vdots & \ddots & \vdots \quad \vdots \quad \vdots \\ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{bmatrix} d_{i-1,t}^{-1} & \dots & -\beta_{i-1,t}d_{i-1,t}^{-1} \\ \vdots & \ddots & \vdots \\ -\beta_{i-1,t}d_{i-1,t}^{-1} & \dots & \beta_{i-1,t}-\beta_{i-1,t}^2 d_{i-1,t}^{-1} \end{bmatrix} \end{bmatrix}.$$

We remove all the rows and columns with zeros which gives us the following matrix

$$\begin{bmatrix} \begin{bmatrix} \beta_{i,j} \end{bmatrix} & \dots & \begin{bmatrix} 0 & \dots & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots \quad \vdots \quad \vdots \\ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{bmatrix} d_{i-1,t}^{-1} & \dots & -\beta_{i-1,t}d_{i-1,t}^{-1} \\ \vdots & \ddots & \vdots \\ -\beta_{i-1,t}d_{i-1,t}^{-1} & \dots & \beta_{i-1,t}-\beta_{i-1,t}^2 d_{i-1,t}^{-1} \end{bmatrix} \end{bmatrix}.$$

Decomposing this matrix once will return $\gamma_{i,j}^2 = \beta_{i,j}$. Thus for every vertex there are two values in our sequence $d_{i,j}^{-1}$ and $\beta_{i,j}$ out of which only the $\beta_{i,j}$ can be negative. Thus the $\kappa_-(L_{V,\beta})$ is obtained by counting the negative β potentials in the delta prime interactions (1.2)

$$\kappa_-(L_{V,\beta}) = \kappa_-(\beta),$$

where $\beta = \{\beta_{i,j}$ such that $i \in \mathbb{N} \cup 0$ and $j \in \mathbb{N}\}$. The above equation holds for all tree graphs.

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