International Journal of Analysis and Applications
ISSN 2291-8639
Volume 6, Number 2 (2014), 170-177
http://www.etamaths.com

HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS RELATED WITH LEMNISCATE OF BERNOULLI

ASHOK KUMAR SAHOO[1] AND JAGANNATH PATEL[2,*]

Abstract. The object of the present investigation is to solve Fekete-Szegö problem and determine the sharp upper bound to the second Hankel determinant for a new class $\widetilde{\mathscr{R}}$ of analytic functions in the unit disk.

1. Introduction and preliminaries

Let $\mathscr{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathscr{A}$ is said to be starlike of order ρ and convex of order ρ, if and only if $\text{Re}\{zf'(z)/f(z)\} > \rho$ and $\text{Re}\{(1 + zf''(z))/f'(z)\} > \rho$ for $0 \le \rho < 1$ and $z \in \mathcal{U}$. By usual notations, we write these classes of functions by $\mathscr{S}^\star(\rho)$ and $\mathscr{K}(\rho)$, respectively. We denote $\mathscr{S}^\star(0) = \mathscr{S}^\star$ and $\mathscr{K}(0) = \mathscr{K}$, the familiar subclasses of starlike and convex functions in $\mathcal{U}$.

Further, we say that a function $f \in \mathscr{A}$ is in the class $\mathscr{R}(\rho)$, if it satisfies the inequality:

$$\text{Re}\{f'(z)\} > \rho \quad (z \in \mathcal{U}) \tag{1.2}$$

We note that $\mathscr{R}(\rho)$ is a subclass of close-to-convex functions order $\rho (0 \le \rho < 1)$ in $\mathcal{U}$. We write $\mathscr{R}(0) = \mathscr{R}$, the familiar class functions in $\mathscr{A}$ whose derivatives have a positive real part in $\mathcal{U}$.

A function f is said to be subordinate to a function g, written as $f \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), z \in \mathcal{U}$. In particular, if g is univalent in $\mathcal{U}$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $\mathscr{P}$ denote the class of analytic functions ϕ normalized by

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathcal{U}) \tag{1.3}$$

such that $\text{Re}\{\phi(z)\} > 0$ in $\mathcal{U}$.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic function; Subordination; Fekete-Szegö problem; Hankel determinant.

Definition. A function $f \in \mathscr{A}$ is said to be in the class $\widetilde{\mathscr{R}}$, if it satisfies the condition

$$\left|(f'(z))^2 - 1\right| < 1 \quad (z \in \mathcal{U}). \tag{1.4}$$

It follows from (1.4) and the definition of subordination that a function $f \in \widetilde{\mathscr{R}}$ satisfies the following subordination relation

$$f'(z) \prec \sqrt{1+z} \quad (z \in \mathcal{U}). \tag{1.5}$$

To bring out the geometrical significance of the class $\widetilde{\mathscr{R}}$, we set

$$h(z) = \sqrt{1+z}, \ z \in \mathcal{U}$$

and note that

$$\omega = h(e^{i\theta}) = \sqrt{1+e^{i\theta}} \ (0 \le \theta \le 2\pi).$$

which yields $\omega^2 - 1 = e^{i\theta}$ or $|\omega^2 - 1| = 1$. Letting $\omega = u + iv$, we deduce that

$$(u^2+v^2)^2 = 2(u^2 - v^2).$$

Thus, $h(\mathcal{U})$ is the region bounded by the right half of the lemniscate of Bernoulli given by $\left\{u + iv \in \mathbb{C} : (u^2+v^2)^2 = 2(u^2-v^2)\right\}$, which implies that the derivative of functions in $\widetilde{\mathscr{R}}$ have a positive real part and hence univalent in $\mathcal{U}$ [1].

Noonan and Thomas [12] defined the q-th Hankel determinant of the function f, given by (1.1) by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1, n, q \in \mathbb{N}). \tag{1.6}$$

The determinant given in (1.6) has been studied by several authors with the subject of inquiry ranging from the rate of growth of $H_q(n)$ (as $n \to \infty$) [13] to the determination of precise bounds with specific values of n and q for certain subclasses of analytic functions in the unit disc $\mathcal{U}$.

For $n = 1, q = 2, a_1 = 1$ and $n = q = 2$, the Hankel determinant simplifies to $H_2(1) = |a_3 - a_2^2|$ and $H_2(2) = |a_2a_4 - a_3^2|$. We refer to $H_2(2)$ as the second Hankel determinant. It is known [1] that if the function f, given by (1.1) is analytic and univalent in $\mathcal{U}$, then the sharp inequality $H_2(1) = |a_3 - a_2^2| \le 1$ holds. For a family $\mathcal{F}$ of functions in $\mathscr{A}$ of the form (1.1), the more general problem of finding the sharp upper bounds for the functionals $|a_3 - \mu\, a_2^2|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as Fekete-Szegö problem for the class $\mathcal{F}$. The Fekete-Szegö problem for the known classes of univalent functions, starlike functions, convex functions and close-to-convex functions has been completely settled ([2], [5], [6], [7]). Recently, Janteng et al. [3, 4] have obtained the sharp upper bounds to the second Hankel determinant $H_2(2)$ for the family $\mathscr{R}$. For initial work on the class $\mathscr{R}$ one may refer to the paper by MacGregor [11].

In our present investigation, by following the techniques devised by Libera and Zlotkiewicz [8, 9], we solve the Fekete-Szegö problem and also determine the sharp upper bound to the second Hankel determinant $H_2(1)$ for the class $\widetilde{\mathscr{R}}$.

To establish our main results, we shall need the followings lemmas.

Lemma 1.1. *Let the function ϕ, given by* (1.3) *be a member of the class $\mathscr{P}$. Then*

$$|p_k| \leq 2 \quad (k \geq 1) \tag{1.7}$$

and

$$\left|p_2 - \nu\, p_1^2\right| \leq 2\max\{1, |2\nu - 1|\}. \tag{1.8}$$

The estimate (1.7) *is sharp for the function $\varphi(z) = (1+z)/(1-z), z \in \mathcal{U}$, whereas the estimate* (1.8) *is sharp for the functions given by φ and $\psi(z) = (1+z^2)/(1-z^2)$, $z \in \mathcal{U}$.*

We note that the estimate (1.7) is contained in [1] and the estimate (1.8) is obtained in [10].

Lemma 1.2 ([9],see also [8])**.** *If the function ϕ, given by* (1.3) *belongs to the class $\mathscr{P}$, then*

$$p_2 = \frac{1}{2}\left\{p_1^2 + (4 - p_1^2)x\right\} \tag{1.9}$$

and

$$p_3 = \frac{1}{4}\left\{p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z\right\} \tag{1.10}$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$.

2. Main results

Now, we determine an upper bound for the Fekete-Szegö problem of the class $\widetilde{\mathscr{R}}$.

Theorem 2.1. *If the function f, given by* (1.1) *belongs to the class $\widetilde{\mathscr{R}}$, then for any $\mu \in \mathbb{C}$*

$$|a_3 - \mu\, a_2^2| \leq \frac{1}{6}\max\left\{1, \frac{|2 + 3\,\mu|}{8}\right\}. \tag{2.1}$$

The estimate in (2.1) *is sharp.*

Proof. From (1.5), it follows that

$$f'(z) = \sqrt{1 + w(z)} \quad (z \in \mathcal{U}), \tag{2.2}$$

where w is analytic and satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$. Setting

$$\chi(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + \cdots \quad (z \in \mathcal{U}), \tag{2.3}$$

we see that $\chi \in \mathscr{P}$. From (2.3), we get

$$w(z) = \frac{\chi(z) - 1}{\chi(z) + 1} \quad (z \in \mathcal{U}) \tag{2.4}$$

so that by (2.2) and (2.4), we get

$$f'(z) = \left(\frac{2\chi(z)}{1 + \chi(z)}\right)^{\frac{1}{2}} \quad (z \in \mathcal{U}). \tag{2.5}$$

Now, by substituting the series expansion of χ from (2.3) in (2.5), it is easily seen that

$$\left(\frac{2\chi(z)}{1+\chi(z)}\right)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right) z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3 + \cdots . \tag{2.6}$$

Differentiating the series expansion of f given by (1.1) with respect to z and comparing the coefficients of z, z^2 and z^3 in (2.6), we deduce that

$$a_2 = \frac{1}{8}p_1 \tag{2.7}$$

$$a_3 = \frac{1}{12}\left(p_2 - \frac{5}{8}p_1^2\right) \tag{2.8}$$

$$a_4 = \frac{1}{16}\left(p_3 - \frac{5}{4}p_1 p_2 + \frac{13}{32}p_1^3\right). \tag{2.9}$$

Thus, by using (2.7) and (2.8), we get

$$\left|a_3 - \mu\, a_2^2\right| = \frac{1}{12}\left|p_2 - \frac{1}{16}(10+3\mu)p_1^2\right| \tag{2.10}$$

The expression in (2.10) with the aid of (1.8) yields the required estimate (2.1).

The estimate in (2.1) is sharp for the function $f_0 \in \mathscr{A}$ defined by

$$f_0'(z) = \begin{cases} \sqrt{1+z^2}, & |2+3\,\mu| \le 8 \\ \sqrt{1+z}, & |2+3\,\mu| > 8. \end{cases} \tag{2.11}$$

This completes the proof of Theorem 2.1. □

Letting $\mu = 0$(or $\mu = 1$ respectively) in Theorem 2.1, we get

Corollary 2.1. *If the function f, given by* (1.1) *belongs to the class $\widetilde{\mathscr{R}}$, then*

$$|a_3| \le \frac{1}{6} \quad \textit{and} \quad |a_3 - a_2^2| \le \frac{1}{6}. \tag{2.12}$$

The estimates in (2.12) *are sharp for the function $f_0 \in \mathscr{A}$ defined by*

$$f_0'(z) = \sqrt{1+z^2} \quad (z \in \mathcal{U}). \tag{2.13}$$

If $\mu \in \mathbb{R}$, then Theorem 2.1 reduces to

Corollary 2.2. *Let $\mu \in \mathbb{R}$. If the function f, given by* (1.1) *belongs to the class $\widetilde{\mathscr{R}}$, then*

$$\left|a_3 - \mu\, a_2^2\right| \le \begin{cases} -\dfrac{2+3\mu}{48}, & \mu \le -\dfrac{10}{3} \\ \dfrac{1}{6}, & -\dfrac{10}{3} \le \mu \le 2 \\ \dfrac{2+3\mu}{48}, & \mu > 2. \end{cases} \tag{2.14}$$

The estimates in (2.14) *are sharp.*

Proof. First, we assume that $\mu < -10/3$. Then, $(2+3\mu)/8 < -1$ so that $|2+3\mu|/8 > 1$. Hence by using (2.1), we get

$$|a_3 - \mu a_2^2| \leq \frac{|2+3\mu|}{48} = -\frac{2+3\mu}{48}. \tag{2.15}$$

Next, if $-10/3 \leq \mu \leq 2$, then $|2+3\mu| \leq 1$ so that

$$|a_3 - \mu a_2^2| \leq \frac{1}{6} \tag{2.16}$$

again by the use of (2.1). Finally, if $\mu > 2$, then $(2+3\mu)/8 > 1$. Thus, by (2.1)

$$|a_3 - \mu a_2^2| \leq \frac{2+3\mu}{48}. \tag{2.17}$$

The estimates are sharp for the function f_1 defined in $\mathcal{U}$ by $f_1'(z) = \sqrt{1+z}$, for $\mu < -10/3$ or $\mu > 2$, and for the function f_0 given by (2.13) in the case $-10/3 \leq \mu \leq 2$. □

In the following theorem, we find the sharp upper bound to the second Hankel determinant for the class $\widetilde{\mathscr{R}}$.

Theorem 2.2. *Let the function f, given by* (1.1) *be a member of the family $\widetilde{\mathscr{R}}$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}. \tag{2.18}$$

The estimate in (2.18) *is sharp.*

Proof. From (2.7), (2.8) and (2.9), we have

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &= \left|\frac{1}{128}\left(p_1p_3 - \frac{5}{4}p_1^2p_2 + \frac{13}{32}p_1^4\right) - \frac{1}{144}\left(p_2^2 - \frac{5}{4}p_1^2p_2 + \frac{25}{64}p_1^4\right)\right| \\ &= \frac{1}{16}\left|\frac{1}{8}p_1p_3 - \frac{5}{288}p_1^2p_2 - \frac{1}{9}p_2^2 + \frac{17}{2304}p_1^4\right|. \qquad (2.19)\end{aligned}$$

Since the function χ, given by (2.3) and the function $\chi(e^{i\theta}z)\,(\theta \in \mathbb{R})$ are in the class $\mathscr{P}$ simultaneously, we assume without loss of generality that $p_1 > 0$. For convenience of notation, we write $p_1 = p\,(0 \leq p \leq 2)$. Now, by using Lemma 2.2 in (2.19), we get

$$\begin{aligned}&\left|a_2a_4 - a_3^2\right| \\ &= \frac{1}{16}\left|\left(\frac{1}{32}p^4 + \frac{1}{16}(4-p^2)p^2x - \frac{1}{32}(4-p^2)p^2x^2 + \frac{1}{16}(4-p^2)p(1-|x|^2)z\right)\right. \\ &\quad - \left(\frac{5}{576}p^4 + \frac{5}{576}(4-p^2)p^2x\right) \\ &\quad \left. - \left(\frac{1}{36}p^4 + \frac{1}{18}(4-p^2)p^2x + \frac{1}{36}(4-p^2)^2x^2\right) + \frac{17}{2304}p^4\right| \\ &= \frac{1}{16}\left|\frac{5}{2304}p^4 - \frac{1}{576}(4-p^2)p^2x - \frac{1}{288}\{8(4-p^2)+9p^2\}(4-p^2)x^2\right. \\ &\quad \left. + \frac{1}{16}(4-p^2)p(1-|x|^2)z\right| \qquad (2.20)\end{aligned}$$

for some $x\,(|x| \leq 1)$ and for some $z\,(|z| \leq 1)$. Applying the triangle inequality in (2.20) and replacing $|x|$ by y in the resulting equation, we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leq \frac{1}{16}\Big\{ &\frac{5}{2304}p^4 + \frac{1}{576}(4-p^2)p^2y \\ &+\frac{1}{288}(4-p^2)(2-p)(16-p)y^2 + \frac{1}{16}(4-p^2)p\Big\} \\ = \mathcal{G}(p,y)\; &(0 \leq p \leq 2, 0 \leq y \leq 1)\,(\text{say}). \end{aligned} \tag{2.21}$$

We next maximize the function $\mathcal{G}(p,y)$ on the closed rectangle $[0,2]\times[0,1]$. Differentiating the function $\mathcal{G}$, given in (2.21) with respect to y, we deduce that

$$\frac{\partial \mathcal{G}}{\partial y} = \frac{1}{9216}(4-p^2)p^2 + \frac{1}{2304}(4-p^2)(2-p)(16-p)y > 0 \tag{2.22}$$

for $0 < p < 2$ and $0 < y < 1$. Thus, in view of (2.22), the function $\mathcal{G}(p,y)$ cannot have a maximum in the interior on the closed rectangle $[0,2]\times[0,1]$. Therefore, for fixed $p \in [0,2]$

$$\max_{0\leq y\leq 1} \mathcal{G}(p,y) = \mathcal{G}(p,1) = F(p)\,(\text{say}), \tag{2.23}$$

where

$$\begin{aligned} F(p) = \frac{1}{16}\Big\{ &\frac{5}{2304}p^4 + \frac{1}{576}(4-p^2)p^2 \\ &+\frac{1}{288}(4-p^2)(2-p)(16-p) + \frac{1}{16}(4-p^2)p\Big\}\; (0 \leq p \leq 2). \end{aligned} \tag{2.24}$$

On differentiating the function F, given by (2.24) followed by a simple calculation yields

$F'(p) = -\dfrac{1}{9216}(7p^2+104)p < 0$ which implies that the function F is a decreasing function of p so that $\max_{0\leq p\leq 2} F(p)$ occurs at $p = 0$. Thus, the upper bound in (2.21) corresponds to $p = 0$ and $y = 1$ from which we get the required estimate (2.18).

Equality holds in (2.18) for the function $f_0 \in \mathscr{A}$, given by (2.13) and the proof of Theorem 2.2 is thus completed. □

Next, we determine the upper bound for the fourth coefficient of functions belonging to the class $\widetilde{\mathscr{R}}$.

Theorem 2.3. *If the function f, given by* (1.1) *belongs to the class $\widetilde{\mathscr{R}}$, then*

$$|a_4| \leq \frac{1}{8} \tag{2.25}$$

and the estimate is sharp.

Proof. Using Lemma 1.1 in (2.9) and following the lines of proof of Theorem 1.2, we deduce that

$$\begin{aligned} |a_4| &\leq \frac{1}{32}\left\{\frac{p^3}{16} + \frac{(4-p^2)p}{2}y + \frac{(4-p^2)p}{2}y^2 + (4-p^2)(1-y^2)\right\} \\ &= \frac{1}{32}\left\{\frac{p^3}{16} + \frac{(4-p^2)p}{2}t + \frac{(4-p^2)(p-2)}{2}t^2 + (4-p^2)\right\} \\ &= G(p,t)\,(\text{say}), \end{aligned} \tag{2.26}$$

where $p \in [0,2]$ and $y \in [0,1]$. We next maximize the function $G(p,y)$ on the closed rectangle $[0,2] \times [0,1]$. Suppose that the maximum of G occurs at the interior point of $[0,2] \times [0,1]$. Differentiating the function G with respect to y, we get

$$\frac{\partial G}{\partial y} = \frac{1}{128}(4-p^2)\{p+4(p-2)y\}.$$

For $y \in (0,1)$ and fixed $p \in (0,2)$, it is easily seen that $\frac{\partial G}{\partial y} > 0$, which shows that G is a decreasing function of y contradicting our assumption. Therefore,

$$\max\{G(p,y)\}_{0\leq y\leq 1} = G(p,0) = \frac{1}{32}\left\{\frac{p^3}{16} + (4-p^2)\right\} = F(p) \text{ (say)}. \tag{2.27}$$

From (2.27), we have

$$F'(p) = \frac{1}{32}\left\{\frac{3}{16}p^2 - 2p)\right\}$$

and

$$F''(p) = \frac{1}{32}\left\{\frac{3}{8}p - 2)\right\} < 0$$

for $p = 0$. This implies that F attains its maximum at $p = 0$. Hence, we get the required result.

The estimate in (2.25) is sharp for the function $f \in \mathscr{A}$, defined by

$$f'(z) = \sqrt{1+z^3} \quad (z \in \mathcal{U}).$$

□

References

[1] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, USA (1983).

[2] M. Fekete and G. Szegö, Eine Bemerkung über ungerede schlichte funktionen, J. London Math. Soc., 8 (1933), 85-89.

[3] A. Janteng, S.A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math., 7 (2006), Article ID 50.

[4] A. Janteng, S.A. Halim and M. Darus, Estimate on the second Hankel functional for functions whose derivative has a positive real part, J. Quality Measurement and Analysis, 4 (2008), 189-195.

[5] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20 (1969), 8-12.

[6] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions-II, Arch. Math.(Basel), 49 (1987), 420-433.

[7] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions, Proc. Amer. Math. Soc., 101 (1987), 89-95.

[8] R.J. Libera and E.J. Zlotkiewicz, Early coefficient of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (2) (1982), 225-230.

[9] R.J. Libera and E.J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, Proc. Amer. Math. Soc. 87 (2) (1983), 251-257.

[10] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), Int. Press, Cambridge, MA,(1994) 157-169.

[11] T.H. MacGregor, Functions whose derivative have a positive real part. Trans. Amer. Math. Soc. 104(3) (1962), 532-537.

[12] J.W. Noonan and D.K. Thomas, On the second Hankel determinant of areally mean p-valent functions, Trans. Amer. Math. Soc., 223 (1976), 337-346.

[13] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., 28 (1983), no. 8, 731 - 739.

[1]Department of Mathematics, Veer Surendra Sai University of Technology, Sidhi Vihar, Burla-768 018, India

[2]Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar-751004, India

[*]Corresponding author