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Linear and Non-linear Contractions in Triple Controlled J Metric Spaces

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Abstract. In this article, we define a new generalization of J metric spaces, defined as Triple controlled J metric spaces where a constant k is replaced by three different function α, β, γ. We prove the existence and the uniqueness of linear and nonlinear contractions like Kannan's, weak and generlized contractions.

1. Introduction

The fixed-point theorem is a cornerstone in various disciplines such as mathematics, economics, and computer science, due to its wide applicability in solving complex problems. It provides powerful tools for addressing questions in optimization, game theory [1], dynamical systems [2] , algorithm design [3] and [4], and data science [5], establishing itself as a vital component in both theoretical research and practical applications. Its versatility ensures that the concept continues to evolve, with researchers expanding its scope across multiple dimensions and generalizations.
When Banach introduced the definition of the fixed point theorem in 1922 [6], it laid the foundation for a whole field of study. The early formulations of the theorem were revolutionary and inspired mathematicians to explore its applications in new and innovative ways. The extension of this field could occur in two ways by extension of the metric space or generalization for the contractions or both [7] [8] [9] [10]. In 2015, Jleli and Samet [11] have introduced the JS metric spaces where a sequence is used in the right triangle instead of the point in the set.

In 2022, Souayah et al. [12]introduced the J metric spaces that consider as a generalization of S^{SJ} metric space where the sequence $\{\varrho_n\}$ used instead of z in the triangle inequality, which was very useful in many application in computer science, Souayah defined the three dimensions S^{SJ}

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metric spaces, then S.Aiadi in [13] introduced Controlled J metric spaces where the constant in the J metric space was replaced by a function $\theta(\varrho, \rho, z)$.

Mlaiki in 2020 [14]introduced the concept of double control metric-type spaces, employing two control functions to refine the right-hand side of the B-triangle inequality. This was a significant leap as it extended the scope of fixed-point theory to controlled spaces. Later, in 2021, Tasneem et al. [15]expanded this framework further by defining triple control metric-type spaces in two dimensions. Most recently, Azmi in 2024 [16]introduced a groundbreaking development: the triple control metric space in three dimensions, setting the stage for further exploration in this domain.

This article introduces the concept of triple control J metric space, extending the framework of J metric spaces. We prove the existence and uniqueness of linear and nonlinear contraction as weak, generalized, and generalized contraction with a control function.

2. Preliminaries

In this section of the article, we recall the work of some important scientific authors that led us to our work on Triple control J metric spaces, we will start with Jleli and Samet [11], who, who have defined the JS- metric space in two dimensions. After that, Souayah introduced the definition of the J metric space, which can be considered an extension of the JS metric spaces. Then Aiadi introduced the controlled J metric space where the constant k is replaced by a function θ.
On the other hand, in 2020 Mlaiki et al. introduced the concept of double controlled metric type spaces, by using a control function $\theta(x, y)$and $\beta(\varrho, \rho)$ of the right-hand side of the b triangle inequality, after that in 2021 Tasneem et al. [15] defined triple controlled metric-type spaces in two dimensions, then in 2024 Azmi [16]introduced triple controlled S metric space in three dimensions by adding three functions to the right side of the triangle inequality.

Definition 2.1. *[11] Consider a nonempty set X, and a function, $d : X^2 \to [0, \infty)$, the set $C(d, X, x)$ is defined*

$$C(d, \Xi, \varrho) = \{\{\varrho_n\} \subset \Xi : \lim_{n\to\infty} d(\varrho, \varrho_n) = 0\}$$

for all $\varrho \in \Xi$.

Definition 2.2. *[11] Let $d : \Xi \times \Xi \to [0, \infty)$ be a mapping which satisfies the following;*

(1) $d(\varrho, \rho) = 0$ *implies* $\varrho = y$, $\forall \varrho, \rho \in \Xi$
(2) *For every* $(\varrho, \rho) \in \Xi \times \Xi$ *and* $\{\varrho_n\} \in C(d, \Xi, \varrho)$ *then*
$d(\varrho, \rho) \leq k \limsup_{n\to\infty} d(\varrho_n, \rho)$, *for some* $k > 0$

Then d is called SJ metric space

Definition 2.3. *[12] Consider a nonempty set Ξ, and a function $J : \Xi^3 \to [0, \infty)$. Let us define the set,*

$$S(J, \Xi, \varrho) = \{\{\varrho_n\} \subset \Xi : \lim_{n\to\infty} J(\varrho, \varrho, \varrho_n) = 0\}$$

for all $\varrho \in \Xi$.

Definition 2.4. *[12] Let Ξ be nonempty set and, $J : \Xi^3 \to [0, \infty)$ that satisfies the mentioned below conditions:*

(i) $J(\varrho, \rho, z) = 0$ *implies* $\varrho = \rho = z$ *for any* $\varrho, \rho, z \in \Xi$.

(ii) *There are some* $K > 0$, *where for each* $(\varrho, \rho, z) \in \Xi^3$ *and* $\{w_n\} \in S(J, \Xi, w)$

$$J(\varrho, \rho, z) \leq k \limsup_{n \to \infty} \Big(J(\varrho, \varrho, w_n) + J(\rho, \rho, w_n) + J(z, z, w_n) \Big).$$

Then, (Ξ, J) *is called as a J-metric space. In addition, if* $J(\varrho, \varrho, \rho) = J(\rho, \rho, \varrho$ *for each* $\varrho, \rho \in \Xi$, *the pair* (Ξ, J) *is called as a symmetric J-metric space.*

Definition 2.5. *[13] Let Ξ be a non empty set and $C_J : \Xi^3 \to [0, \infty)$ fulfill the following conditions:*

(i) $C_J(\varrho, \rho, z) = 0$ *implies* $\varrho = \rho = z$ *for all* $\varrho, \rho, z \in \Xi$

(ii) *There exist a function* $\theta : \Xi^3 \to [0, \infty)$.

(iii) $C_J(\varrho, \rho, z) \leq \theta(\varrho, \rho, z). \limsup_{n \to \infty} \Big(C_J(\varrho, \varrho, w_n) + C_J(\rho, \rho, w_n) + C_J(z, z, w_n) \Big)$

Then (Ξ, C_J) *is defined as* C_J*-metric space. In addition, if*

$$C_J(\varrho, \varrho, \rho) = C_J(\rho, \rho, \varrho)$$

for each $\varrho, \rho \in \Xi$, *then* (Ξ, C_J) *is defined as symmetric* C_J*-metric space.*

Definition 2.6. *[14] Given a non-empty set Ξ and $\theta : \Xi \times \Xi \to [1, \infty)$. The function $d : \Xi \times \Xi \to [0, \infty)$ is called a controlled metric type if,*

(1) $d(\varrho, \rho) = 0$ *if and only if* $\varrho = \rho$.

(2) $d(\varrho, \rho) = d(\rho, \varrho)$.

(3) $d(\varrho, \rho) \leq \theta(\varrho, z) d(\varrho, z) + \theta(z, \rho) d(z, \rho)$. *For all* $\varrho, \rho, z \in \Xi$.*The pair*(Ξ, d) *is called a controlled metric type space.*

Definition 2.7. *[15] Let Ξbe a non-empty set and $P, Q, R : \Xi \times \Xi \to [1, \infty)$. A function $dt : \Xi \times \Xi \to [0, \infty)$ is called a triple controlled metric type if it satisfies:*

(1) $dt(\varrho, \rho) = 0$ *if and only if* $\varrho = \rho$ *for all* $\varrho, \rho \in \Xi$.

(2) $dt(\varrho, \rho) = dt(\rho, \varrho) forall \varrho, \rho \in \Xi$.

(3) $dt(\varrho, \rho) \leq P(\varrho, z) dt(\varrho, z) + Q(z, w) dt(z, w) + R(\rho, w) d(\rho, w)$
for all $\varrho, \rho \in \Xi$, *and for all distinct points*$z, w \in \Xi$

Definition 2.8. *[16] Let $d : \Xi^3 \to [0, \infty)$ be a mapping where Ξ be a nonempty set and suppose $\theta, \beta. \gamma : \Xi^2 \to [1, \infty)$ are mapping such that for all $\varrho, \rho, z, w \in \Xi$, these conditions are satisfied:*

(1) $d(\varrho, \rho, z) = 0$ *if* $\varrho = \rho = z$

(2) $d(\varrho, \varrho, \rho) = d(\rho, \rho, \varrho)$ *for all* $\varrho, \rho \in \Xi$

(3) $d(\varrho, \rho, z) \leq \theta(\varrho, w) d(\varrho, \varrho, w) + \beta(\rho, w) d(\rho, \rho, w) + \gamma(z, w) d(z, z, w)$.

The pair (Ξ, d) *is referred to as a triple controlled S metric type space.*

3. Main Result

3.1. **Triple Controlled J Metric Spaces.** In this part, we will define triple controlled J metric space T_J-metric spaces and prove the existence and the uniqueness of the some fixed point theorems that include Weak contraction, generalized contraction, and generalized contraction with control function.

Definition 3.1. *Let X be a nonempty set and a function $T_J : \Xi^3 \to [0, \infty)$. Then the set is defined as follows*

$$S(T_J, X, \varrho) = \{\{\varrho_n\} \subset X : \lim_{n\to\infty} T_J(\varrho, \varrho, \varrho_n) = 0\}$$

for each $\varrho \in \Xi$

Definition 3.2. *Let X be a non-empty set and $T_J : X^3 \to [0, \infty)$ satisfying the following conditions:*

(i) *$T_J(\varrho, \rho, z) = 0$ implies $\varrho = \rho = z$ for all $\varrho, \rho, z \in \Xi$*

(ii) *There are $\theta, \beta, \gamma : \Xi^2 \to [0, \infty)$ functions, and $w_n \in S(T_J, \Xi, w)$.*

$$T_J(\varrho, \rho, z) \leq \theta(\varrho, \varrho) \limsup_{n\to\infty} T_J(\varrho, \varrho, w_n) + \beta(\rho, \rho) \limsup_{n\to\infty} T_J(\rho, \rho, w_n) + \gamma(z, z) \limsup_{n\to\infty} T_J(z, z, w_n)$$

Then,the pair (Ξ, T_J) is referred to as a triple controlled J-metric space. If

$$T_J(\varrho, \varrho, \rho) = T_J(\rho, \rho, \varrho)$$

$\forall \varrho, \rho \in \Xi$,then (X, T_J) is known as symmetric triple controlled J-metric space.

In the study of metric spaces, sequences and their properties play a critical role in understanding the structure and behavior of these spaces. Convergence and Cauchy sequences, in particular, are essential tools for analyzing the completeness within such spaces.

Definition 3.3. (1) *Let (Ξ, T_J) be metric space. A sequence $\{\varrho_n\} \subset \Xi$ be convergent to $\varrho \in \Xi$ iff $\lim_{n\to\infty} \varrho_n = \varrho$,for $\{\varrho_n\} \in S(T_J, \Xi, \varrho)$.*

(2) *Let (Ξ, T_J) is triple controlled J metric space. A sequence $\{\varrho_n\} \subset \Xi$ is called Cauchy iff*

$$\lim_{n,m\to\infty} T_J(\varrho_n, \varrho_n, \varrho_m) = 0.$$

(3) *A T_J-metric space is complete if each Cauchy Sequence in Ξ is convergent.*

Proposition 3.1. *In the triple controlled J metric space, if $\{\varrho_n\}$ is convergent then it is convergent to a unique element in X*

Proof. Assume that $\{\varrho_n\}$ is convergent to two elements in X, ϱ and ρ, where $\varrho_n \in S(T_J, \Xi, \rho)$

$$\begin{aligned} T_J(\varrho, \varrho, \rho) &\leq \limsup_{n\to\infty} \theta(\varrho, \varrho) T_J(\varrho, \varrho, \varrho_n) + \beta(\varrho, \varrho) T_J(\varrho, \varrho, \varrho_n) + \gamma(\rho, \rho) T_J(\rho, \rho, \varrho_n) \\ &= 0. \end{aligned}$$

Since,$\{\varrho_n\}$ is convergent to ϱ and ρ, and by the definition of $S(T_J, \Xi, \varrho)$, so $T_J(\varrho, \varrho, \varrho_n) = 0$ and $T_J(\rho, \rho, \varrho_n) = 0$ now by the definition of the triple controlled J metric spaces $\varrho = \rho$ which is

contradiction.
Then $\{\varrho_n\}$ is convergent to a unique element in Ξ. □

Example 3.1. *Let $\Xi = [1, \infty)$ and, $T_J : \Xi^3 \to [0, \infty)$ and let*

$$T_J(\varrho, \rho, z) = max\{|\varrho - z|, |\rho - z|\},$$

for all $\varrho, \rho, z, w \in \Xi$, and let $w_n = w + \dfrac{1}{n}$. Let $\theta(\varrho, \rho) =| \varrho + \rho |$,
$\beta(\varrho, \rho) = 2 | \varrho + \rho |$, and $\gamma(\varrho, \rho) = 3 | \varrho + \rho |$.
This is an example of the existence of T_J metric spaces and here is the proof.
lets start with $\{w_n\} = w + \dfrac{1}{n}$, and for each $w \in \Xi, \lim_{n\to\infty} w_n = w$, and
$T_J(w, w, w_n) = max\{|w - w_n|, |w - w_n|\}$
as $n \to \infty, T_J(w, w, w_n) = 0$.
Finally, we will take the conditions of the triple controlled J metric space

(1) $T_J(\varrho, \rho, z) = 0 = max\{| \varrho - z |, | \rho - z |\} = 0$ *,which means that $| \varrho - z |= 0$ and $| \rho - z |= 0$ implies that $\varrho = \rho = z$.*
(2)

$$T_J(\varrho, \rho, z) = max\{|\varrho - z|, |\rho - z|\},$$

$$\theta(\varrho, \varrho)T_J(\varrho, \varrho, w_n) = 2 | x || \varrho - w_n |$$

$$\beta(\rho, \rho)T_J(\rho, \rho, w_n) = 4 | y || \rho - w_n |$$

$$\gamma(z, z)T_J(z, z, w_n) = 6 | z | | z - w_n |$$

$$max\{|\varrho - z|, |\rho - z|\} \leq \limsup_{n\to\infty} 2 | \varrho || \varrho - w_n | +4 | \rho || \rho - w_n | +6 | z || z - w_n |$$

So, this is an example of the triple controlled J metric space.

3.2. **Fixed point theorems in Controlled *J* Metric spaces.** In this part of our article, we will prove the existence and the uniqueness of generalized contraction, generalized with control function, and Weak contraction.

Theorem 3.1. *Let (Ξ, C_J) be a triple controlled J complete symmetric metric space, and $\Lambda : \Xi \to \Xi$ is a continuous map satisfies*

$$T_J(\Lambda\varrho, \Lambda\rho, \Lambda z) \leq P(T_J(\varrho, \rho, z)) \quad \text{for all } \varrho, \rho, z \in \Xi. \tag{3.1}$$

Where,$P : [0, +\infty) \to [0, +\infty)$ is a strictly increasing continuous function. And,

$$\lim_{n\to\infty} P^n(t) = 0 \tag{3.2}$$

for each fixed $t \geq 0$ and for every $\varrho \in \Xi$, let $M(T_J, g, \varrho) = \sup\left\{T_J(\varrho, \varrho, \Lambda^j\varrho) : j \in \mathbb{N} \cup \{0\}\right\}$, If there exists $\varrho_0 \in \Xi$ such that $M(T_j, \Lambda, \varrho_0)$ is finite. Then, Λ has a unique fixed point in Ξ.

Proof. To prove this theorem we need to define the iteration $\varrho_0 \in \Xi$, and $\{\varrho_n\}_{n\geq 0} \subset \Xi$ where,

$$\varrho_1 = \Lambda\varrho_0, \varrho_2 = \Lambda\varrho_1.........\varrho_n = \Lambda^n\varrho_0, \quad n = 1, 2, ... \tag{3.3}$$

let's start with Cauchy:

$$\begin{aligned} T_J(\varrho_n, \varrho_n, \varrho_m) = T_J(\Lambda\varrho_{n-1}, \Lambda\varrho_{n-1}, \Lambda\varrho_{m-1}\Big) &\leq P\Big(T_J(\varrho_{n-1}, \varrho_{n-1}, \varrho_{m-1})\Big) \\ &= P\Big(T_J(\Lambda\varrho_{n-2}, \Lambda\varrho_{n-2}, \Lambda\varrho_{m-2})\Big) \\ &\leq \vdots \\ &\leq P^n\Big(T_J(\varrho_0, \varrho_0, \varrho_{m-n})\Big) \end{aligned}$$

After applying (3.1) n times and with assumption that $m = n + q$ for some constant $q \in \mathbb{N}$ to get

$$T_J(\varrho_n, \varrho_n, \varrho_m) \leq P^n\Big(T_J(\varrho_0, \varrho_0, \varrho_q)\Big) \tag{3.4}$$

By applying the limit in (3.4) as $n \longrightarrow \infty$, and by using the fact that there is a $\varrho_0 \in M(T_j, \Lambda, \varrho_0)$ is finite ,we get

$$\lim_{n\to\infty} T_J(\varrho_n, \varrho_n, \varrho_m) = 0. \tag{3.5}$$

Accordingly, $\{\varrho_n\}$ is a Cauchy sequence in Ξ. The metric space is complete which means that each Cauchy sequence is convergent, so, there is $\varrho \in \Xi$ such that $\varrho_n \to \varrho$ as $n \to \infty$.
In addition, $\varrho = \lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \varrho_{n+1} = \lim_{n\to\infty} \Lambda\varrho_k = \Lambda\varrho$. Thus, Λ has ϱ as a fixed point.
Assume that ϱ and ρ are two fixed points of Λ.

$$\begin{aligned} T_J(\varrho, \varrho, \rho) = T_J(\Lambda\varrho, \Lambda\varrho, \Lambda\rho) &\leq P\Big(T_J(\varrho, \varrho, \rho)\Big) \\ &\leq P^2\Big(T_J(\varrho, \varrho, \rho)\Big) \\ &\vdots \\ &\leq P^n\Big(T_J(\varrho, \varrho, \rho)\Big) \end{aligned}$$

as $n \longrightarrow \infty$ we have $T_J(\varrho, \varrho, \rho) = 0$ and $\varrho = \rho$.
Thus, Λ has a unique fixed point in Ξ as desired. □

Example 3.2. (1) *Let $\Xi = [1, \infty)$, and $T_J : \Xi^3 \to [0, \infty)$*

$$T_J(\varrho, \rho, z) = max\{| \varrho - z |, | z - \rho |\}$$

Let $\theta(\varrho, \rho) =| \varrho + \rho |$, $\beta(\varrho, \rho) = 2 | \varrho + \rho |$, and $\gamma(\varrho, \rho) = 3 | \varrho + \rho |$.
This is the same example of 3.1, and it satisfies all the triple controlled J metric space.

(2) *Let $P : [0, \infty) \to [0, \infty)$, $P(t) = \dfrac{t}{2}$, where P is strickly increasing and continuous function, Moreover,*

$$\lim_{n\to\infty} P^n(t) = 0$$

(3) *Take $\Lambda : \Xi \to \Xi$,and $g(\varrho) = \dfrac{\varrho}{2}$, Λ is continuous function.*

For verification the theorem conditions

- *We need to verify*

$$T_J(\varrho,\rho,z) \le P(T_J(\varrho,\rho,z)), \forall \varrho,\rho,z \in \Xi$$

$$T_J(\Lambda\varrho,\Lambda\rho,\Lambda z) = \frac{1}{2} max\{\mid \varrho - z \mid, \mid z - \rho \mid\}$$

On the other hand:

$$P(T_J(\varrho,\rho,z)) = \frac{1}{2} max\{\mid \varrho - z \mid, \mid z - \rho \mid\}$$

So, the condition satisfies

$$T_J(\varrho,\rho,z) \le P(T_J(\varrho,\rho,z)), \forall \varrho,\rho,z \in \Xi$$

- *The existence and the uniqueness of the fixed point of Λ, we solve*

$$\Lambda(\varrho) = \varrho$$

$$\frac{\varrho}{2} = \varrho \Rightarrow \varrho = 0.$$

Thus,$\varrho = 0$ is the unique fixed point of Λ in Ξ.

Theorem 3.2. *Let (Ξ, T_J) be a triple controlled J complete symmetric metric space and $\Lambda : \Xi \to \Xi$ be a continuous mapping that satisfies,*

$$T_J(\Lambda\varrho,\Lambda\rho,\Lambda z) \le \phi(\varrho,\rho,z) T_J(\varrho,\rho,z) \quad \forall \varrho,\rho,z \in \Xi, \tag{3.6}$$

where $\phi : \Xi^3 \to (0,1)$, is a continuous mapping such that

$$\phi((\Lambda\varrho,\Lambda\rho,\Lambda z)) \le \phi(\varrho,\rho,z)$$

let $M(T_J,\Lambda,\varrho) = \sup\left\{T_J(\varrho,\varrho,g^j\varrho) : j \in \mathbb{N} \cup \{0\}\right\}$ and

$$M'(T_J,\phi,\Xi) = \sup\left\{\phi(\Lambda^i\varrho,\Lambda^i\varrho,\Lambda^j\varrho)) : i,j \in \mathbb{N} \cup \{0\}\right\}.$$

If there exist $\varrho_0 \in \Xi$ such that $M(T_j,\Lambda,\varrho_0)$ is finite, and $M'(P_j,\phi,\varrho_0) < 1$. Then Λ has a unique fixed point in Ξ.

Proof. Let ϱ_0 be an element in Ξ and $\{\varrho_n = \Lambda^n\varrho_0\}$.
We begin by showing that $\{\varrho_n\}$ is a Cauchy sequence. For all n,m are natural numbers, we assume that $n < m$ and there is $q \in N$ where $m = n + q$.

$$\begin{aligned} T_J(\varrho_n,\varrho_n,\varrho_m) =& T_J(\Lambda\varrho_{n-1},\Lambda\varrho_{n-1},\Lambda\varrho_{m-1}) \\ \le& \phi(\varrho_{n-1},\varrho_{n-1},\varrho_{m-1}) T_J(\varrho_{n-1},\varrho_{n-1},\varrho_{m-1}) \\ & \vdots \\ \le& \phi^n(\varrho_0,\varrho_0,\varrho_q) T_J(\varrho_0,\varrho_0,\varrho_q). \end{aligned}$$

Since $T_J(\varrho_0,\varrho_0,\varrho_q) \le M(T_J,\Lambda,\varrho_0) < +\infty$ and $\phi(\varrho_0,\varrho_0,\varrho_q) \le M'(T_J,\Lambda,\varrho_0) < 1$ and by taking $n \to +\infty$ and noting that $(M'(T_J,\Lambda,\varrho_0))^n \to 0$, we get

$\lim_{n,m\to\infty} T_J(\varrho_n, \varrho_n, \varrho_m) = 0$, which implies $\{\varrho_n\}$ is a Cauchy sequence.
Now, by the completeness definition, each Cauchy sequence is convergent.

$$\varrho = \lim_{n\to\infty} \varrho_{n+1} = \lim_{n\to\infty} \Lambda\varrho_n = \Lambda x. \tag{3.7}$$

This proves the existence of the fixed point. Next, we proceed to prove the uniqueness. Assume that there are two fixed points ϱ and ρ, $\Lambda(\varrho) = \varrho$ and $\Lambda(\rho) = \rho$

$$\lim_{n\to\infty} T_J(\varrho, \varrho, \varrho_n) = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty} T_J(\Lambda\varrho, \Lambda\varrho, \varrho_n) = 0. \tag{3.9}$$

By using the triangle inequality we get:

$$T_J(\Lambda\varrho, \Lambda\varrho, \varrho) \leq \lim_{n\to\infty} \sup \theta(\Lambda\varrho, \Lambda\varrho) T_J(\Lambda\varrho, \Lambda\varrho, \varrho_n) + \beta(\Lambda\varrho, \Lambda\varrho) T_J(\Lambda\varrho, \Lambda\varrho, \varrho_n) + \gamma(\varrho, \varrho) T_J(\varrho, \varrho, \varrho_n)\Big] \tag{3.10}$$

By applying (3.8) in (3.9) we obtain that $T_J(g\varrho, g\varrho, \varrho) = 0$, and we could repeat the same for $T_J(\Lambda\rho, \Lambda\rho, \rho)$ to get zero also

$$\begin{aligned} T_J(\varrho, \varrho, \rho) =& T_J(\Lambda\varrho, \Lambda\varrho, \Lambda\rho) \\ &\leq \lim_{n\to\infty} \sup \theta(\varrho, \varrho) T_J(\varrho, \varrho, \varrho_n) + \beta(\varrho, \varrho) T_J(\varrho, \varrho, \varrho_n) + \gamma(\rho, \rho) T_J(\rho, \rho, \varrho_n)\Big] = 0. \end{aligned}$$

Which implies that $\varrho = \rho$, that means Λ has a unique fixed point.

□

Example 3.3. (1) *Let $\Xi = [1, \infty)$, and $T_J : \Xi^3 \to [0, \infty)$*

$$T_J(\varrho, \rho, z) = max\{\mid \varrho - \rho \mid, \mid z - \rho \mid\}$$

Let $\theta(\varrho, \rho) = \mid \varrho + \rho \mid$, $\beta(\varrho, \rho) = 2 \mid \varrho + \rho \mid$, and $\gamma(\varrho, \rho) = 3 \mid \varrho + \rho \mid$.
It satisfies all the triple controlled J metric space conditions. And

(2) *Let $\phi : X^3 \to (0, 1), \phi(\varrho, \rho, z) = \dfrac{1}{3 + \mid \varrho - \rho - z \mid}$, where ϕ is continuous function.*

(3) *Take $\Lambda : \Xi \to \Xi$,and $\Lambda(\varrho) = \dfrac{\varrho}{3}$, Λ is continuous function.*
For verifying the theorem conditions

(4) *we need to verify*

$$T_J(g\varrho, g\rho, gz) \leq \phi(\varrho, \rho, z) T_J(\varrho, \rho, z), \forall \varrho, \rho, z \in \Xi$$

$$T_J(\Lambda\varrho, \Lambda\rho, \Lambda z) = \frac{1}{3} max\{\mid \varrho - z \mid, \mid z - \rho \mid\}$$

On the other hand:

$$\phi(\varrho, \rho, z)(T_J(\varrho, \rho, z) = \frac{1}{6}(\phi(\varrho, \rho, z) \mid \varrho - z \mid + \mid z - \rho \mid).$$

Since $\phi(\varrho,\rho,z)$ less than 1 then,

$$T_J(\Lambda\varrho,\Lambda\rho,\Lambda z) \leq \phi(\varrho,\rho,z)T_J(\varrho,\rho,z)$$

So, the condition satisfies

$$T_J(\varrho,\rho,z) \leq \phi(\varrho,\rho,z)T_J(\varrho,\rho,z), \forall \varrho,\rho,z \in \Xi$$

(5) *the existence and the uniqueness of the fixed point of Λ, we solve*

$$\Lambda(\varrho) = \varrho$$

$$\frac{\varrho}{3} = \varrho \Rightarrow \varrho = 0.$$

Thus,$\varrho = 0$ is the unique fixed point of Λ in Ξ.

Theorem 3.3. *Let (Ξ, T_J) is a triple controlled J complete symmetric metric spaces, $\Lambda : X \to X$is a continuous map where :*

$$T_J(\Lambda\varrho,\Lambda\rho,\Lambda z) \leq aT_J(\varrho,\rho,z) + bT_J(\varrho,\Lambda\varrho,\Lambda\varrho) + cC_J(\rho,\Lambda\rho,\Lambda\rho) + dT_J(z,\Lambda z,\Lambda z) \tag{3.11}$$

For each $\varrho,\rho,z \in \Xi$ where

$$0 < a + b < 1 - c - d \tag{3.12}$$

$$0 < a,b,c,d < 1. \tag{3.13}$$

$$0 < a < \frac{a+b}{1-c-d} \tag{3.14}$$

For every $\varrho \in \Xi$, let $M(T_J,\Lambda,\varrho) = \sup\left\{T_J(\varrho,\varrho,\Lambda^j\varrho) : j \in \mathbb{N} \cup \{0\}\right\}$. If there exists $\varrho_0 \in X$ such that $M(T_J,\Lambda,\varrho_0) < +\infty$, Then, there is a unique fixed point of Λ.

Proof. Let $\varrho_0 \in \Xi$,and let $\{\varrho_n = \Lambda^n$
$varrho_0\}$ be any sequence in Ξ. We will start with $T_J(\varrho_n,\varrho_{n=1},\varrho_{n+1})$ to prove that this equals to zero.

$$\begin{aligned}
T_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) = T_J&(\Lambda\varrho_{n-1},\Lambda\varrho_n,\Lambda\varrho_n) \\
\leq &aT_J(\varrho_{n-1},\varrho_n,\varrho_n) + bT_J(\varrho_{n-1},\varrho_n,\varrho_n) \\
&+ cT_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) + dT_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) \\
\leq &(a+b)T_J(\varrho_{n-1},\varrho_n,\varrho_n) + (c+d)T_J(\varrho_n,\varrho_{n+1},\varrho_{n+1})
\end{aligned}$$

And,

$$T_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) \leq \frac{a+b}{1-c-d}T_J(\varrho_{n-1},\varrho_n,\varrho_n).$$

let, $u = \frac{a+b}{1-c-d}$, and then by using (3.12), $0 < u < 1$.

$$T_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) \leq u^n T_J(\varrho_0,\varrho_1,\varrho_1).$$

That implies,

$$\lim_{n\to\infty} T_J(\varrho_n,\varrho_{n+1},\varrho_{n+1}) = 0. \tag{3.15}$$

We denote $T_{J_n} = T_J(\varrho_n, \varrho_{n+1}, \varrho_{n+1})$. For each $n, m \in N, n < m$,and let $m = n + q$.

$$\begin{aligned}
T_J(\varrho_n, \varrho_n, \varrho_m) &= T_J(\varrho_n, \varrho_n, \varrho_{n+q}) = T_J(\Lambda\varrho_{n-1}, \Lambda\varrho_{n-1}, \Lambda\varrho_{n+q-1}) \\
&\leq aT_J(\varrho_{n-1}, \varrho_{n-1}, \varrho_{n+q-1}) + bT_J(\varrho_{n-1}, \varrho_n, \varrho_n) + cT_J(\varrho_{n-1}, \varrho_n, \varrho_n) \\
&+ dT_J(\varrho_{n+q-1}, \varrho_{n+q}, \varrho_{n+q}) \\
&= aT_J(\varrho_{n-1}, \varrho_{n-1}, \varrho_{n+q-1}) + (b + c)T_{J_{n-1}} + dT_{J_{n+q-1}} \\
&\leq a[aT_J(\varrho_{n-2}, \varrho_{n-2}, \varrho_{n+p-2}) + (c + b)T_{J_{n-2}} + dT_{J_{n+q-2}}] + (b + c)T_{J_{n-1}} \\
&+ dT_{J_{n+q-1}} \\
&= a^2T_J(\varrho_{n-2}, \varrho_{n-2}, \varrho_{n+q-2}) + a(c + b)T_{J_{n-2}} + adT_{J_{n+q-2}} + (b + c)T_{J_{n-1}} \\
&+ dT_{J_{n+q-1}} \\
&\vdots \\
&\leq a^nT_J(\varrho_0, \varrho_0, \varrho_q) + (b + c)\sum_{k=1}^{n} a^{k-1}T_{J_{(n-k)}} + d\sum_{k=1}^{n} a^{k-1}T_{J_{(n+q-k)}}.
\end{aligned} \tag{3.16}$$

By using that

$$T_J(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \leq u^nT_J(\varrho_0, \varrho_1, \varrho_1).$$

And

$$T_J(\varrho_{n+q_k}, \varrho_{n+q-k+1}, \varrho_{n+q-k+1}) \leq u^{n+q-k}T_J(\varrho_0, \varrho_1, \varrho_1).$$

$$\begin{aligned}
T_J(\varrho_n, \varrho_n, \varrho_m) &= T_J(\varrho_n, \varrho_n, \varrho_{n+q}) = T_J(\Lambda\varrho_{n-1}, \Lambda\varrho_{n-1}, \Lambda\varrho_{n+q-1}) \\
&\leq a^nT_J(\varrho_0, \varrho_0, \varrho_q) + (b + c)\sum_{k=1}^{n} a^{k-1}T_{J_{(n-k)}} + d\sum_{k=1}^{n} a^{k-1}T_{J_{(n+q-k)}}. \qquad (3.17) \\
&\leq a^nT_J(\varrho_0, \varrho_0, \varrho_q) + (b + c)T_J(\varrho_0, \varrho_1, \varrho_1)u^{n-1}\sum_{k=1}^{n}(\frac{a}{u})^{k-1} \\
&+ dT_J(\varrho_0, \varrho_1, \varrho_1)u^{m-1}\sum_{k=1}^{n}(\frac{a}{u})^{k-1} \\
&\leq a^nT_J(\varrho_0, \varrho_0, \varrho_q) + (b + c)T_J(\varrho_0, \varrho_1, \varrho_1)u^{n-1}(\frac{1 - (\frac{a}{u})^n}{1 - \frac{a}{u}}) \\
&+ dT_J(\varrho_0, \varrho_1, \varrho_1)u^{m-1}(\frac{1 - (\frac{a}{u})^n}{1 - \frac{a}{u}})
\end{aligned}$$

By applying the limit in (3.16) as $n, m \to \infty$, using (3.13) and (3.15), and consider that there is ϱ_0,such that$M(T_J, \Lambda, \varrho_0) < +\infty$, and because $\frac{a}{u} < 1$,then $\lim_{n\to\infty}(\frac{a}{u})^n = 0$ we get

$$\lim_{n,m\to\infty} T_J(\varrho_n, \varrho_n, \varrho_m) = 0.$$

Then,for any $\{\varrho_n\}$ in X, $\{\varrho_n\}$ is Cauchy sequence. By the completeness definition, each Cauchy is Convergent, which means $\varrho_n \to \varrho$ as $n \to \infty$ and

$$\lim_{n\to\infty} T_J(\varrho_n, \varrho_n, \varrho) = \lim_{n\to\infty} T_J(\varrho_n, \varrho_n, \varrho_n) = 0. \tag{3.18}$$

In addition, $\varrho = \lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \varrho_{n+1} = \lim_{n\to\infty} \Lambda\varrho_n = \Lambda\varrho$.
Therefore, Λ has Ξ as a fixed point.
Let $\varrho, \rho \in X$ are two fixed point of g, $\varrho \neq y$,where, $g\varrho = \varrho$, $g\rho = \rho$.

$$\begin{aligned} T_J(\varrho, \varrho, \rho) =& T_J(\Lambda\varrho, \Lambda\varrho, \Lambda\rho) \\ \leq& aT_J(\varrho, \varrho, \rho) + (b + c)T_J(\varrho, \Lambda\varrho, \Lambda\varrho) + dT_J(\rho, \Lambda\rho, \Lambda\rho) \\ =& aT_J(\varrho, \varrho, \rho) + (b + c)T_J(\varrho, \varrho, \varrho) + dT_J(\rho, \rho, \rho). \end{aligned}$$

Then, $(1 - a)T_J(\varrho, \varrho, \rho) \leq 0$. Using (3.13) so this gives $T_J(\varrho, \varrho, \rho) = 0$ that is $\varrho = \rho$,which means that Λ has a unique a fixed point.

□

4. Conclusion

We introduced a novel extension of the J metric space, termed the Triple Controlled J metric space. In this framework, we established the existence and uniqueness of several significant fixed-point theorems, including weak contraction, generalized contraction, and generalized contraction with a control function. This advancement incorporates three control functions θ, β, and γ marking a groundbreaking development for J metric spaces. A key question arises from this work: how would fixed-point theorems behave if the control functions were defined over sequences rather than individual points? This intriguing prospect opens avenues for further exploration and deeper insights into the field.

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