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Geometry of Moving Spacelike Curves and their Evolution Equations in de Sitter 3-Space

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Abstract. In this paper, we study the geometry of moving spacelike curves in the three-dimensional de Sitter space S^3_1. Then, the evolution equations of the pseudo-orthonormal frame and the curvatures for these curves are derived. Moreover, some conditions for an inelastic curve flow in S^3_1 are presented. Finally, interesting illustrative examples of the obtained results are given and plotted.

1. Introduction

One of the important topics related to curves is the geometry of curves evolution. It is a quite area of the differential geometry which deals with curves where the time plays the fundamental rule. Geometrically, curves evolution means that deforming a curve into another curve in a continuous manner. When we study some properties of curves in $\mathbb{R}^3$, we find some links between the geometry of the studied curves and integrable equations [1]. Such this study has been considered by Lamb [2] who introduced a formalism in which certain exceptional kinds of movement of curves can be planned to be completely integrable, solution-supporting [1], nonlinear partial differential equations (NLPDEs) such as the nonlinear Schrödinger (NLS) equation, the Sine-Gordon equation, the Hirota equation, the modified kdv equations, etc., indicating that the corresponding curve motions are also integrable. This formalism arose as an extension of Hasimoto's earlier work [3]. After the work of Hasimoto's, some interesting classes of moving space curves in a three-dimensional space with soliton equations have been investigated [4, 5]. The analysis is extended to more general types of motion and other integrable systems [6, 7]. For more details, one can see [8–16].

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In this work, we shall present a study of evolving spacelike curves in de Sitter 3−space S^3_1 to illustrate their behaviors during the evolution process.
The outline of this paper is as follows: In section 2, we have information about the Lorentzian differential geometry of spacelike curves in S^3_1. Section 3 describes the motion for the considered model and provides some conditions for an inelastic curve flow in S^3_1 that represent the main results. Section 4 is devoted to some examples as an application of our main results. Finally, section 5 contains conclusion.

2. Basic Notions of de Sitter Space S^3_1

We start with a brief review for some basic concepts related to the theory of curves in Minkowski 4−space in for later use, for more details see [17].
Let E^4_1 be Minkowski 4-space with the metric of signature $(-,+,+,+)$. We say that a non-zero vector $\mathbf{v} \in E^4_1$ is spacelike, lightlike (null) or timelike if $\langle \mathbf{v},\mathbf{v}\rangle > 0, \langle \mathbf{v},\mathbf{v}\rangle = 0$ or $\langle \mathbf{v},\mathbf{v}\rangle < 0$, respectively. In the light of this, we can define the signature of a vector $\mathbf{v}$ as:

$$sign(\mathbf{v}) = \begin{cases} 1 & \mathbf{v} \text{ is spacelike,} \\ 0 & \mathbf{v} \text{ is lightlike,} \\ -1 & \mathbf{v} \text{ is timelike.} \end{cases}$$

Similarly, an arbitrary curve $\mathbf{r} = \mathbf{r}(s) : I \longrightarrow E^4_1$ can locally be spacelike, timelike or null, if all of its velocity vectors $\mathbf{r}'(s)$ are, respectively spacelike, timelike or null. So, $\mathbf{r}(s)$ is a unit speed curve if $\langle \mathbf{r}'(s), \mathbf{r}'(s)\rangle = 1$, where s is the arc length parameter of $\mathbf{r}$ and dash $\prime = \frac{d}{ds}$. Two vectors $\mathbf{v}$ and $\mathbf{w} \in E^4_1$ are called orthogonal if $\langle \mathbf{v},\mathbf{w}\rangle = 0$. The norm $\parallel \mathbf{v} \parallel$ of $\mathbf{v} \in E^4_1$ is defined as $\parallel \mathbf{v} \parallel = \sqrt{|\langle \mathbf{v},\mathbf{v}\rangle|}$.
We define the de Sitter 3-space (deS) of constant sectional curvature in E^4_1 by

$$deS = S^3_1 = \{< x, x >= 1 : x \in E^4_1\}. \tag{2.1}$$

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in E^4_1$, we define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x^1_1 & x^2_1 & x^3_1 & x^4_1 \\ x^1_2 & x^2_2 & x^3_2 & x^4_2 \\ x^1_3 & x^2_3 & x^3_3 & x^4_3 \end{vmatrix}, \tag{2.2}$$

where $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of E^4_1. This vector is pseudo- orthogonal to any $x_i (i = 1, 2, 3)$.
Let us consider $\mathbf{r} : I \subset R \to S^3_1$ be a spacelike unit speed curve in S^3_1. So we have the tangent vector $\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}$ with $\parallel \mathbf{T}(s) \parallel = 1$. In the case when $\langle \frac{d\mathbf{T}}{ds}, \frac{d\mathbf{T}}{ds}\rangle \neq 1$, we can define a unit spacelike vector $\mathbf{P}(s)$ by

$$\mathbf{P}(s) = \frac{\mathbf{T}'(s) + \mathbf{r}(s)}{\parallel \mathbf{T}'(s) + \mathbf{r}(s) \parallel},$$

and call it principal normal vector of $\mathbf{r}$. Moreover, we define a vector $\mathbf{Q}(s) = \mathbf{r} \wedge \mathbf{T} \wedge \mathbf{P}$ and call it binormal vector of $\mathbf{r}$.

In this situation, we have a pseudo-orthonormal frame $\{\mathbf{r}(s), \mathbf{T}(s), \mathbf{P}(s), \mathbf{Q}(s)\}$ of E_1^4 along $\mathbf{r}$. The vector $\mathbf{T}$ is a spacelike vector and tangent to each of $\mathbf{r}(s)$ and S_1^3. On the other hand, since $\mathbf{T}$ is a unit spacelike vector, its derivative $\frac{d\mathbf{T}}{ds}$ will be normal to $\mathbf{T}$. The remaining vector $\mathbf{Q}$ of the pseudo-orthonormal frame is taken as a timelike vector. Depending on the causal character of the vectors $\mathbf{T}, \mathbf{P}$ and $\mathbf{Q}$, we have

$$\langle \mathbf{T}, \mathbf{T}\rangle = \langle \mathbf{P}, \mathbf{P}\rangle = 1,\ \langle \mathbf{Q}, \mathbf{Q}\rangle = -1, \langle \mathbf{T}', \mathbf{T}'\rangle > 1,$$

and all the other products vanish. By the standard arguments, under the above assumption that $\langle \frac{d\mathbf{T}}{ds}, \frac{d\mathbf{T}}{ds}\rangle > 1$, we have the following Frenet formulas that describe the geometry of the differentiable curve $\mathbf{r}(s)$ in S_1^3 as

$$\left.\begin{array}{c} \mathbf{r}'(s) = \mathbf{T}(s), \\ \mathbf{T}'(s) = -\mathbf{r}(s) + \kappa_g(s)\mathbf{P}(s), \\ \mathbf{P}'(s) = \kappa_g(s)\delta(\mathbf{r}(s))\mathbf{T}(s) + \tau_g(s)\mathbf{Q}(s), \\ \mathbf{Q}'(s) = \tau_g(s)\mathbf{P}(s), \end{array}\right\} \tag{2.3}$$

where $\delta(\mathbf{r}(s)) = -\text{sign }(\mathbf{P}(s))$. As $\mathbf{P}(s)$ is spacelike, we have $\delta(\mathbf{r}(s)) = -1$.
The functions $\kappa_g(s)$ and $\tau_g(s)$ are defined to be the geodesic curvature and the geodesic torsion scalars of the curve $\mathbf{r}(s)$, respectively, where

$$\kappa_g(s) =\| \mathbf{T}'(s) + \mathbf{r}(s) \|, \tag{2.4}$$

$$\tau_g(s) = -\frac{\delta(\mathbf{r}(s))}{\kappa_g^2(s)} \det(\mathbf{r}(s), \mathbf{r}'(s), \mathbf{r}''(s), \mathbf{r}'''(s)), \kappa_g(s) =\| \mathbf{T}'(s) + \mathbf{r}(s) \|> 0. \tag{2.5}$$

Note that the condition $\langle \frac{d\mathbf{T}}{ds}, \frac{d\mathbf{T}}{ds}\rangle > 1$ is equivalent to the condition $\kappa_g(s) \neq 0$. In the rest of the paper, we suppose everywhere that $\kappa_g =\| \mathbf{T}'(s) + \mathbf{r}(s) \|> 0$ and $\tau_g \neq 0$. In a matrix form, Eqs. (2.3) can be written as

$$\Omega_s = E \cdot \Omega, \tag{2.6}$$

where

$$\Omega = \begin{pmatrix} \mathbf{r} \\ \mathbf{T} \\ \mathbf{P} \\ \mathbf{Q} \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g(s) & 0 \\ 0 & -\kappa_g(s) & 0 & \tau_g(s) \\ 0 & 0 & \tau_g(s) & 0 \end{pmatrix}. \tag{2.7}$$

3. Geometry of inelastic flow of spacelike curves

In this section, we study the motion of spacelike curves in the three-dimensional de Sitter space to establish the kinematics of these curves in terms of their intrinsic geometric formulas.
We assume that $\mathbf{r} = \mathbf{r}(u) : I \subset \mathbb{R} \to S_1^3$ is a differentiable spacelike curve parameterized by an arbitrary parameter $u \in I$ moving in the de Sitter 3−space S_1^3. Let r be given at any second in time t by the position vector $\mathbf{r} = \mathbf{r}(u, t)$, with initial curve $\mathbf{r}_o = \mathbf{r}(u, 0)$. The time parameter t is the parameter for the deformation $\mathbf{r}(u, t)$ of the curve.

The metric on $\mathbf{r}(u,t)$ is expressed as

$$g(u,t) = \langle \mathbf{r}_u, \mathbf{r}_u \rangle; \ \mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}. \tag{3.1}$$

Further, the arc length of the moving curve $\mathbf{r}$ of metric $g(u,t)$ is given by

$$s(u,t) = \int_0^L \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u \rangle}\, du = \int_0^L \sqrt{g}\, du, \frac{\partial}{\partial s} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u}, \ s_u = \sqrt{g}, \tag{3.2}$$

where $\sqrt{g} = \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u \rangle}$. Thus the element of arc length is $ds = \sqrt{g(u,t)}\, du$. With this metric, when the curve is given as $\mathbf{r} = \mathbf{r}(s(u,t))$, then Eq. (2.6) can be written as

$$\Omega_u = \sqrt{g}\, E \cdot \Omega, \tag{3.3}$$

where Ω and E are given as in Eq. (2.7).

The requirement that the curve not be subject to any elongation or compression can be expressed by the condition:

$$s_t(u,t) = \int_0^u \frac{\partial}{\partial t} \sqrt{g(\varrho,t)}\, d\varrho = 0, \ \varrho \in [0,u], u \in [0, u_{\max}]. \tag{3.4}$$

The change (motion) of the curve with respect to the parameter t is specified by the velocity fields:

$$\mathbf{r}_t = \lambda \mathbf{T}(u,t) + \mu \mathbf{P}(u,t) + \nu \mathbf{Q}(u,t), \tag{3.5}$$

where λ, μ, ν are the tangent, normal and bi-normal of $\mathbf{r}_t$ velocities.

To study the motion of the considered curve $\mathbf{r}$ which described by the three functions μ, ν and λ, we seek to get the partial differential equations (PDEs), which describe the evolution of the frame and the curvatures of the evolving curve. Each choice of these functions gives a different class of curves in S_1^3.

Remark 3.1. *The derivatives with respect to u and t commute; whereas the derivatives with respect to s and t in general do not commute.*

For this study it is important to give the following definition.

Definition 3.1. *The curve $\mathbf{r}(u,t)$ is said to be inelastic curve if its length is preserved, i.e., it doesn't evolve in time. Then, we have*

$$s_t = 0, \ i.e., \ g_t = 0. \tag{3.6}$$

3.1. **The time evolution of metric and length.** For a given spacelike curve in S_1^3 we calculate the evolution equations for its metric and the length of the curve. Our results can be stated as follows:

Proposition 3.1. *Let $\mathbf{r} = \mathbf{r}(u,t)$ be a unit speed spacelike curve with $\kappa_g \neq 0$, moving in S_1^3. Then the evolution equation for the metric g of $\mathbf{r}$ is given by*

$$g_t = 2g(\lambda_s - \mu \kappa_g(s)). \tag{3.7}$$

Proof. Since u and t are independent coordinates, for the derivatives in u amd t to commute, the congruence of curves $r(u,t)$ must form a regular surface of type C^2, so that the Schwarz theorem can be applied. So, by differentiating Eq. (3.1), we have

$$\begin{aligned} g_t = \frac{\partial}{\partial t}\langle \mathbf{r}_u, \mathbf{r}_u\rangle = 2\langle \mathbf{r}_u, \frac{\partial}{\partial t}\mathbf{r}_u\rangle = 2g\langle \mathbf{r}_s, \frac{\partial}{\partial s}\mathbf{r}_t\rangle \\ = 2g\langle \mathbf{r}_s, (\lambda\mathbf{T} + \mu\mathbf{P} + \nu\mathbf{Q})_s\rangle. \end{aligned}$$

Frenet equations together with the last expression for the g_t allow us to obtain

$$g_t = 2g\langle \mathbf{T}, [(\lambda_s - \mu\kappa_g)\mathbf{T} + \Theta]\rangle,$$

where

$$\Theta = -\lambda\mathbf{r} + (\mu_s + \lambda\kappa_g + \nu\tau_g)\mathbf{P} + (\nu_s + \mu\tau_g)\mathbf{Q}.$$

Therefore, we get

$$g_t = 2g(\lambda_s - \mu\kappa_g).$$

Hence, the result is clear. □

Proposition 3.2. *Suppose that* $\mathbf{r} : I \subset \mathbb{R} \to S^3_1$ *is a unit speed spacelike curve with* $\langle \frac{d\mathbf{T}}{ds}, \frac{d\mathbf{T}}{ds}\rangle > 1$, *then the time evolution of the arc length of* $\mathbf{r}$ *is given by*

$$L_t = \int_0^L (\lambda_s - \mu\kappa_g)\, ds, \quad s \in [0, L]. \tag{3.8}$$

Proof. Since,

$$L(t) = \int_0^{u_{\max}} \sqrt{g(u,t)}\, du = \int_0^L ds,$$

so, by differentiating concerning time the last integral definition and using Eq. (3.7) and Eq. (3.9), we obtain

$$L_t = \int_0^{u_{\max}} \frac{\partial}{\partial t}(\sqrt{g(u,t)})\, du = \int_0^{u_{\max}} \frac{g_t(u,t)}{2\sqrt{g(u,t)}}\, du = \int_0^L (\lambda_s - \mu\kappa_g)\, ds, \tag{3.9}$$

where $ds = \sqrt{g(u,t)}\, du$, then the proof is completed. □

In the light of Proposition 3.2, we can present the following result which is the main result of this work.

Theorem 3.1. *Let* $\frac{\partial \mathbf{r}}{\partial t}$ *be a smooth flow of* $\mathbf{r}(u,t)$ *in* S^3_1. *Then, the flow of the curve is inelastic if and only if*

$$\lambda_s = \mu\kappa_g. \tag{3.10}$$

Proof. Firstly, let the curve be an inelastic, then from Eq. (3.2), the time variation of the arc length is

$$s_t = \int_0^{u_{\max}} \frac{g_t}{2\sqrt{g}} du. \tag{3.11}$$

Substituting from Eq. (3.7) into Eq. (3.11), we find

$$L_t = \int_0^L (\lambda_s - \mu\kappa_g) ds.$$

According to Definition (3.1), we get

$$\lambda_s = \mu\kappa_g.$$

Secondly, the argument can be reversed by direct computation to show sufficiency, completing the proof. □

3.2. **The time evolution for curve frame and curvatures.**

Theorem 3.2. *Let $\mathbf{r}(u,t)$ be an elastic spacelike curve in the de Sitter space S_1^3. Then*
(i) The time evolution equations of the pseudo-orthonormal frame of the curve can be expressed in a matrix form

$$\Omega_t = F\cdot\Omega, \tag{3.12}$$

where F is the evolution matrix and Ω is the frame matrix, they are given by

$$F = \begin{pmatrix} 0 & \lambda & \mu & \nu \\ -\lambda & 0 & \alpha & \beta \\ -\mu & -\alpha & 0 & \Gamma \\ \nu & \beta & \Gamma & 0 \end{pmatrix}, \ \Omega = \begin{pmatrix} \mathbf{r} \\ \mathbf{T} \\ \mathbf{P} \\ \mathbf{Q} \end{pmatrix},$$

Taking into account:

$$\begin{aligned} \alpha &= \mu_s + \lambda\kappa_g + \nu\tau_g,\ \beta = \nu_s + \mu\tau_g, \\ \Gamma &= \frac{1}{\kappa_g}(\beta_s + \alpha\tau_g + \nu). \end{aligned}$$

(ii) The time evolution equations of $\mathbf{r}(u,t)$ satisfy the following matrix equation

$$\begin{pmatrix} \kappa_g \\ \tau_g \end{pmatrix}_t = \begin{pmatrix} \frac{-g_t}{2g} & \beta \\ \beta & \frac{-g_t}{2g} \end{pmatrix}\begin{pmatrix} \kappa_g \\ \tau_g \end{pmatrix} + \begin{pmatrix} \alpha_s + \mu \\ \Gamma_s \end{pmatrix}. \tag{3.13}$$

Proof. Since the unit tangent vector $\mathbf{T} = \frac{1}{\sqrt{g}}\frac{d\mathbf{r}(u)}{du}$, then

$$\mathbf{r}_{ut} = (\sqrt{g}\,\mathbf{T})_t = \sqrt{g}\,(\frac{g_t}{2g}\mathbf{T} + \mathbf{T}_t), \tag{3.14}$$

where we used $\partial_u = \sqrt{g}\,\partial_s$. On the other side, we have

$$\mathbf{r}_{tu} = \sqrt{g}\,\mathbf{r}_{ts} = \sqrt{g}[-\lambda\mathbf{r} + (\lambda_s - \mu\kappa_g)\mathbf{T} + \alpha\mathbf{P} + \beta\mathbf{Q}], \tag{3.15}$$

where α and β are as in the above.
The following compatibility condition of Eqs. (3.14) and (3.15),

$$\frac{\partial}{\partial t}\,\mathbf{r}_u = \frac{\partial}{\partial u}\,\mathbf{r}_t, \tag{3.16}$$

gives rise to the following equations:

$$\mathbf{T}_t = -\lambda\mathbf{r} + \alpha\mathbf{P} + \beta\mathbf{Q},$$

$$g_t = 2g(\lambda_s - \mu\kappa_g). \tag{3.17}$$

The second equation of (3.17) is the same as we have obtained in Eq. (3.7).

Later on, by differentiating the second equation of (2.3) concerning time t, furthermore, change the $s-$derivatives into $u-$derivatives through $\partial_u = \sqrt{g}\,\partial_s$, we acquire

$$g^{-\frac{1}{2}}\mathbf{T}_{ut} = [-\frac{g_t}{2g}\mathbf{r} - \lambda\mathbf{T} + ((-\mu + (\kappa_g)_t + \frac{g_t}{2g}\kappa_g))\mathbf{P} - \nu\mathbf{Q} + \kappa_g\mathbf{P}_t]. \tag{3.18}$$

Using the commutativity between derivatives, then substituting $\mathbf{T}_t$ from Eq. (3.17), and utilizing again Frenet frame, we find

$$g^{-\frac{1}{2}}\mathbf{T}_{tu} = [-\lambda_s\mathbf{r} + (-\lambda - \alpha\kappa_g)\mathbf{T} + (\alpha_s + \beta\tau_g)\mathbf{P} + (\beta_s + \alpha\tau_g)\mathbf{Q}]. \tag{3.19}$$

By recognizing of Eqs. (3.18) and (3.19) and using Eq. (3.7), we get

$$\mathbf{P}_t = -\mu\mathbf{r} - \alpha\mathbf{T} + \Gamma\mathbf{Q}, \tag{3.20}$$

and

$$(\kappa_g)_t = \alpha_s + \beta\tau_g + \mu - \frac{g_t}{2g}\kappa_g. \tag{3.21}$$

Likewise, we are going to obtain the time evolution equations for the bi-normal vector $\mathbf{Q}$ and the geodesic torsion τ_g of $\mathbf{r}$ as follows:
Since $\mathbf{Q}(s) = \mathbf{r}(s) \wedge \mathbf{T}(s) \wedge \mathbf{P}(s)$, then we have

$$\mathbf{Q}_t = \mathbf{r}_t \wedge \mathbf{T} \wedge \mathbf{P} + \mathbf{r} \wedge \mathbf{T}_t \wedge \mathbf{P} + \mathbf{r} \wedge \mathbf{T} \wedge \mathbf{P}_t. \tag{3.22}$$

By using Eqs. (3.5), (3.17) and (3.20) and substituting in Eq. (3.22)

$$\mathbf{Q}_t = \nu\mathbf{r} + \beta\mathbf{T} + \Gamma\mathbf{P}. \tag{3.23}$$

The compatibility condition: $\mathbf{Q}_{tu} = \mathbf{Q}_{ut}$, where

$$\mathbf{Q}_{tu} = \sqrt{g}[(\nu_s - \beta)\mathbf{r} + (\nu + \beta_s - \Gamma\kappa_g)\mathbf{T} + (\beta\kappa_g + \Gamma_s)\mathbf{P} + \Gamma\tau_g\mathbf{Q}], \tag{3.24}$$

and

$$\mathbf{Q}_{ut} = \sqrt{g}[-\mu\tau_g\mathbf{r} - \alpha\tau_g\mathbf{T} + ((\tau_g)_t + \frac{g_t}{2g}\tau_g)\mathbf{P} + \Gamma\tau_g\mathbf{Q}], \tag{3.25}$$

leads to

$$(\tau_g)_t = \beta\kappa_g + \Gamma_s - \frac{g_t}{2g}\tau_g, \tag{3.26}$$

Eqs. (3.17), (3.20), (3.21), (3.23) and (3.26) give the required result and complete the proof. □

The following lemma is clear from Theorem 3.2.

Lemma 3.1. *Let $\mathbf{r}(u,t)$ be an inelastic spacelike curve in S^3_1, then the evolution equations for the $\kappa_g(s)$ and $\tau_g(s)$ of r are given by*

$$\begin{pmatrix} \kappa_g \\ \tau_g \end{pmatrix}_t = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \kappa_g \\ \tau_g \end{pmatrix} + \begin{pmatrix} \alpha_s + \mu \\ \Gamma_s \end{pmatrix}. \tag{3.27}$$

Proof. Because the curve is inelastic, $g_t = 0$ i.e., $\lambda_s - \mu\kappa_g = 0$. Then, Eq. (3.13) together Definition 3.1 complete the proof. □

3.3. **Zero curvature condition.** In this subsection, we discuss an interesting property for spacelike curves $\mathbf{r}(s,t)$ in S_1^3, so called the linear problem integrability of Eqs. (2.6) and (3.12), and we are considered that in the following theorem.

Theorem 3.3. *Let $\mathbf{r}(s,t)$ be an inelastic spacelike curve in S_1^3 and Ω be Frenet matrix that satisfies Eqs.* (3.3) *and* (3.12)*. Then, we have the integrability condition:*

$$E_t - F_s + [E,F] = 0, \tag{3.28}$$

where

$$[E,F] = E\cdot F - F\cdot E,$$

is the Lie bracket of E and F.

Proof. Let $\{\mathbf{r},\mathbf{T},\mathbf{P},\mathbf{Q}\}$ be the pseudo-orthonormal frame of the given inelastic curve $\mathbf{r}$ such that the frame satisfies Eqs. (3.3) and (3.12). Since $\mathbf{r}$ is inelastic, so $(g_t = 0)$ and the ordering of derivatives commute, i.e.,

$$\frac{\partial}{\partial t}\Omega_u = \frac{\partial}{\partial u}\Omega_t. \tag{3.29}$$

By differentiating Eq. (3.3) with respect to t and (3.12) with respect to u and using $g_t = 0$, we get respectively,

$$\Omega_{ut} = \sqrt{g}(E_t + E\cdot F)\cdot\Omega, \tag{3.30}$$

$$\Omega_{tu} = \sqrt{g}(F_s + F\cdot E)\cdot\Omega. \tag{3.31}$$

By means of the obtained equations, the theorem holds. □

Lemma 3.2. *According to Theorem* 3.3*, if the integrability condition of Eq.* (3.28) *is satisfied, then there exists a system as a partial differential equation (PDE) can be derived as follows:*

As we mentioned above, the matrices E and F are

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g(s) & 0 \\ 0 & -\kappa_g(s) & 0 & \tau_g(s) \\ 0 & 0 & \tau_g(s) & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & \lambda & \mu & \nu \\ -\lambda & 0 & \alpha & \beta \\ -\mu & -\alpha & 0 & \Gamma \\ \nu & \beta & \Gamma & 0 \end{pmatrix}. \tag{3.32}$$

The t derivative of E and the s derivative of F are respectively, given by

$$E_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\kappa_g)_t & 0 \\ 0 & (-\kappa_g)_t & 0 & (\tau_g)_t \\ 0 & 0 & (\tau_g)_t & 0 \end{pmatrix} \text{ and } F_s = \begin{pmatrix} 0 & \lambda_s & \mu_s & \nu_s \\ -\lambda_s & 0 & \alpha_s & \beta_s \\ -\mu_s & -\alpha_s & 0 & \Gamma_s \\ \nu_s & \beta_s & \Gamma_s & 0 \end{pmatrix}. \tag{3.33}$$

Thus the Lie bracket $[E,F]$ is obtained

$$[E,F] = \left[a_{ij}\right], \quad i,j = 1,2,3,4, \tag{3.34}$$

where

$$
\begin{aligned}
a_{ii} &= 0 \text{ for all } i, \\
a_{12} &= \mu\kappa_g,\ a_{13} = F_1 - \lambda\kappa_g + \nu\tau_g(s),\ a_{14} = F_2 - \mu\tau_g, \\
a_{21} &= -\mu\kappa_g,\ a_{23} = -\mu - F_2\tau_g,\ a_{24} = -\nu + \Gamma\kappa_g - F_1\tau_g, \\
a_{31} &= \lambda\kappa_g - \nu\tau_g - F_1,\ a_{32} = -F_2\tau_g + \mu,\ a_{34} = -F_2\kappa_g, \\
a_{41} &= -\mu\tau_g - F_2,\ a_{42} = -F_1\tau_g + \nu + \Gamma\kappa_g, a_{43} = F_2\kappa_g.
\end{aligned}
$$

Substituting from Eqs. (3.32), (3.33) and (3.34) in Eq. (3.28), then the required system is achieved.

4. Applications

We consider the motion of an inelastic spacelike curve in S^3_1 (see Figure 1), to give some examples of time evolution equations for this one. Different choices for the tangent, normal and bi-normal components of the curve velocities will be considered to give various sorts of nonlinear equations in terms of the geodesic curvatures of the curve.

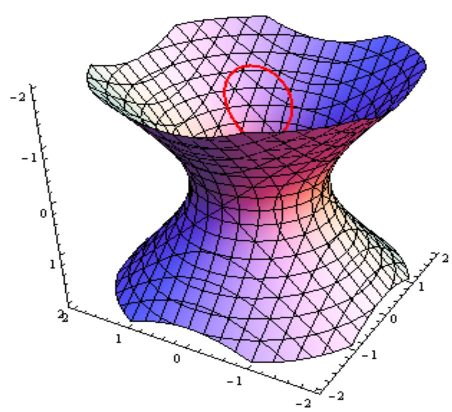


Figure 1. A spacelike curve in S^3_1 (de Sitter space described by the hyperboloid of one sheet).

Example 4.1. *Consider an inelastic spacelike curve* $\mathbf{r}(s,t)$ *in* S^3_1. *If we choose the tangential component of the curve velocity with* $A = \lambda = const. \neq 0$, *zero normal component* $\mu = 0$, *and bi-normal component* $\nu = \kappa_g(s,t) + A$, *then the dynamical equations for the geodesic curvature and the geodesic torsion are given by*

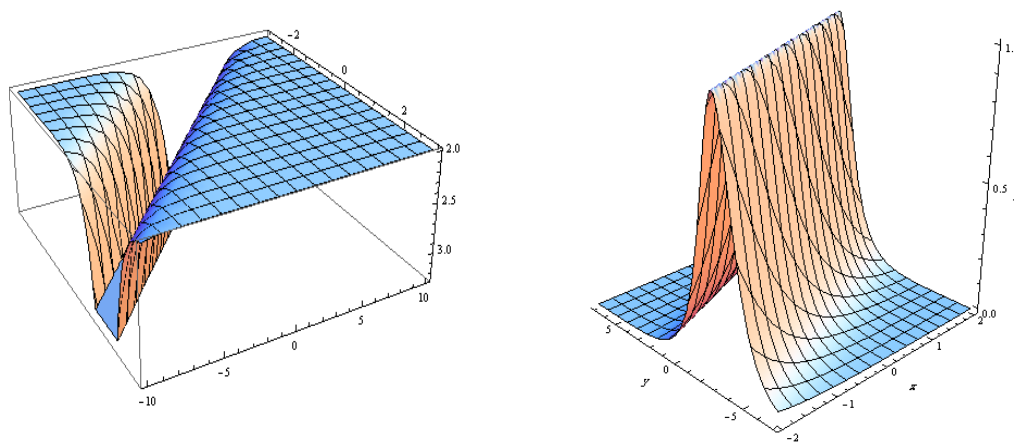
$$(\kappa_g)_t = (A + 2\tau_g)(\kappa_g)_s + (\kappa_g + A)(\tau_g)_s,$$

$$(\tau_g)_t = -\kappa_g(\kappa_g)_s + \frac{\partial}{\partial s}\Big[\frac{1}{\kappa_g}(1+\tau_g^2)(\kappa_g + A) + A\tau_g + \frac{(\kappa_g)_{ss}}{\kappa_g}\Big].$$

The solution of these PDEs is

$$\kappa_g(s,t) = -A,$$
$$\tau_g(s,t) = c_3 + c_4 \operatorname{sech}(\frac{c_1 s}{A} + c_1 t + c_2),$$

where c_1, c_2, c_3 and c_4 are constants. The time evolution of the geodesic curvatures for $s \in [0,7], t \in [0,3]$, $A = 0.4, c_1 = 1.2, c_2 = 1.1, c_3 = 1.3$ *and* $c_4 = 1.6.$, are shown in Figures (2a, 2b).



(A) (B)

FIGURE 2. (a) The time evolution of $\kappa_g(s,t)$, (b) The time evolution of $\tau_g(s,t)$.

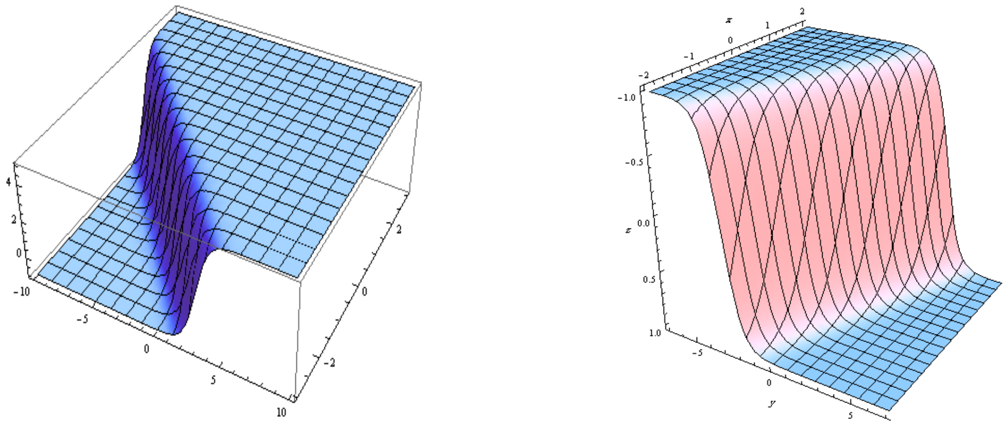
Example 4.2. *Let* $\mathbf{r}(s,t)$ *be an inelastic spacelike curve in* S_1^3*. Another possible choice for the tangential, normal and bi-normal velocities is given as follows:*
$\lambda = A$, $\mu = 0$, $\nu = \frac{\kappa_g(s,t)}{A}$, *the differential equations for* κ_g *and* τ_g *are obtained as follows:*

$$(\kappa_g)_t = \kappa_g(\tau_g)_s + 2\tau_g(\kappa_g)_s,$$
$$(\tau_g)_t = -\kappa_g(\kappa_g)_s + \frac{\partial}{\partial s}[(1+\tau_g^2) + \frac{(\kappa_g)_{ss}}{\kappa_g}],$$

and the resulting solution equations for $(\kappa_g)_t$ and $(\tau_g)_t$ are

$$\kappa_g(s,t) = 2c_1 \tanh(c_1 s + c_2 t + c_3),$$
$$\tau_g(s,t) = \frac{A(-c_1 A + c_2)}{2c_1},$$

where c_1, c_2, c_3 are constants. The time evolution of the geodesic curvatures for $s \in [0,5], t \in [0,4]$, $A = 0.8, c_1 = 1.3, c_2 = 1.5$ *and* $c_3 = 1.7$, are shown in Figures (3a, 3b).



(A) (B)

Figure 3. (a) The time evolution of $\kappa_g(s,t)$, (b) The time evolution of $\tau_g(s,t)$.

Example 4.3. *For a given an inelastic spacelike curve* $\mathbf{r}(s,t)$ *in de Sitter* S_1^3*, choose* $\lambda = s$ *and* $\mu = \nu = 0$*, we obtain the equations:*

$$(\kappa_g)_t = s(\kappa_g)_s,$$
$$(\tau_g)_t = \frac{\partial}{\partial s}(s\tau_g).$$

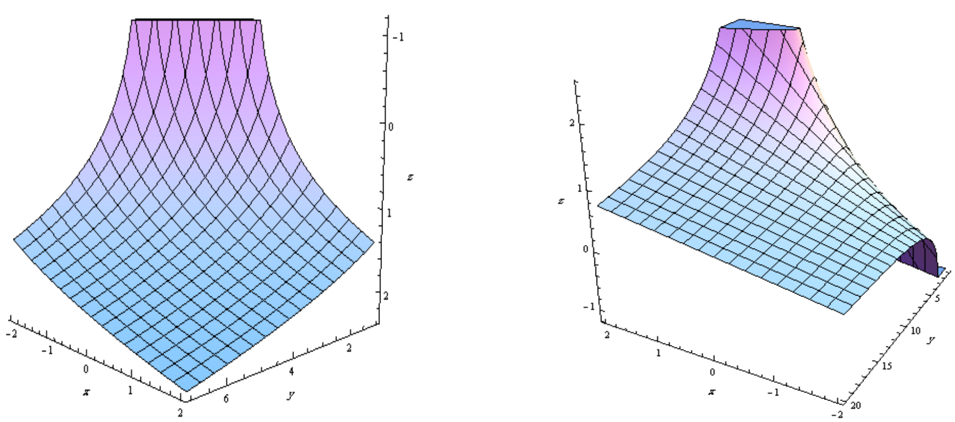
They have the solutions:

$$\kappa_g(s,t) = c_1(t + \ln s),$$

and

$$\tau_g(s,t) = \frac{c_1(t + \ln s)}{s},$$

respectively, where c_1 is a constant. The time evolution of the geodesic curvatures for $s \in [1,4], t \in [0, 2.5]$ and $c_1 = 1.9$, are shown in Figures (4a, 4b).



(A) (B)

Figure 4. (a) The time evolution of $\kappa_g(s,t)$, (b) The time evolution of $\tau_g(s,t)$.

5. Conclusion

The study of curves in Lorentz geometry represents one of the important subjects, where the curves have interesting uses in many fields such as computer vision, robotics and physical science. This paper aims at studying a special type of curves, namely spacelike curves in de Sitter 3-space. Then, a set of partial differential equations that characterize the time evolution equations of the meant curves has been derived. Necessary and sufficient conditions, such as the curve flow is inelastic and the integrability conditions for the evolutions, have been obtained. Finally, some examples of motions of inelastic spacelike curves have been given and plotted.
In future works, we plan to study the geometry of moving spacelike and timelike curves in different spaces for some queries and further improve the results in this paper, combined with the techniques and results in [18–24].

Conflict of Interests: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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