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Spinor Formulation of Frenet Normal Spherical Image in Euclidean and Pseudo-Euclidean Spaces

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Abstract. In this paper, we introduce one of the spherical images of a regular curve by translating Frenet frame vectors to the center of the unit sphere (Lorentizian sphere) of the Euclidean 3-space $\mathbb{E}^3$ (pseudo-Euclidean 3-space $\mathbb{E}^{1,2}$). Especially, Frenet formulas for the normal spherical image of a regular curve can be obtained in terms of spinors. As a result of this study, we found that Frenet equations for that one can be simplified to a single equation with two complex components. Finally, interesting illustrative examples of the obtained results are given and plotted.

1. Introduction

Clifford Algebra is a significant subject across various disciplines. Geometry and its related aspects of mathematics, as well as other subjects, undoubtedly warrant further discussion. In mathematics, spin representations are specific types of projective representations associated with orthogonal or special orthogonal groups in any dimension. Although they are typically explored using real or complex numbers, they can also be formulated over other mathematical structures [1, 2]. The elements of a spin representation, known as spinors, were first introduced in a geometric context by the French mathematician E. Cartan in 1913, [3]. He represented spinors, consisting of two complex components, using vectors in three-dimensional Euclidean space. Additionally, P. Ehrenfest coined the term "spinors" in his research on quantum physics [4]. W. Pauli later introduced spin matrices and was the first to apply spinors in mathematical physics. Subsequently, P. A. M. Dirac established the relationship between spinors and the Lorentz groups. Subsequently, physicists determined that spinors are essential for defining the inherent angular momentum, or spin, of electrons and other subatomic particles within the realm of Quantum Mechanics. In geometry and physics, spinors are elements of a complex vector space that are associated with

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Euclidean space. Similar to geometric vectors or, more broadly, tensors, spinors undergo linear transformations when Euclidean space experiences an infinitesimal rotation, and this property is what defines spinors. In the present time, spinors have a broad range of applications, especially in the fields of physics and mathematics, particularly in the theory of relativity. [2,3,5,6]. Several researchers [7–9] have explored the connection between spinors and orthogonal triads of vectors refers to how spinors can represent or encode sets of three mutually perpendicular vectors. for certain curves using their Frenet formulas in both Euclidean and semi-Euclidean spaces. Additionally, in [10], Şenyurt formulated spinor representations of the Serret-Frenet formulae for a curve based on the Sabban frame. Furthermore, Abdel-Aziz investigated spinor representations of the Frenet and Darboux equations within three-dimensional pseudo-Galilean space [11]. Moreover, in [12], the authors introduced a novel approach to hyperbolic spinor $b-$Darboux equations. The spherical images of a regular curve in Euclidean space are determined using Serret-Frenet frame vector fields, making this classical topic a well-established concept in the differential geometry of curves, see for more details [13,14].
The aim of this work, Frenet frame of a normal spherical image of a space curve is symbolized by the aid of spinors with two Euclidean and pseudo-Euclidean components, respectively. In other words, spinors are studied as a means to describe or represent orthogonal triads of vectors (Serret-Frenet formulae) of that one via a single spinor equation.

The organization of this paper is as follows: In section 2, we have introduced some geometric properties concerning the central themes of Euclidean and pseudo-Euclidean 3-spaces. Section 3 is devoted to present some important characteristics and facts related to the research topic. Spinor approach which equivalent to Frenet equations for a normal image of a space curve in E^3 and $E^{1,2}$ spaces are obtained in section 4. Finally, we concluded this work with a summary of our findings in Section 5.

2. Fundamental Concepts

Let us review the fundamental concepts of three-dimensional Euclidean and pseudo-Euclidean spaces (see for more details [12–16]).

In Euclidean 3-space $\mathbb{E}^3$, it is well known that for any unit-speed curve with at least four continuous derivatives, three mutually orthogonal unit vector fields can be associated: the tangent vector ζ, the normal vector n, and the binormal vector F [13]. We consider the standard metric in Euclidean 3-space $\mathbb{E}^3$, which is given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$.

The sphere of radius $r > 0$ centered at the origin in the space $\mathbb{E}^3$ is given by

$$S^2 = \{m = (m_1, m_2, m_3) \in \mathbb{E}^3 : \langle m, m \rangle = r^2\}.$$

Let $\alpha : I \subset R \to \mathbb{E}^3, \alpha = \alpha(s)$ be an arbitrary space curve in $\mathbb{E}^3$. The curve α is called a unit speed curve (or arclength parameterized) if $\langle \alpha\prime(s), \alpha\prime(s)\rangle = 1$ for any $s \in I$. Throughout this work, we will assume that α is a unit speed curve.

Let$\{\zeta(s)$,$n(s)$,$F(s)\}$ be the moving Frenet frame along α, where the vectors ζ, n and F are mutually orthogonal and satisfy the following conditions:

$$\langle \zeta(s), \zeta(s)\rangle = \langle n(s), n(s)\rangle = \langle F(s), F(s)\rangle = 1.$$

The Frenet equations corresponding to α are expressed as follows [15]:

$$\begin{pmatrix} \zeta(s) \\ n(s) \\ F(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \zeta(s) \\ n(s) \\ F(s) \end{pmatrix}. \tag{2.1}$$

Now, we present the following definition which is of great relevance to the research topic.

Definition 2.1. Consider α as a unit-speed regular curve in three-dimensional Euclidean space with associated Frenet vectors ζ, n and F. The normal vectors along the curve $\alpha(s)$ trace out a curve $\alpha_n = n$ positioned on the unit sphere centered at the origin. This curve α_n is known as the spherical image of n or more commonly, α_n is called normal image of the curve α. If $\alpha = \alpha\ (s)$ is a natural representation of the curve α , then $\alpha_n(s) = n(s)$ serves as a representation of α_n. Similarly, one can define the tangent spherical image $\alpha_\zeta = \zeta(s)$ and binormal spherical image $\alpha_F = F(s)$.

Denote by $\{\zeta_n, n_n, F_n\}$ the moving Frenet frame along the normal image $\alpha_n(s) = n(s)$, then we have Frenet formula:

$$\begin{pmatrix} \zeta_n(s_n) \\ n_n(s_n) \\ F_n(s_n) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_n & 0 \\ -\kappa_n & 0 & \tau_n \\ 0 & -\tau_n & 0 \end{pmatrix} \begin{pmatrix} \zeta_n(s_n) \\ n_n(s_n) \\ F_n(s_n) \end{pmatrix}, \tag{2.2}$$

where

$$\zeta_n = \frac{-\zeta + \Omega F}{\sqrt{1+\Omega^2}}, \ n_n = \frac{\sigma}{\sqrt{1+\sigma^2}}\left(\frac{\Omega\zeta + F}{\sqrt{1+\Omega^2}} - \frac{n}{\sigma}\right), \ F_n = \frac{1}{\sqrt{1+\sigma^2}}\left(\frac{\Omega\zeta + F}{\sqrt{1+\Omega^2}} + \sigma n\right), \tag{2.3}$$

and

$$s_n = \int \kappa(s)\sqrt{1+\Omega^2(s)}ds, \ \kappa_n = \sqrt{1+\sigma^2}, \ \tau_n = \varpi\sqrt{1+\sigma^2},$$

$$\Omega = \frac{\tau}{\kappa}, \ \sigma = \frac{\Omega'(s)}{\kappa(s)((1+\Omega^2(s))^{\frac{3}{2}}}, \ \varpi = \frac{\sigma'(s)}{\kappa(s)\sqrt{1+\Omega^2(s)}(1+\sigma^2(s))^{\frac{3}{2}}}. \tag{2.4}$$

The parameter s_n is natural representation of the normal image $\alpha_n = n$ and κ_n and τ_n are the curvature and torsion of α_n. Therefore we have:

$$\frac{\tau_n}{\kappa_n} = \varpi. \tag{2.5}$$

The pseudo-Euclidean 3-space $\mathbb{E}^{1,2}$ is the real vector space equipped with the standard flat Lorentzian metric defined by

$$\langle,\rangle_L = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) represents a Cartesian coordinate system of $\mathbb{E}^{1,2}$. We denote the Clifford algebra on $\mathbb{E}^{1,2}$ by $Cl_{1,2}(\mathbb{R})$.

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are randomly chosen vectors in $\mathbb{E}^3$, the Lorentzian cross product is defined for u and v as follows [15,16]:

$$(u \times v)_L = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where e_1, e_2, e_3 is a standard orthonormal basis for $\mathbb{E}^{1,2}$ which consists of three vectors perpendicular to each other, two of which have norm $(+1)$ and the other has norm (-1).

In $\mathbb{E}^{1,2}$, the vectors are defined by the Lorentzian inner product. For a vector $v \in \mathbb{E}^{1,2}$, the vector v is said to be a spacelike vector, lightlike (or null) vector or a timelike vector if $\langle v, v\rangle_L > 0$ or $v = 0$, $\langle v, v\rangle_L = 0$ or $\langle v, v\rangle_L < 0$, respectively. Additionally, curves are categorized based on the nature of their tangent vectors. A curve is classified as spacelike, timelike, or lightlike (null) if its tangent vector remains spacelike, timelike, or lightlike, respectively, at all points. For $v \in \mathbb{E}^{1,2}$, the norm of the vector v is defined by

$$\| v \|_L = \sqrt{| \langle v, v\rangle |},$$

and v is called a unit vector if $\| v \|_L = 1$, [16,17].

The Lorentzian sphere of radius 1 in $\mathbb{E}^{1,2}$ is given by

$$S_1^2 = \{O = (o_1, o_2, o_3) \in \mathbb{E}^{1,2} : \langle O, O\rangle_L = 1\}.$$

Given a curve $\beta = \beta(s)$ with pseudo arclength parameter s in $\mathbb{E}^{1,2}$. The Frenet frame of β is defined by the set $\{\xi, P, Q\}$, where

$$\xi(s) = \beta'(s), \ \ P(s) = \frac{\beta''(s)}{\| \beta''(s) \|_L}, \ \ Q(s) = \xi(s) \times_L P(s),$$

correspond to the tangent, normal, and binormal vector fields, in that order. Consequently, the Frenet equations for the curve β are expressed as follows:

$$\begin{pmatrix} \xi(s) \\ P(s) \\ Q(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\varepsilon_\xi \varepsilon_P \kappa(s) & 0 & \tau(s) \\ 0 & -\varepsilon_P \varepsilon_Q \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \xi(s) \\ P(s) \\ Q(s) \end{pmatrix}, \tag{2.6}$$

where $\varepsilon_\xi = \langle \xi, \xi\rangle_L$, $\varepsilon_P = \langle P, P\rangle_L$ and $\varepsilon_Q = \langle Q, Q\rangle_L$), κ and τ denote the curvatures of the curve β, respectively [18].

As part of curve analysis, the pseudo-spherical image of a closed space curve $\beta(s)$ with its Frenet quantities $\{\xi(s), P(s), Q(s), \kappa(s), \tau(s)\}$ consists of curves on the unit pseudo-sphere that are closely associated with the curvatures of the primary curve in $\mathbb{E}^{1,2}$. The tangent image $\beta_\xi = \xi(s)$ of the curve β is defined as the trajectory on the unit pseudo-sphere outlined by the tangent vector $\xi(s)$ of the curve. It can then be expressed as follows:

$$\beta_\xi(s_\xi) = \xi(s) = \frac{\beta'(s)}{\| \beta'(s) \|_L}, \quad s_\xi = \int \kappa(s)ds.$$

Similarly, the normal image $\beta_P = P(s)$ of the curve β is defined as the path on the unit pseudo-sphere outlined by the normal vector $P(s)$ of the curve. Hence, it can be represented as follows:

$$\beta_P(s_P) = P(s) = \xi(s) \times_L Q(s), \quad s_P = \int \sqrt{(\kappa(s))^2 + (\tau(s))^2}ds.$$

Similarly, binormal image $\beta_Q = Q(s)$ of the curve β is defined as a curve on the unit pseudo sphere outlined by the binormal $Q(s)$ of the curve. The curve $\beta_Q = Q(s)$ is represented by

$$\beta_Q(s_Q) = Q(s) = \frac{\beta'(s) \times_L \beta''(s)}{\| \beta'(s) \times_L \beta''(s) \|_L}, \; s_Q = \int \tau(s)ds.$$

In what follows, we take into account the Frenet frame of the normal image $\beta_P = P(s)$ of a nonnull curve in $\mathbb{E}^{1,2}$ is $\{\xi_P(s_P), P_P(s_P), Q_P(s_P)\}$. Consequently, the derivative equations for this frame are expressed as follows:

$$\begin{pmatrix} \xi_P(s_P) \\ P_P(s_P) \\ Q_P(s_P) \end{pmatrix}' = \begin{pmatrix} 0 & \varepsilon_1 \kappa_P(s_P) & 0 \\ -\varepsilon_0\, \kappa_P(s_P) & 0 & -\varepsilon_0\varepsilon_1\tau_P(s_P) \\ 0 & -\varepsilon_0\varepsilon_1\tau_P(s_P) & 0 \end{pmatrix} \begin{pmatrix} \xi_P(s_P) \\ P_P(s_P) \\ Q_P(s_P) \end{pmatrix}, \tag{2.7}$$

where the vector fields ξ_P , P_P and Q_P are

$$\begin{aligned} \xi_P &= -\frac{\varepsilon_0(\xi + \varepsilon_1\Omega\, Q)}{\sqrt{| 1 - \varepsilon_1\Omega^2 |}}, \; P_P = \frac{\varepsilon_0\varepsilon_1\varepsilon_2\sigma}{\sqrt{| 1 + \varepsilon_0\varepsilon_2\sigma^2 |}}\Big(\frac{\Omega\, \xi + \varepsilon_1 Q}{\sqrt{1 - \varepsilon_1\Omega^2}} + \frac{P}{\sigma}\Big), \\ Q_P &= \frac{\sigma}{\sqrt{| 1 + \varepsilon_0\varepsilon_2\sigma^2 |}}\Big(\frac{\Omega\, \xi + \varepsilon_1 Q}{\sqrt{1 - \varepsilon_1\Omega^2}} + \varepsilon_1\varepsilon_2\sigma P\Big), \end{aligned}$$

with the notion

$$\varepsilon_2 = 1 \text{ if } \; \varepsilon_1\Omega^2 - 1 > 0, \;\; \varepsilon_2 = -1 \text{ if } \; \varepsilon_1\Omega^2 - 1 < 0,$$

with noting that the arclength s_P, the curvature κ_P and torsion τ_P for the curve $\beta_P = P$ are given as follows:

$$\begin{aligned} s_P &= \int \sqrt{(\kappa^2(s) + \tau^2(s))}ds, \;\; \kappa_P = \sqrt{| 1 + \varepsilon_0\varepsilon_2\sigma^2 |}, \; \sigma = \frac{\Omega'(s)}{\kappa(s) \mid 1 - \Omega^2(s) \mid^{\frac{3}{2}}}, \;\; \Omega = \frac{\tau}{\kappa}, \\ \tau_P &= \frac{[\kappa^3(1 - \varepsilon_1\Omega^2)[\Omega\, \kappa'' - \varepsilon_1\, \tau'' + \varepsilon_0(1 - \varepsilon_1)\kappa^3\Omega(1 - \varepsilon_1\Omega^2)] + 3\varepsilon_0\kappa^2\Omega'(-\varepsilon_1\kappa\kappa' + \tau\tau')]}{(1 - \varepsilon_1\Omega^2)^3(1 + \varepsilon_0\varepsilon_2\sigma^2)}, \\ \varpi &= \frac{\sigma'(s)}{\kappa(s) \sqrt{| 1 - \varepsilon_1\Omega^2(s) |} \mid (1 + \varepsilon_0\varepsilon_2\sigma^2(s)) \mid^{\frac{3}{2}}}. \end{aligned}$$

3. Spinors Geometry

In this section, we present the fundamental concepts and certain properties of spinors that were introduced in [17].

In Euclidean 3-space, the property of homomorphism represents a close connection between two groups; the first is the rotation group centered at the origin, represented by $SO(3)$ and the second is the group $SU(2)$ consists of unitary 2×2 complex matrices with a determinant of one. Consequently, there exists a two-to-one homomorphism from $SU(2)$ onto $SO(3)$. While elements of $SU(2)$ act on two-component complex vectors, known as spinors, elements of $SO(3)$ operate on real vectors in three dimensions, representing points in Euclidean 3-space [18].

We take into account numbers in the form of $\varpi_1 + J\varpi_2$ where ϖ_1 and ϖ_2 real numbers and J is a commutative element that fulfills the relation $J^2 = \mp 1$ in $\mathbb{E}^3$ and $\mathbb{E}^{1,2}$, respectively.

Assume that the three dimensional space $\mathbb{E}^3$ or $\mathbb{E}^{1,2}$ pertains to a system of mutually perpendicular coordinates; let $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ be an isotropic vector, i.e., has zero length ($\Gamma \neq 0$). We can correspond this vector to the spinor:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.1}$$

So, we have

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = (\phi_1^2 - \phi_2^2, J(\phi_1^2 + \phi_2^2), -2\phi_1\phi_2).$$

Via three vectors $a, b, c \in \mathbb{E}^3$ or $\mathbb{E}^{1,2}$, one can express this as

$$a + Jb = \phi^t \varrho\, \phi, \quad c = -\hat{\phi}^t \varrho\, \phi, \; J^2 = \mp 1, \tag{3.2}$$

where the superscript t gives transposition and $\hat{\phi}$ is the mate of ϕ [3], $\overline{\phi}$ is the complex conjugation of ϕ [19].

Let $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ be the vector composed of symmetric matrices, with its entries defined as

$$\varrho_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varrho_2 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad \varrho_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

Then the following equation holds

$$\hat{\phi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\phi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} = \begin{pmatrix} \overline{-\phi}_2 \\ \overline{\phi}_1 \end{pmatrix} \tag{3.4}$$

and the vectors a, b, c are explicitly defined as

$$a + Jb = (\phi_1^2 - \phi_2^2, J(\phi_1^2 + \phi_2^2), -2\phi_1\phi_2),$$

$$c = (\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2, J\,(\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2), |\,\phi_1\,|^2 - |\,\phi_2\,|^2).$$

Because the vector $a + Jb \in C^3$ is isotropic, explicit computation shows that a, b and c are mutually orthogonal, and their magnitudes satisfy $\|a\| = \|b\| = \|c\| = \bar{\phi}^t\phi$. Furthermore, the relation $\langle a \times b, c\rangle = \det(a, b, c) > 0$ holds. Conversely, if the vectors $a, b, c \in \mathbb{E}^3$ or $\mathbb{E}^{1,2}$ are mutually orthogonal with equal magnitudes and satisfy $(\langle a\times b, c\rangle) > 0$, then there exists a spinor, defined up to sign, such that equation (3.1) is satisfied.

Based on the preceding information, for any two arbitrary spinors ϕ and ψ, the following equalities hold [3]:

$$\overline{\phi^t \varrho\, \psi} = -\widehat{\phi}^t \varrho\, \widehat{\psi},$$
$$\widehat{(\lambda\phi+\mu\psi)} = \bar{\lambda}\, \widehat{\phi} + \bar{\mu}\, \widehat{\psi}.$$

Furthermore,

$$\widehat{\widehat{\phi}} = -\phi,$$

where λ and μ are complex numbers, the connection between spinors and orthogonal bases, as described by equation (3.2), follows a two-to-one correspondence. Owing to the fact that the spinors ϕ and $-\phi$ align with the same sequence of orthogonal basis $\{a, b, c\}$, where $\|a\| = \|b\| = \|c\|$ and $\langle a, b, c\rangle > 0$. Additionally, different spinors can be defined using the ordered triads $\{a, b, c\}$, $\{b, c, a\}$ and $\{c, b, a\}$. Furthermore, the equation $\psi^t \varrho\, \phi = \phi^t \varrho\, \psi$ holds for any pair of spinors ϕ and ψ since the matrices ϱ, as defined in equation (3.3), are symmetric. Moreover, the set $\{\phi, \hat{\phi}\}$ forms a basis for the space of two-component spinors.

4. Main Results

This section investigates the fundamental aim of this work which is the spinor representations of Frenet equations of a normal image for a regular space curve in each of $\mathbb{E}^3$ and $\mathbb{E}^{1,2}$.

4.1. **Spinor representation for normal image in $\mathbb{E}^3$.** Here, we introduce the Euclidean spinor representation of Frenet frame for a normal image of a given regular curve α.

Let $\alpha_n = n : I \subset \mathbb{R} \to \mathbb{E}^3$ be a normal image of α with arclength parameter s_n in the Euclidean space $\mathbb{E}^3$. Let $\{\zeta_n, n_n, F_n\}$ be its Frenet vector fields.

According to the properties presented above about spinors, we know that the each triad of $\mathbb{E}^3$ corresponds to the one Euclidean spinor ψ. Thus, we can write

$$n_n + JF_n = \psi^t \varrho\, \psi \;\text{ and }\; \zeta_n = -\hat{\psi}^t \varrho\, \psi,\; J^2 = -1, \tag{4.1}$$

with $\bar{\psi}^t\, \psi = 1$. Hence, the spinor ψ represents the triad $\{n_n, F_n, \zeta_n\}$ and the Frenet formulae of the curve α_n should relate to a particular expression for $\frac{d\psi}{ds_n}$.

As stated above, since the pair $\{\psi, \hat{\psi}\}$ forms a basis for the Euclidean spinors $(\psi \neq 0)$, there exist two functions p and q such that

$$\frac{d\,\psi}{ds_n} = p\,\psi + q\,\hat{\psi}, \tag{4.2}$$

where p and q may be Euclidean two functions.

By computing the derivative of the first equation in (4.1) according to s_n and taking into account Eq. (2.2), we can straightforwardly obtain that

$$\frac{d\,n_n}{ds_n} + J\frac{d\,F_n}{ds_n} = (\frac{d\,\psi}{ds_n})^t \varrho\,\psi + \psi^t\;\varrho\,(\frac{d\,\psi}{ds_n}),$$

which leads to

$$-\kappa_n\,\zeta_n+\tau_n\,F_n+J(-\tau_n\,n_n)=(\frac{d\,\psi}{ds_n})^t\varrho\,\psi+\psi^t\varrho(\frac{d\,\psi}{ds_n}). \tag{4.3}$$

Thus, considering the Eqs. (4.1), (4.2) and (4.3), we get

$$\begin{aligned}-J\tau_n(n_n+J\,F_n)-\kappa_n\,\zeta_n &= (\,p\,\psi+q\,\hat{\psi}\,)^t\varrho\,\psi+\psi^t\,\varrho\,(p\,\psi+q\,\hat{\psi}\,)\\ &= 2p\,(n_n+J\,F_n)-2q\,\zeta_n.\end{aligned}$$

From the last equation, we can obtain

$$p=-\frac{J\,\tau_n}{2},\; q=\frac{\kappa_n}{2}, \tag{4.4}$$

with

$$\begin{aligned}\kappa_n &= \sqrt{1+\sigma^2},\;\; \tau_n=\varpi\sqrt{1+\sigma^2},\;\; \sigma=\frac{\Omega'(s)}{\kappa(s)((1+\Omega^2(s))^{\frac{3}{2}}},\\ \Omega &= \frac{\tau(s)}{\kappa(s)},\;\; \varpi=\frac{\sigma'(s)}{\kappa(s)\,\sqrt{1+\Omega^2(s)}(1+\sigma^2(s))^{\frac{3}{2}}}.\end{aligned}$$

The same result can be obtained from the second part in the Eq. (4.1).
With assistance from the Eqs. (4.2) and (4.4), the following result can be given.

Theorem 4.1. Let $\alpha_n=n$ be a normal image of a given space curve α and let $\{\zeta_n,n_n,F_n\}$ be its Frenet frame in $\mathbb{E}^3$. If the Euclidean spinor ψ describes the oriented triad $\{n_n,F_n,\zeta_n\}$ of α_n, thus the Frenet formulae can be represented as a single spinor equation:

$$\frac{d\,\psi}{ds_n}=-(\frac{J\,\tau_n}{2})\psi+(\frac{\kappa_n}{2})\hat{\psi}, \tag{4.5}$$

where, κ_n and τ_n represent the curvatures of α_n, respectively. Equation (4.5) is referred to as the spinor Frenet equation in the context of three-dimensional Euclidean space.

4.2. **Spinor representation for normal image in $\mathbb{E}^{1,2}$.** Let $\beta_P=P:I\subset\mathbb{R}\to\mathbb{E}^{1,2}$ be a normal image of a nonnull curve β with pseudo arclength parameter s_P in pseudo-Euclidean 3-space $\mathbb{E}^{1,2}$. Let $\{\xi_P,P_P,Q_P\}$ be its Serret-Frenet fields of vectors.

Based on the properties presented about spinors, one can find a spinor ϕ such that

$$P_P+J\,Q_P=\phi^t\varrho\,\phi\;\text{ and }\;\xi_P=-\,\hat{\phi}^t\varrho\,\phi,\;J^2=1, \tag{4.6}$$

with $\bar{\phi}^t\phi=1$. Hence, the spinor ϕ represents the triad $\{P_P,Q_P,\xi_P\}$ and the Frenet equations of the curve β_P should be associated with a certain expression for $\frac{d\,\phi}{ds_P}$.

As we mentioned above, there exist two functions f and g satisfying the condition that

$$\frac{d\,\phi}{ds_P}=f\,\phi+g\;\hat{\phi}, \tag{4.7}$$

where f and g are two possibly pseudo-Euclidean functions.

Taking the derivative of the first equation in Eqs. (4.6) with respect to s_P, then by considering Eq. (2.7), one can obtain

$$\frac{d\,P_P}{ds_P}+J\,\frac{d\,Q_P}{ds_P}=(\frac{d\,\phi}{ds_P})^t\,\varrho\,\phi+\phi^t\,\varrho\,(\frac{d\,\phi}{ds_P}),$$

which leads to

$$-\varepsilon_0\,\kappa_P\,\xi_P-\varepsilon_0\varepsilon_1\tau_P(s_P)\,Q_P+J\,(-\varepsilon_0\varepsilon_1\tau_P(s_P)\,P_P)=(\frac{d\,\phi}{ds_P})^t\varrho\,\phi+\phi^t\,\varrho\,(\frac{d\,\phi}{ds_P}). \tag{4.8}$$

Thus, considering the Eqs. (4.6), (4.7) and (4.8), we get

$$\begin{aligned}-\varepsilon_0\varepsilon_1 J\,\tau_P\,(P_P+J\,Q_P)-\varepsilon_0\,\kappa_P\,\xi_P &= (f\;\;\phi+g\,\hat{\phi}\;)^t\varrho\,\phi+\phi^t\,\varrho\,(f\;\;\phi+g\,\hat{\phi}\;)\\ &= f\,(\phi^t\varrho\,\phi+\phi^t\varrho\,\phi)+g(\hat{\phi}^t\varrho\,\phi+\phi^t\,\varrho\,\hat{\phi})\\ &= 2f(P_P+J\;\;Q_P)-2g\,\xi_P.\end{aligned}$$

Thus, the equation

$$-\varepsilon_0\varepsilon_1 J\tau_P(P_P+J\;\;Q_P)-\varepsilon_0\kappa_P\xi_P=2f(P_P+J\;\;Q_P)-2g\,\xi_P,$$

holds.

Finally, we can see easily that

$$f=-\frac{\varepsilon_0\varepsilon_1 J\,\tau_P}{2},\;g=\frac{\varepsilon_0\,\kappa_P}{2}, \tag{4.9}$$

where

$$\begin{aligned}\kappa_P &= \sqrt{|\,1+\varepsilon_0\varepsilon_2\sigma^2\,|},\;\;\sigma=\frac{\Omega'(s)}{\kappa(s)\,|\,1-\Omega^2(s)\,|^{\frac{3}{2}}},\;\;\Omega=\frac{\tau}{\kappa},\\ \tau_P &= \frac{[\kappa^3(1-\varepsilon_1\Omega^2)[\Omega\,\kappa''-\;\varepsilon_1\,\tau''+\varepsilon_0(1-\;\varepsilon_1)\kappa^3\Omega(1-\varepsilon_1\Omega^2)]+3\varepsilon_0\kappa^2\Omega'(-\varepsilon_1\kappa\kappa'+\tau\tau')]}{(1-\varepsilon_1\Omega^2)^3(1+\varepsilon_0\varepsilon_2\sigma^2)},\end{aligned}$$

$$\varpi=\frac{\sigma'(s)}{\kappa(s)\,\sqrt{|\,1-\varepsilon_1\Omega^2(s)\,|}\,|\,(1+\varepsilon_0\varepsilon_2\sigma^2(s))\,|^{\frac{3}{2}}}.$$

With the help of equations Eqs. (4.7) and (4.9), we have proved the theorem stated below.

Theorem 4.2. Let $\beta_P=P$ be a nonnull normal image of a given space curve β and assume that $\{\xi_P,P_P,Q_P\}$ be its Frenet frame in $\mathbb{E}^{1,2}$. If the pseudo-Euclidean spinor ϕ is equivalent to the oriented triad $\{P_P,Q_P,\xi_P\}$ of β_P, then the Frenet derivative equations have single pseudo-Euclidean spinor equation:

$$\frac{d\phi}{ds_P}=-(\frac{\varepsilon_0\varepsilon_1 J\,\tau_P}{2})\,\phi+(\frac{\varepsilon_0\,\kappa_P}{2})\,\hat{\phi}\,. \tag{4.10}$$

where τ_P and κ_P are the curvatures of the curve β_P, respectively.

From this point of view, the equation (4.10) is referred to as the spinor Frenet equation in pseudo-Euclidean 3-space.

5. Computational Examples

Based on the results we obtained during this work, we now provide the following examples to demonstrate the key results.

Example 5.1. *Let $\alpha = \alpha(u)$, be Viviani's curve in $\mathbb{E}^3$ with the parameterization (see Fig. 1a):*

$$\alpha = \left(1 + \cos(u), \sin(u), 2\sin\left(\frac{u}{2}\right)\right),$$

then, we obtain the Frenet apparatus of this curve as follows:

$$\begin{aligned}
\zeta &= \left(-\frac{\sqrt{2}\sin(u)}{\sqrt{3+\cos(u)}}, \frac{\sqrt{2}\cos(u)}{\sqrt{3+\cos(u)}}, \frac{\sqrt{2}\cos\left(\frac{u}{2}\right)}{\sqrt{3+\cos(u)}}\right),\\
n &= \begin{pmatrix} -\frac{4\left(\cos^4\left(\frac{u}{2}\right)+\cos(u)\right)}{\sqrt{3+\cos(u)}\sqrt{13+3\cos(u)}}, \\ \frac{(6+\cos(u))\sin(u)}{\sqrt{3+\cos(u)}\sqrt{13+3\cos(u)}}, \\ -\frac{2\sin\left(\frac{u}{2}\right)}{\sqrt{3+\cos(u)}\sqrt{13+3\cos(u)}} \end{pmatrix},\\
F &= \left(\frac{3\sin\left(\frac{u}{2}\right)+\sin\left(\frac{3u}{2}\right)}{\sqrt{26+6\cos(u)}}, -\frac{2\sqrt{2}\cos^3\left(\frac{u}{2}\right)}{\sqrt{13+3\cos(u)}}, \frac{2\sqrt{2}}{\sqrt{13+3\cos(u)}}\right),\\
\kappa &= \frac{\sqrt{13+3\cos(u)}}{(3+\cos(u))^{\frac{3}{2}}},\ \tau = \frac{6\cos\left(\frac{u}{2}\right)}{13+3\cos(u)}.
\end{aligned}$$

The arclength function of Viviani's curve is given by

$$s(u) = 2\sqrt{2}E\left(\frac{u}{2}, \frac{\sqrt{2}}{2}\right),$$

where $E(u,k)$ is an incomplete elliptic integral of the second kind [20].

From the above, we can write the equation of the spherical image of the normal of α in the following form (see Fig. 1b):

$$\alpha_n = n(u) = \frac{1}{\mu_1}\left(-4\left(\cos^4\left(\frac{u}{2}\right)+\cos(u)\right), (6+\cos(u))\sin(u), -2\sin\left(\frac{u}{2}\right)\right),$$

where

$$\mu_1 = \sqrt{(3+\cos(u))(13+3\cos(u))}.$$

After some calculations, we derive the curvature and torsion of α_n, respectively as

$$\begin{aligned}
\kappa_n = \ & \frac{1}{\mu_2}\{20702144458 + 9\cos(u)(6294477918 + \cos(u)(8106357591\\
& + \cos(u)(6525783681 + \cos(u)(3686172912\\
& + \cos(u)(1553651704 + \cos(u)(504539210\\
& + 3\cos(u)(42442258 + 3\cos(u)(2738678\\
& + 3\cos(u)(131030 + 3\cos(u)(4379 + 273\cos(u) + 4\cos(2u))))))))))\};
\end{aligned}$$

$$\mu_2=\left(2\sqrt{2}(3+\cos(u))^3(13+3\cos(u))^3\left(\begin{array}{c}1+\frac{2}{(3+\cos(u))^2}+\frac{3}{2(3+\cos(u))}\\+\frac{40}{(13+3\cos(u))^2}-\frac{14}{13+3\cos(u)}\end{array}\right)^{3/2}\right)$$

$$\tau_n=\frac{1}{\mu_3}\left(3(3+\cos(u))^{3/2}(13+3\cos(u))^{3/2}\left(\begin{array}{c}2496243604\cos\left(\frac{u}{2}\right)+1552233406\cos\left(\frac{3u}{2}\right)\\+495707778\cos\left(\frac{5u}{2}\right)\\+70452762\cos\left(\frac{7u}{2}\right)+396342\cos\left(\frac{9u}{2}\right)\\-1153098\cos\left(\frac{11u}{2}\right)\\-140022\cos\left(\frac{13u}{2}\right)-6291\cos\left(\frac{15u}{2}\right)\\-81\cos\left(\frac{17u}{2}\right)\end{array}\right)\right);$$

$$\begin{aligned}\mu_3 \;=\; &\{128(20702144458+9\cos(u)(6294477918+\cos(u)(8106357591+\cos(u)(6525783681\\&+\cos(u)(3686172912+\cos(u)(1553651704+\cos(u)(504539210+3\cos(u)(42442258\\&+3\cos(u)(2738678+3\cos(u)(131030+3\cos(u)(4379+273\cos(u)+4\cos(2u)))))))))))\}.\end{aligned}$$

Since the Euclidean spinor represents the oriented triad $\{\zeta_n, n_n, F_n\}$ of α_n, then from Eq. (4.5), the Frenet derivative equations can be expressed as a single spinor equation:

$$\frac{d\psi}{du_n}\frac{du_n}{ds_n}=-\left(\frac{J\,\tau_n}{2}\right)\psi+\left(\frac{\kappa_n}{2}\right)\widehat{\psi},\ \textit{such that}\ \frac{d\widehat{\psi}}{du_n}\frac{du_n}{ds_n}=-\left(\frac{\tau_n}{2}\right)\psi+\left(\frac{J\,\kappa_n}{2}\right)\widehat{\psi}.$$

By solving these two differential equations, we obtain

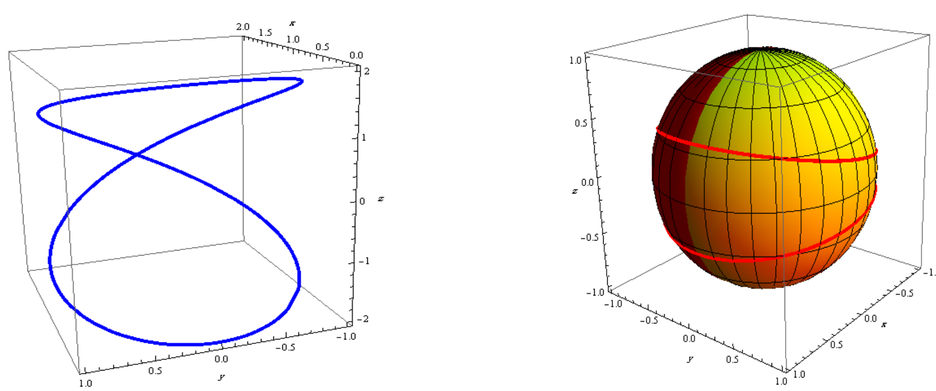
$$\psi(u(s))=c_1e^{\sigma_1u(s)}V_{1,1}+c_2e^{\sigma_2u(s)}V_{1,2},$$

and

$$\widehat{\psi}(u(s))=c_3e^{\sigma_1u(s)}V_{2,1}+c_4e^{\sigma_2u(s)}V_{2,2},$$

where $c_1,...,c_4\in\mathbb{R}$, and $V_{1,1},V_{1,2},V_{2,1},V_{2,2}$ are the components of the eigenvectors corresponding to σ_1 and σ_2, such that:

$$\sigma_{1,2}=\frac{\kappa_n-\tau_n}{4}\pm\frac{1}{2}\sqrt{\left(\frac{\kappa_n+\tau_n}{2}\right)^2+J\,\kappa_n\tau_n}.$$



(A) (B)

FIGURE 1. (A) The Viviani's curve α in $\mathbb{E}^3$, (B) The normal spherical image α_n lies on S^2.

Example 5.2. *Consider the unit speed spacelike curve $\beta = \beta(s)$ in $\mathbb{E}^{1,2}$ with timelike normal vector in the parametric form (see Fig. 2a)*

$$\beta(s) = \frac{1}{\sqrt{2}}(\cosh(s), 1, \sinh(s)).$$

This curve has the following Frenet vectors

$$\begin{aligned} \xi &= \frac{1}{\sqrt{2}}(\sinh(s), 1, \cosh(s)), \\ P &= (-\cosh(s), 0, -\sinh(s)), \\ Q &= \frac{1}{\sqrt{2}}(-\sinh(s), 1, -\cosh(s)). \end{aligned}$$

The curvature and torsion of β are respectively, given by

$$\kappa = \frac{1}{\sqrt{2}}, \tau = -\frac{1}{\sqrt{2}}.$$

From the aforementioned data, the normal spherical image of β is obtained (see Fig. 2b):

$$\beta_P\,(s_P) = P(s) = (-\cosh(s), 0, -\sinh(s)).$$

For this curve, the Frenet apparatus is calculated as

$$\begin{aligned} \xi_P &= (-\sinh(s), 0, -\cosh(s))\,, \\ P_P &= (\cosh(s), 0, \sinh(s)), \\ Q_P &= (0, 1, 0), \\ \kappa_P &= 1, \tau_P = 0. \end{aligned}$$

As the pseudo-Euclidean spinor depicts the oriented triad $\{\xi_P, P_P, Q_P\}$ of β_P, then from Eq. (4.10), the Frenet derivative equations are equivalent to the single spinor equation:

$$\frac{d\phi}{ds_P} = -\left(\frac{\varepsilon_0\varepsilon_1 J\,\tau_P}{2}\right)\phi + \left(\frac{\varepsilon_0\,\kappa_P}{2}\right)\hat{\phi} = \left(\frac{\varepsilon_0}{2}\right)\hat{\phi};$$

such that

$$\frac{d\hat{\phi}}{ds_P} = -\left(\frac{\varepsilon_0\varepsilon_1\tau_P}{2}\right)\phi + \left(\frac{\varepsilon_0\,J\,\kappa_P}{2}\right)\hat{\phi} = \left(\frac{J\,\varepsilon_0}{2}\right)\hat{\phi},$$

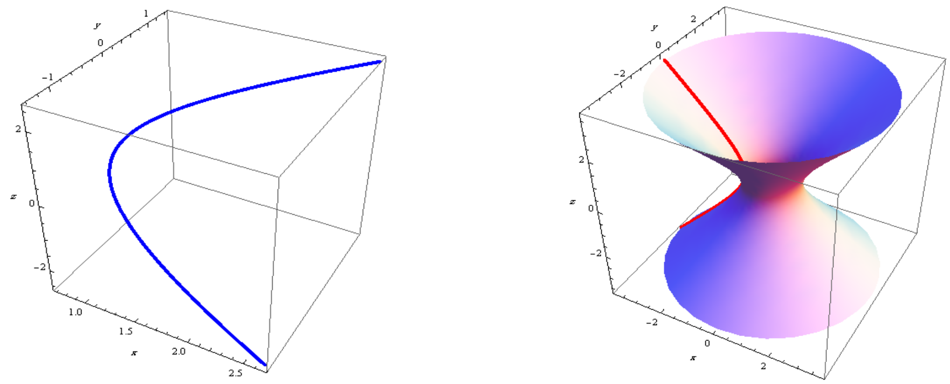
solving these two equations gives

$$\phi\,(s) = \frac{c_1}{J}e^{\frac{\varepsilon_0 J}{2}s} + c_2,$$

and

$$\hat{\phi}\,(s) = c_1 e^{\frac{\varepsilon_0 J}{2}s},$$

where $c_1, c_2 \in \mathbb{R}$.



(A) (B)

Figure 2. (A) The spacelike curve β in $\mathbb{E}^{1,2}$, (B) The normal spherical image β_P lies on S_1^2.

6. Conclusion

In geometry and physics, spinors are defined as elements of a complex vector space linked to Euclidean space. On the other hand, spinors can be defined as elements of the vector space that carries a linear representation of the Clifford algebra which is an associative algebra. In this work, we have studied spinors which represent the vital subject in mathematics, where relates curves theory in the differential geometry with algebra. We have shown that how can be Frenet equations of normal spherical image expressed in terms of Euclidean and pseudo-Euclidean spinors. More clearly, we have proven that in both Euclidean and pseudo-Euclidean 3-spaces, the Frenet equation system is correspond to a single spinor equation, resulting from the connection between spinors and orthogonal triads of the given vectors, as well as the use of complex quantities. Finally, we provided two computational examples to support our main results.

In future works, we plan to study spinors in Galilean and pseudo-Galilean spaces for different queries and further improve the results in this paper, combined with the techniques and results in [21–27].

Conflict of Interests

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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