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Fractal Attractor via Controlled Strong $b-$Kannan Iterated Function System

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Abstract. Nowadays, most of the real time problems have been attempted by using well-known fixed point theorems. Especially the Banach contraction theorem is a well-posted tool to solve many dynamical problems of applied mathematics. This paper explores an idea in generalizing fixed point theorem to generate a proposed fractal type set called Controlled Strong $b-$Kannan Fractal (CSbK-Fractal) through the dynamical system of Kannan contractivity function in the Controlled Strong $b-$Metric Space (CSbMS). Furthermore, the collage type theorem is proved on CSbK-Fractal. In this context, the interesting results and consequences of newly developing iterated function system and its fractal attractor in the controlled strong $b-$metric space are discussed with examples. This theory can provide a novel direction to construct a new kind of fractal set in generalized spaces.

1. Introduction

The concept of fractals was initially proposed by Mandelbrot in his vital book "The Fractal Geometry of Nature" illustrates the non-linearity of many scientific events and real-life objects. The fractal geometry has been demonstrated as extremely effective tool for modelling complex structures with infinite details in the real world [1]. The fixed point theory is an essential technique in the theory of Hutchinson's iterated function systems (IFS). The construction of deterministic fractals was studied by Barnsley in detail ([2], [3], [4]). For generating various types of fractals, it has been made IFS an invaluable tool. The few examples of IFS applications are image processing, random dynamical systems, and stochastic growth models. The existence of an attractor or deterministic fractal of IFS in a Complete Metric Space (CMS) follows the widely recognized Banach contraction principle ([5], [6], [7]). Researchers nowadays frequently use Fractals in various scientific fields, such as Sierpinski-type fractal structures, fractal-time derivative operators,

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topological insulators, fractionally-perturbed systems, kinetic energy, quantum mechanics, time evolution of quantum fractals and other physical problems applications ([8], [9], [10], [11], [12], [13]).

The theory of Hutchinson's IFS is immensely extended to numerous generalized contractions ([14], [15], [16], [17]) and various types of generalized spaces ([18], [19], [20]), as well as to multifunction systems and infinite IFS for generating generalized fractal sets ([21], [22], [23]).

The Controlled Strong $b-$Metric Space (CSbMS) is a generalization of Strong $b-$Metric Space, by introducing the variable control function instead of constant in the triangle inequality [24]– [25]. In the current scenario, the above described extensions indicate us in the direction of introducing the concept of fractal structures in CSbMS using the Kannan contraction mapping. As a general case, it also inspired us to discuss Hutchinson–Barnsley (HB) theory, and develop a novel type of fractal attractor on CSbMS by constructing a system called Controlled Strong $b-$Kannan Iterated Function System (CSbK-IFS).

The research work is expressed in the following manner. Section 2 explores fundamental concepts of contraction, Hausdorff metric space, and IFS, which is vital to the present study. In Section 3, the Hausdorff version of CSbMS is defined and the completeness of Hausdorff controlled strong $b-$metric space (HCSbMS) is proved. In Section 4, it is shown that fractals exist in controlled strong $b-$Metric space via the IFS of Kannan contractions, and some exciting outcomes were also presented. Lastly, Section 5 summarizes the results obtained in the research work.

2. Preliminaries

In this preliminary section, the fundamental concepts of iterated function systems are discussed, which are essential to the present study.
Let (M, d) be a Metric Space (MS). Let $\Gamma : M \to M$ be a contraction, where the function Γ satisfies $d(\Gamma(\xi), \Gamma(\lambda)) \leq \beta d(\xi, \lambda), \forall \xi, \lambda \in M,\ \beta \in [0, 1)$, where β represents contraction factor.

Theorem 2.1. *Let $\Gamma : M \to M$ be a contraction mapping on CMS. Then Γ has a unique fixed point. Then $\{\Gamma^n(\xi)\}_{n=1}^{\infty}$ converges to ξ^*, That is, $\lim_{n\to\infty} \Gamma^n(\xi) = \xi^*$ for all $\xi \in M$.*

The Theorem 2.1 demonstrates the existence and uniqueness of fixed points of Banach contraction.

Definition 2.1. *(Hausdorff Metric Space) Let (M, d) be a CMS and $K_0(M)$ be a collection of all nonempty compact subsets of M. For $\xi \in M$ and $A, B \in K_0(M)$, define*

$$d(\xi, B) = \inf \{d(\xi, \lambda) : \lambda \in B\}$$

and

$$d(A, B) = \sup \{d(\xi, B) : \xi \in A\}.$$

Then the Hausdorff metric between A and B is defined as
$H_d : K_0(M) \times K_0(M) \to [0, \infty)$ *such that*

$$H_d(A, B) = \max\left\{\sup_{\xi \in A} d(\xi, B), \sup_{\lambda \in B} d(\lambda, A)\right\}.$$

The pair $(K_0(M), H_d)$ *is called a Hausdorff metric space.*

Theorem 2.2. *If* (M, d) *is a complete metric space then* $(K_0(M), H_d)$ *is also a complete Hausdorff metric space.*

Theorem 2.3 ([3]). *A (hyperbolic) IFS is contained in CMS* (M, d)*, together with a finite collection of continuous contraction mapping* $\Gamma_n : M \to M$ *with contractivity factor* $\{\beta_n\}_{n=1}^{N}$*. The system* $\{M; \Gamma_n, n = 1, 2, ..., N\}$ *is the hyperbolic IFS and* $G : K_0(M) \to K_0(M)$*, which is defined by*

$$G(B) = \bigcup_{n=1}^{N} \Gamma_n(B), \forall\, B \in K_0(M),$$

where $\Gamma_n(B) = \{\Gamma_n(\lambda) : \lambda \in B\}$ *and G is a contraction mapping on CMS* $(K_0(M), H_d)$ *with contractivity factor* β*.*
i.e, $H_d(G(B), G(C)) \leq \beta H_d(B, C)$*, where* $\beta = \max_{n=1}^{N} \beta_n$*. Here,* $A^* \in K_0(M)$ *is the one and only fixed point for the set valued map G and such a unique fixed point is called as an attractor.*
Furthermore, $A^* = \lim_{n\to\infty} \Gamma^n(B)$ *for any* $B \in K_0(M)$*, where* $\Gamma^n = \underbrace{\Gamma \circ \Gamma \circ \Gamma \circ \cdots \circ \Gamma}_{n \text{ times}}$*.*

Further, $G(A^*) = A^*$ *and hence* A^* *may be called an invariant set.*

Definition 2.2 (Controlled Strong $b-$Metric Space (CSbMS) [25]). *Let M be a nonempty set,* $d : M \times M \to [0, \infty)$ *and* $\alpha : M \times M \to [1, \infty)$ *satisfies the given conditions*

(a). $d(\xi, \lambda) = 0$ *iff* $\xi = \lambda$*,*
(b). $d(\xi, \lambda) = d(\lambda, \xi)$*,*
(c). $d(\xi, \lambda) \leq d(\xi, z) + \alpha(z, \lambda) d(z, \lambda)\ \forall\ \xi, \lambda, z \in M$*.*

The pair (M, d) *is called a CSbMS.*

Definition 2.3 (Complete Controlled Strong $b-$Metric Space (CCSbMS) [25]). *The CSbMS* (M, d) *is said to be complete if for all Cauchy sequence, it is convergent.*

Definition 2.4 (Kannan contraction [14]). *Let* (M, d) *be a CMS. Let* $\Gamma : M \to M$ *if*

$$d(\Gamma(\xi), \Gamma(\lambda)) \leq \beta[d(\xi, \Gamma(\xi)) + d(\lambda, \Gamma(\lambda))], \text{ where } \beta \in \left(0, \frac{1}{2}\right) \text{ and } \xi, \lambda \in M.$$

Then Γ *has the unique fixed point in M.*

Theorem 2.4 (Controlled Strong $b-$Kannan Fixed Point Theorem [25]). *Let* $\Gamma : M \to M$ *be a Kannan mapping on CCSbMS* (M, d) *with contractivity factor* β *and* $\alpha : M \times M \to [1, \infty)$*. For* $\xi_0 \in M$*, take* $\xi_n = \Gamma^n(\xi_0)$*. Suppose that*

$$\sup_{k\geq 1} \lim_{n\to\infty} \alpha(\xi_{n+1}, \xi_k) < \frac{1-\beta}{\beta}.$$

Also consider that, for all $\xi \in M$, $\lim_{n\to\infty} \alpha(\xi_n, \xi)$ and $\lim_{n\to\infty} \alpha(\xi, \xi_n)$ are exist. Then, Γ has a unique fixed point.

Example 2.1. *Let $M = \{0, 1, 2\}$. Define $d : M \times M \to \mathbb{R}^+$ as $d(\xi, \xi) = 0$ and $d(\xi, \lambda) = d(\lambda, \xi)$ where $\xi, \lambda \in M$ and*

$$d(0,1) = 9,\ d(1,2) = 4,\ d(0,2) = 6.$$

Define $\alpha : M \times M \to [1, \infty)$ as

$$\alpha(\xi,\xi) = 1,\ \alpha(0,1) = \frac{4}{3},\ \alpha(1,2) = \frac{7}{4},\ \alpha(0,2) = \frac{5}{3}.$$

Thus (M, d) is a CSbMS. Consider a self mapping $\Gamma : M \to M$ such that

$$\Gamma(\xi) = \begin{cases} 1; & \text{if} \quad \xi = 0 \\ 2; & \text{if} \quad \xi \in \{1,2\} \end{cases}$$

Choose $\beta \in \left[\frac{2}{5}, \frac{1}{2}\right)$. Thus, Γ follows all conditions of Theorem 2.4, so Γ is a Kannan mapping and Γ has a unique fixed point, that is $\xi = 2$.

3. Hausdorff Controlled Strong b−Metric Space

We denote $\alpha(\xi, C) = \inf_{\eta \in C} \alpha(\xi, \eta)$ and $\alpha_H(B, C) = \sup_{\xi \in B} \alpha(\xi, C)$.

Lemma 3.1. *Let (M, d) be a CSbMS. Then $d(\xi_1, A) \leq d(\xi_1, \xi_2) + \alpha(\xi_2, A) d(\xi_2, A)$ for all $\xi_1, \xi_2 \in M$ and $A \subset M$.*

Proof. From the definition of controlled strong b−triangle inequality,
$d(\xi_1, a) \leq d(\xi_1, \xi_2) + \alpha(\xi_2, a) d(\xi_2, a),\ \forall \xi_1, \xi_2, a \in M$
Taking infimum over A, we get
$\inf_{a\in A} d(\xi_1, a) \leq d(\xi_1, \xi_2) + \inf_{a\in A} \alpha(\xi_2, a) \inf_{a\in A} d(\xi_2, a)$.
Therefore
$d(\xi_1, A) \leq d(\xi_1, \xi_2) + \alpha(\xi_2, A) d(\xi_2, A)\ \forall \xi_1, \xi_2 \in M$. □

Here we can define the Hausdorff controlled strong b−metric space.

Definition 3.1. *(Hausdorff Controlled Strong b-Metric Space) Let (M, d) be a CSbMS. Then the function $H_d : K_0(M) \times K_0(M) \to [0, \infty)$ is defined as*

$$H_d(A, B) = \max\left\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\right\}$$

where $A, B \in K_0(M)$

Then, $(K_0(M), H_d)$ is called as Hausdorff Controlled Strong b−Metric Space (HCSbMS).

Lemma 3.2. *Let $A, B, C \subset K_0(M)$ and $v \in B$, then*

$$H_d(A, C) \leq H_d(A, B) + \max\left\{\alpha(v, C), \sup_{w\in C} \alpha(w, v)\right\} H_d(B, C).$$

Proof. Let $H_d(A,B)$ and $H_d(B,C)$ be finite. By using Lemma 3.1 then,
$d(u,C) \leq d(u,v) + \alpha(v,C)d(v,C)$ where $u \in A, v \in B$.
As $d(v,C) \leq H_d(B,C)$, then
$d(u,C) \leq d(u,v) + \alpha(v,C)H_d(B,C)$,
Taking infimum over $v \in B$ on both sides of the above inequality,
$d(u,C) \leq d(u,B) + \inf_{v\in B} \alpha(v,C)H_d(B,C)$.
$d(u,C) \leq H_d(A,B) + \alpha(v,C)H_d(B,C)$.
Taking supremum over $u \in A$, we have
$\sup_{u\in A} d(u,C) \leq H_d(A,B) + \alpha(v,C)H_d(B,C)$.
Analogously,
$\sup_{w\in C} d(w,A) \leq H_d(A,B) + \sup_{w\in C} \alpha(w,v)H_d(B,C)$.
so
$\max\left\{\sup_{u\in A} d(u,C), \sup_{w\in C} d(w,A)\right\} \leq H_d(A,B) + \max\left\{\alpha(v,C), \sup_{w\in C} \alpha(w,v)\right\} H_d(B,C)$.
Therefore, by Definition 3.1 we get

$$H_d(A,C) \leq H_d(A,B) + \max\left\{\alpha(v,C), \sup_{w\in C} \alpha(w,v)\right\} H_d(B,C).$$

□

Definition 3.2. *Suppose,* $\overline{A} = \left\{a \in A : \exists \{a_n\}_{n=0}^{\infty} \text{ in } A \ni \lim_{n\to\infty} a_n = a\right\}$*, then* $\overline{A}$ *is said to be a closure of a set* $A \subset M$*. Denote for* $\varepsilon > 0$ *and* $A \subset M$*,* $A_\varepsilon = \{\xi \in M : d(\xi, A) \leq \varepsilon\}$.

Lemma 3.3. *If* $\xi \in \overline{A_\varepsilon}$*, then* $d(\xi,A) \leq \varepsilon \lim_{n\to\infty} \alpha(\xi_n, A)$ *where* $\alpha(\xi_n, A) = \inf_{a\in A} \alpha(\xi_n, a)$.

Proof. Let $\xi \in \overline{A_\varepsilon}$, then $\exists \{\xi_n\}$ in A such that $\lim_{n\to\infty} \xi_n = \xi$.
From Lemma 3.1, we have
$d(\xi,A) \leq d(\xi,\xi_n) + \alpha(\xi_n,A)d(\xi_n,A)$
In the above inequality if $n \to \infty$, then
$d(\xi,A) \leq \varepsilon \lim_{n\to\infty} \alpha(\xi_n,A)$. □

Definition 3.3. *Let* (M,d) *be a CSbMS. The upper topological limit of* $\{A_l\}_{l=1}^{\infty}$ *in* M *is denoted as* $\overline{Lt}A_l$ *and defined by* $a \in \overline{Lt}A_l$*, iff* $\lim_{l\to\infty} \inf d(a, A_l) = 0$.

Theorem 3.1. *If there exists a subsequence* $\{\xi_{n_l}\}$ *in* A *is convergent to* ξ *and* $\xi_{n_l} \in A_{n_l}$ *for* $l = 1,2,3,\ldots$*, iff the point* $\xi \in \overline{Lt}A_{n_l}$.

Proof. Let $\xi \in \overline{Lt}A_l$, then $\exists$ a subsequence $\{A_{n_l}\}$ of A_l such that $\lim_{l\to\infty} d(\xi, A_{n_l}) = 0$.
We have a sequence of positive integers $\{P_l\}$ that strictly increases for every l, where

$$d(\xi, A_{n_l}) < \frac{1}{l},\ \forall n \geq P_l.$$

We can find $\{\xi_{n_l}\}$ of points such that $\xi_{n_l} \in A_{n_l}$ and $d(\xi, \xi_{n_l}) < \dfrac{1}{l}$ for $P_l \leq n \leq P_{l+1}$. Hence $\lim\limits_{l\to\infty} \xi_{n_l} = \xi$.
Conversely, suppose that $\xi_{n_l} \to \xi$ and $\xi_{n_l} \in A_{n_l}$, $l = 1, 2, 3, \ldots$.
Hence $d(\xi, A_{n_l}) \leq d(\xi, \xi_{n_l}) \to 0$ and $\lim\limits_{l\to\infty} \inf d(\xi, A_l) = 0$. Thus, $\xi \in \overline{Lt} A_l$. □

Theorem 3.2. *$L = \overline{Lt}\, \overline{A_l}$ is closed.*

Proof. Suppose ξ is a limit point of L. Then $\exists$ a sequence $\xi_m \in L - \{\xi\}$ such that $\xi_m \to \xi$.
By Theorem 3.1, for $\xi_m \in L$, $\exists$ a subsequence $\{\xi_{m_l}\} \subset A$ such that $\lim\limits_{l\to\infty} \xi_{m_l} = \xi_l$ and $\xi_{m_l} \in A_{m_l}$, for $l = 1, 2, 3, \ldots$.
Then by the controlled strong $b-$triangle inequality,

$$d(\xi_{m_l}, \xi) \leq d(\xi_{m_l}, \xi_l) + \alpha(\xi_l, \xi) d(\xi_l, \xi).$$

Clearly $\lim\limits_{l\to\infty} \xi_{m_l} = \xi_l$. It follows that $\{\xi_{m_l}\} \to \xi$ and $\xi_{m_l} \in A_{m_l}$, for $l = 1, 2, 3, \ldots$. From Theorem 3.1, $\xi \in L$. Hence L is closed. □

Corollary 3.1. $\overline{Lt} A_l = \bigcap\limits_{l=1}^{\infty} \overline{\bigcup\limits_{n=0}^{\infty} A_{l+n}}$.

Corollary 3.2. $\lim\limits_{l\to\infty} A_l = \overline{\overline{Lt} A_l} = \overline{Lt A_l}$.

Theorem 3.3. *Let (M, d) be a CCSbMS with $\lim\limits_{n,k\to\infty} \alpha(\xi_n, \xi_k)\beta < 1$ for all $\xi_n, \xi_k \in M$, where $\beta \geq 1$. Then $(K_0(M), H_d)$ is complete.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $K_0(M)$. If $\forall\ \varepsilon > 0$, $\exists$ a positive integer $P \in \mathbb{N}$ then,

$$H_d(A_n, A_k) < \varepsilon,\ \forall\ n, k \geq P. \tag{3.1}$$

Let $A = \overline{Lt} A_n$. We need to prove that $A \in K_0(M)$ and $A_n \to A$. From Theorem 3.2, $L = \overline{Lt} A_l$ is closed and then we get $A \in K_0(M)$.
We will demonstrate that $\{A_n\} \to A$, i.e $\exists\ P$ is a positive integer such that $H_d(A_n, A) < \varepsilon\ \forall\ n \geq P$.
By triangle inequality, $\forall\ n, k \geq P$,
$H_d(A_n, A) \leq H_d(A_n, A_k) + max \left\{ \sup\limits_{a_k \in A_k} \alpha(a_k, a), \alpha(a, A_k) \right\} H_d(A_k, A)$.
For $n, k \geq P$, we have from Eqn. (3.1)

$$H_d(A_n, A) \leq \varepsilon + max \left\{ \sup_{a_k \in A_k} \alpha(a_k, a), \alpha(a, A_k) \right\} H_d(A_k, A). \tag{3.2}$$

Next, we prove that
$H_d(A_k, A) \leq max \left\{ \sup\limits_{a_k \in A_k} \alpha(a_k, a_{n_m}), \alpha(a_{n_m}, A_k) \right\} \varepsilon$.
At first, the following inequalities will be derived for proving the above inequality,

$$d(a_k, a^*) \leq \alpha(a_k, a_{n_m})\varepsilon, \forall a_k \in A_k, \tag{3.3}$$

$$d(a^*, A_k) \leq \alpha(a_{n_m}, A_k)\varepsilon. \tag{3.4}$$

Fix $a^* \in A$. From Eqn. (3.1), we get $A_n \subset A_k$, for all $n > k \geq P$. By Corollary 3.1, we have $A \subset \left(\overline{A_n \cup A_{n+1} \cup \ldots}\right) \subset \overline{A_k}$.

From Lemma 3.3, for $a^* \in A$, we get

$d(a^*, A_k) \leq \alpha(a_n, A_k)\varepsilon$.

Thus, Eqn. (3.4) is proved.

Now, we have to show Eqn. (3.3). since $\{A_n\}$ is Cauchy in $K_0(M)$, we have a sequence of positive integers that strictly increases $\{n_m\}_{m=1}^{\infty} = \{\varepsilon l^{-m}\}_{m=1}^{\infty}$ such that $n_m > P$, where $P \in \mathbb{N}$ and $H_d(A_n, A_k) < \varepsilon l^{-m}$ for all $n, k \geq n_m$.

Take arbitrary $a_k \in A_k$, where $a_k = a_{n_0}$.

Since $H_d(A_n, A_{n_0}) < \varepsilon$ for $n > n_0$, there exists $a_{n_1} \in A_{n_1}$ such that $d(a_{n_0}, a_{n_1}) < \varepsilon$ for $n = n_1 > n_0$.

Similarly, $H_d(A_n, A_{n_1}) < \dfrac{\varepsilon}{l}$, so there exists $a_{n_2} \in A_{n_2}$ such that $d(a_{n_1}, a_{n_2}) < \dfrac{\varepsilon}{l}$, for $n = n_2 > n_1$.

By following the same process, we can form a sequence $\{a_{n_m}\}$ with $a_{n_m} \in A_{n_m}$, for $m = 0, 1, 2, \ldots$ and

$$d(a_{n_m}, a_{n_{m+1}}) < \frac{\varepsilon}{l^m},\ a_{n_0} = a \tag{3.5}$$

Now, we will confirm that $\{a_{n_m}\}$ is Cauchy using the controlled strong $b-$triangle inequality.

$$\begin{aligned}
d(a_{n_m}, a_{n_{m+l}}) &\leq d(a_{n_m}, a_{n_{m+1}}) + \alpha(a_{n_{m+1}}, a_{n_{m+l}})d(a_{n_{m+1}}, a_{n_{m+l}}) \\
&\leq d(a_{n_m}, a_{n_{m+1}}) + \alpha(a_{n_{m+1}}, a_{n_{m+l}})d(a_{n_{m+1}}, a_{n_{m+2}}) \\
&+ \alpha(a_{n_{m+1}}, a_{n_{m+l}})\alpha(a_{n_{m+2}}, a_{n_{m+l}})d(a_{n_{m+2}}, a_{n_{m+l}}) \\
&\leq \ldots \\
&\leq d(a_{n_m}, a_{n_{m+1}}) + \sum_{i=m+1}^{m+l-2}\left(\prod_{j=m+1}^{i}\alpha(a_{n_j}, a_{n_{m+l}})\right)d(a_{n_i}, a_{n_{i+1}}) \\
&+ \prod_{j=m+1}^{m+l-1}\alpha(a_{n_j}, a_{n_{j+1}})d(a_{n_{m+l-1}}, a_{n_{m+l}}) \\
&\leq d(a_{n_m}, a_{n_{m+1}}) + \sum_{i=m+1}^{m+l-1}\left(\prod_{j=m+1}^{i}\alpha(a_{n_j}, a_{n_{m+l}})\right)d(a_{n_i}, a_{n_{i+1}}).
\end{aligned}$$

From Eqn. (3.5), we have

$$d(a_{n_m}, a_{n_{m+1}}) \leq \frac{\varepsilon}{l^m} + \left[\sum_{i=m+1}^{m+l-1}\left(\prod_{j=m+1}^{i}\alpha(a_{n_j}, a_{n_{m+l}})\right)\frac{\varepsilon}{l^i}\right]. \tag{3.6}$$

As $\lim_{n,k\to\infty}\alpha(\xi_n, \xi_k)\beta < 1$, for all $\xi_n, \xi_k \in M$. Using ratio test, the series $\left[\sum_{i=m+1}^{m+l-1}\left(\prod_{j=m+1}^{i}\alpha(a_{n_j}, a_{n_{m+l}})\right)\frac{\varepsilon}{l^i}\right]$ converges. As limit $m \to \infty$ in Eqn. (3.6), we have $\lim_{m\to\infty} d(a_{n_m}, a_{n_m+l}) = 0$.

Hence, we have $\{a_{n_m}\}$ is Cauchy. We have (M,d) is complete, $\exists\, a^* \in M$ where $a_{n_m} \to a^* \in M$ and clearly $a^* \in A$. Using the controlled strong $b-$triangle inequality,

$$\begin{aligned} d(a_{n_0},a_{n_m}) &\le d(a_{n_0},a_{n_1}) + \alpha(a_{n_1},a_{n_m})d(a_{n_1},a_{n_m}) \\ &\le d(a_{n_0},a_{n_1}) + \alpha(a_{n_1},a_{n_m})d(a_{n_1},a_{n_2}) + \alpha(a_{n_1},a_{n_m})\alpha(a_{n_2},a_{n_m})d(a_{n_2},a_{n_m}) \\ &\le \dots \\ &\le d(a_{n_0},a_{n_1}) + \sum_{i=1}^{m-2}\left(\prod_{j=1}^{i}\alpha(a_{n_j},a_{n_m})\right)d(a_{n_i},a_{n_{i+1}}) + \prod_{j=1}^{m-1}\alpha(a_{n_j},a_{n_m})d(a_{n_{m-1}},a_{n_m}) \\ &\le d(a_{n_0},a_{n_1}) + \sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\alpha(a_{n_j},a_{n_m})\right)d(a_{n_i},a_{n_{i+1}}). \end{aligned}$$

By Eqn. (3.5), we get

$$d(a_{n_0},a_{n_m}) \le \varepsilon + \sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\alpha(a_{n_j},a_{n_m})\right)\frac{\varepsilon}{l^i}. \tag{3.7}$$

As $\lim\limits_{n,k\to\infty}\alpha(\xi_n,\xi_k)\beta < 1$ for all $\xi_n,\xi_k \in M$. Using ratio test, the series $\sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\alpha(a_{n_j},a_{n_m})\right)\frac{\varepsilon}{l^i}$ converges. As limit $m \to \infty$ in Eqn. (3.7), we get $\lim\limits_{m\to\infty} d(a_{n_0},a_{n_m}) < \varepsilon$.
From the controlled strong $b-$triangle inequality, we have
$d(a^*,a_k) \le d(a^*,a_{n_m}) + \alpha(a_{n_m},a_k)d(a_{n_m},a_k)$
Hence, $d(a^*,a_k) \le \alpha(a_{n_m},a_k)\varepsilon$, as $m \to \infty$.
Hence from Eqn. (3.2), we obtain

$$H_d(A_n,A) \le \varepsilon + max\left\{\sup_{a_k\in A_k}\alpha(a_k,a),\alpha(a,A_k)\right\}\varepsilon \tag{3.8}$$

Since $\lim\limits_{n,k\to\infty}\alpha(\xi_n,\xi_k)\beta < 1$ for all $\xi_n,\xi_k \in M$, and let $n,k \to \infty$ in Eqn. (3.8), then we have a positive real number. Therefore, A_n approaches A. Hence the proof. □

4. Controlled Strong $b-$Kannan Fractal

In this section, we examined the HB theorem for constructing fractals on CCSbMS.

Theorem 4.1. *Let $\Gamma : M \to M$ be a continuous Kannan mapping on CCSbMS (M,d) with the contractivity factor $\beta \in \left(0,\frac{1}{12}\right)$ and bounded variable control functions $\alpha_0 = \sup\limits_{\xi,\lambda\in M}\alpha(\xi,\lambda)$ and $\alpha_{H_0} = \sup\limits_{B,C\in K_0(M)}\alpha_H(B,C)$, where $|\alpha(\xi,\lambda)| < \frac{5}{2}\ \forall\ \xi,\lambda \in M$ and $|\alpha_H(B,C)| < \frac{5}{2}\ \forall\ B,C \in K_0(M)$. Then $\Gamma : K_0(M) \to K_0(M)$ is defined as $\Gamma(B) = \{\Gamma(\xi) : \xi \in B\}$, $\forall B \in K_0(M)$, a Kannan contraction on $(K_0(M),H_d)$ with the contractivity factor ζ, $0 < \zeta = \frac{\beta}{1-2\beta\alpha_0-2\beta\alpha_{H_0}} < \frac{1}{2}$.*

Proof. Consider that Γ is a continuous map, it maps $K_0(M)$ into itself. Let $\xi, \lambda \in M$. Then,

$$\begin{aligned}
d(\Gamma(\xi),\Gamma(\lambda)) &\leq \beta[d(\xi,\Gamma(\xi)) + d(\lambda,\Gamma(\lambda))],\\
&\leq \beta[d(\xi,\Gamma(\lambda)) + \alpha(\Gamma(\lambda),\Gamma(\xi))d(\Gamma(\lambda),\Gamma(\xi)) + d(\lambda,\Gamma(\xi)) + \alpha(\Gamma(\xi),\Gamma(\lambda))d(\Gamma(\xi),\Gamma(\lambda))],\\
&= \beta[d(\xi,\Gamma(\lambda)) + d(\lambda,\Gamma(\xi))] + 2\beta\alpha(\Gamma(\xi),\Gamma(\lambda))d(\Gamma(\xi),\Gamma(\lambda)).\\
&\leq \beta[d(\xi,\Gamma(\lambda)) + d(\lambda,\Gamma(\xi))] + 2\beta\alpha_0 d(\Gamma(\xi),\Gamma(\lambda)),\ \textit{where}\ \alpha_0 = \sup_{\xi,\lambda\in M}\alpha(\Gamma(\xi),\Gamma(\lambda)).
\end{aligned}$$

$$d(\Gamma(\xi),\Gamma(\lambda)) \leq \frac{\beta}{1-2\beta\alpha_0}[d(\xi,\Gamma(\lambda)) + d(\lambda,\Gamma(\xi))].$$

Then $B, C \in K_0(M)$

$$\begin{aligned}
\sup_{\xi\in B}\inf_{\lambda\in C} d(\Gamma(\xi),\Gamma(\lambda)) &\leq \frac{\beta}{1-2\beta\alpha_0}\left[\sup_{\xi\in B}\inf_{\lambda\in C} d(\xi,\Gamma(\lambda)) + \sup_{\xi\in B}\inf_{\lambda\in C} d(\lambda,\Gamma(\xi))\right].\\
H_d(\Gamma(B),\Gamma(C)) &\leq \frac{\beta}{1-2\beta\alpha_0}\left[H_d(B,\Gamma(C)) + H_d(C,\Gamma(B))\right].\\
H_d(\Gamma(B),\Gamma(C)) &\leq \frac{\beta}{1-2\beta\alpha_0}\Big[H_d(B,\Gamma(B)) + \alpha_H(\Gamma(B),\Gamma(C))H_d(\Gamma(B),\Gamma(C))\\
&\qquad + H_d(C,\Gamma(C)) + \alpha_H(\Gamma(C),\Gamma(B))H_d(\Gamma(C),\Gamma(B))\Big],\\
&\leq \frac{\beta}{1-2\beta\alpha_0}\left[H_d(B,\Gamma(B)) + H_d(C,\Gamma(C)) + 2\alpha_{H_0}H_d(\Gamma(B),\Gamma(C))\right],\\
&\qquad \textit{where}\ \alpha_{H_0} = \sup_{B,C\in K_0(M)}\alpha_H(B,C).
\end{aligned}$$

$$\left[1 - \frac{2\beta\alpha_{H_0}}{1-2\beta\alpha_0}\right]H_d(\Gamma(B),\Gamma(C)) \leq \frac{\beta}{1-2\beta\alpha_0}[H_d(B,\Gamma(B)) + H_d(C,\Gamma(C))].$$

Then, $H_d(\Gamma(B),\Gamma(C)) \leq \zeta[H_d(B,\Gamma(B)) + H_d(C,\Gamma(C))]$,
where $\zeta = \dfrac{\beta}{1-2\beta\alpha_0-2\beta\alpha_{H_0}} < \dfrac{1}{2}\ \textit{text for}\ 0 < \beta < \dfrac{1}{12}$. Hence the proof. □

Definition 4.1. *(CSbK-IFS) If (M,d) is CCSbMS, and $\Gamma_n : M \to M$, $n = 1,2,3,\ldots,N$ $(N \in \mathbb{N})$ are Kannan contractive functions in CCSbMS with the contractivity factors β_n, $n = 1,2,3,\ldots,N$. Then, $\{M;\Gamma_n, n = 1,2,3,\ldots,N\}$ is known as CSbK-IFS of Kannan map with the contractivity factor $\beta = \max_{n=1}^{N}(\beta_n)$.*

Example 4.1. *Let $M = \{0,1,2\}$ and $d : M\times M \to \mathbb{R}^+$. Thus, d is a CSbMS on M from Example (2.1). We consider the self mappings $\Gamma_n : M \to M$ for $n = 1,2$.*

$$\Gamma_1(\xi) = \begin{cases} 1, & \text{if} \quad \xi = 0\\ 2, & \text{if} \quad \xi \in \{1,2\}\end{cases}$$

$$\Gamma_2(\xi) = \begin{cases} 0, & \text{if} \quad \xi \in \{0,1,2\}\end{cases}$$

Thus the system with Kannan contractivity maps $\{M;\Gamma_1,\Gamma_2\}$ is a Controlled Strong $b-$Kannan IFS.

Theorem 4.2. *Let (M,d) be a CCSbMS. Let $\{M;\Gamma_n, n=1,2,...,N\}$ be a CSbK-IFS of continuous Kannan mappings on $(K_0(M),H_d)$, with the contractivity factor $\beta=\max\{\beta_n\}_{n=1}^{N}$, $0<\beta_n<\frac{1}{12}$, for all n. Consider $\alpha_0=\sup_{\xi,\lambda\in M}\alpha(\xi,\lambda)$ and $\alpha_{H_0}=\sup_{B,C\in K_0(M)}\alpha_H(B,C)$, where $|\alpha(\xi,\lambda)|<\frac{5}{2}$ $\forall$ $\xi,\lambda\in M$ and $|\alpha_H(B,C)|<\frac{5}{2}$ $\forall$ $B,C\in K_0(M)$. Define $G:K_0(M)\to K_0(M)$ by $G(B)=\bigcup_{n=1}^{N}\Gamma_n(B)$, $\forall$ $B\in K_0(M)$. Then, G is a Kannan map with the contractivity factor $\zeta=\max\{\zeta_n\}_{n=1}^{N}$, where $\zeta_n=\frac{\beta_n}{1-2\beta_n\alpha_0-2\beta_n\alpha_{H_0}}$.*

Proof. Let $B,C\in K_0(M)$, then

$$\begin{aligned}H_d(G(B),G(C))&=H_d\Big(\Gamma_1(B)\cup\Gamma_2(B)\cup\ldots\cup\Gamma_N(B),\Gamma_1(C)\cup\Gamma_2(C)\cup\ldots\cup\Gamma_N(C)\Big)\\&\le\max\Big\{H_d(\Gamma_1(B),\Gamma_1(C)),H_d(\Gamma_2(B),\Gamma_2(C)),\ldots,H_d(\Gamma_N(B),\Gamma_N(C))\Big\}\end{aligned}$$

By using the Theorem 4.1, we obtain

$$\begin{aligned}H_d(G(B),G(C))=\max\Big\{&\frac{\beta_1}{1-2\beta_1\alpha_0-2\beta_1\alpha_{H_0}}\Big[H_d(B,\Gamma_1(B))+H_d(C,\Gamma_1(C))\Big],\\&\frac{\beta_2}{1-2\beta_2\alpha_0-2\beta_2\alpha_{H_0}}\Big[H_d(B,\Gamma_2(B))+H_d(C,\Gamma_2(C))\Big],\\&\vdots\\&\frac{\beta_N}{1-2\beta_N\alpha_0-2\beta_N\alpha_{H_0}}\Big[H_d(B,\Gamma_N(B))+H_d(C,\Gamma_N(C))\Big]\Big\}\end{aligned}$$

$$\begin{aligned}&\le\max_{1\le n\le N}\Big\{\frac{\beta_n}{1-2\beta_n\alpha_0-2\beta_n\alpha_{H_0}}\Big\}\Big[\max\Big\{H_d(B,\Gamma_1(B)),H_d(B,\Gamma_2(B)),\ldots,H_d(B,\Gamma_N(B))\Big\}\\&\qquad+\max\Big\{H_d(C,\Gamma_1(C)),H_d(C,\Gamma_2(C)),\ldots,H_d(C,\Gamma_N(C))\Big\}\Big]\\&\le\max_{1\le n\le N}\{\zeta_n\}\Big[H_d(B,\Gamma_1(B)\cup\Gamma_2(B))\ldots\cup\Gamma_N(B)+H_d(C,\Gamma_1(C)\cup\Gamma_2(C))\ldots\cup\Gamma_N(C)\Big]\\&\le\zeta\Big[H_d(B,G(B))+H_d(C,G(C))\Big]\end{aligned}$$

where $\zeta=\max_{1\le n\le N}\Big\{\zeta n\Big\}=\max_{1\le n\le N}\Big\{\frac{\beta_n}{1-2\beta_n\alpha_0-2\beta_n\alpha_{H_0}}\Big\}<\frac{1}{2}$.

Hence, G is a Kannan map with the contractivity factor ζ. □

Theorem 4.3. *If $\{M;(\Gamma_1,\Gamma_2,\cdots,\Gamma_n)\}$, is an CSbK-IFS with the contractivity factor $\beta=\max\{\beta_n\}_{n=1}^{N}$. Then, the transformation $G:K_0(M)\to K_0(M)$, defined by $G(B)=\bigcup_{n=1}^{N}\Gamma_n(B),\forall B\in K_0(M)$, is a continuous Kannan mapping on CCSbMS $(K_0(M),H_d)$ with the contractivity factor ζ. Furthermore, G*

has a (unique) fixed point $A \in K_0(M)$, so it shows that

$$A = G(A) = \bigcup_{n=1}^{N} \Gamma_n(A),$$

given by $A = \lim_{n\to\infty} G^{\circ}(B)$, $\forall\, A \in K_0(M)$.

Proof. Since (M, d) is CCSbMS, the $(K_0(M), H_d)$ is also a CCSbMS. The HB operator, G by Theorem 4.2 is a Kannan contraction map on CSbMS. We conclude G is a fixed point (unique) from Theorem 2.4. This, completes the proof. □

Definition 4.2 (Controlled Strong $b-$Kannan Fractals)**.** The fixed point $A \in K_0(M)$ obtained in Theorem 4.3 for CSbK-IFS of G is called Controlled Strong $b-$Kannan Fractal or Controlled Strong $b-$Kannan Attractor in CSbMS. Thus $A \in K_0(M)$ is an attractor constructed by a CSbK-IFS on CSbMS.

Theorem 4.4 (Collage Type Theorem)**.** *Let (M, d) be a CCSbMS. Let $J \in K_0(M)$ and $\varepsilon \geq 0$. Suppose $\{M; \Gamma_n, n = 1, 2, ...N\}$ is a CSbK-IFS, with the contractivity factor $\beta = max\{\beta_n\}_{n=1}^{N}$, and $\alpha_0 = \sup_{\xi,\lambda\in M} \alpha(\xi, \lambda)$, $\alpha_{H_0} = \sup_{B,C\in K_0(M)} \alpha_H(B, C)$, where $|\alpha(\xi, \lambda)| < \dfrac{5}{2}$ $\forall\ \xi, \lambda \in M$ and $|\alpha_H(B, C)| < \dfrac{5}{2}$ $\forall\, B, C \in K_0(M)$. Consider the continuous Kannan map on CCSbMS $(K_0(M), H_d)$, $G : K_0(M) \to K_0(M)$ such that $G(B) = \bigcup_{n=1}^{N} \Gamma_n(B), \forall B \in K_0(M)$, with the contractivity factor $\zeta = \max\{\zeta_n\}_{n=1}^{N}$, where $\zeta_n = \dfrac{\beta_n}{1 - 2\beta_n\alpha_0 - 2\beta_n\alpha_{H_0}}$, then*

$$H_d(J, G(J)) = H_d(J, \cup_{n=1}^{N} \Gamma_n(J)) \leq \varepsilon.$$

Then,

$$H_d(J, A) \leq \left[\frac{1}{1 - \psi\alpha_{H_0}}\right]\varepsilon,$$

where A is the attractor of CSbK-IFS and $\psi = \dfrac{\zeta}{1 - \zeta}$.

Proof. From triangular inequality of CSbMS, we have

$$\begin{aligned}
H_d(J, G^n(J)) &\leq H_d(J, G(J)) + \alpha_H(G(J), G^n(J))H_d(G(J), G^n(J)),\\
&\leq H_d(J, G(J)) + \alpha_H(G(J), G^n(J))H_d(G(J), G^2(J))\\
&\quad + \alpha_H(G(J), G^n(J))\alpha_H(G^2(J), G^n(J))H_d(G^2(J), G^n(J)),\\
&\quad \vdots\\
&\leq H_d(J, G(J)) + \alpha_H(G(J), G^n(J))H_d(G(J), G^2(J))\\
&\quad + \alpha_H(G(J), G^n(J))\alpha_H(G^2(J), G^n(J))H_d(G^2(J), G^3(J)) + \dots
\end{aligned}$$

$$
\begin{aligned}
&+\prod_{i=1}^{n-1}\alpha_H(G^i(J),G^n(J))H_d(G^{n-1}(J),G^n(J)).\\
&\leq H_d(J,G(J))+\alpha_{H_0}H_d(G(J),G^2(J))+(\alpha_{H_0})^2H_d(G^2(J),G^3(J))+\ldots\\
&+(\alpha_{H_0})^{n-1}H_d(G^{n-1}(J),G^n(J)),\text{ where }\alpha_{H_0}=\sup_{B,C\in K_0(M)}\alpha_H(B,C).
\end{aligned}
$$

Since G is Kannan mapping, then

$$
\begin{aligned}
H_d(G^n(J),G^{n+1}(J)) &\leq \zeta\Big[H_d(G^{n-1}(J),G^n(J))+H_d(G^n(J),G^{n+1}(J))\Big],\\
&\leq \frac{\zeta}{1-\zeta}H_d(G^{n-1}(J),G^n(J)),\\
&\leq \psi H_d(G^{n-1}(J),G^n(J)),\text{ where }\psi=\frac{\zeta}{1-\zeta}.\\
H_d(J,G^n(J)) &\leq \Big[1+\psi\alpha_{H_0}+\psi^2(\alpha_{H_0})^2+\ldots+\psi^{n-1}(\alpha_{H_0})^{n-1}\Big]H_d(J,G(J)),\\
&\leq \frac{1-\psi^n(\alpha_{H_0})^n}{1-\psi\alpha_{H_0}}H_d(J,G(J)),\text{ for }\psi(\alpha_{H_0})<1,
\end{aligned}
$$

Taking limit as $n\to\infty$, then

$$
H_d(J,A)\leq\frac{1}{1-\psi\alpha_{H_0}}H_d(J,\cup_{n=1}^N\Gamma_n(J)).
$$

□

The distance function in strong $b-$metric space is continuous, whereas $b-$metric function need not be continuous. The controlled strong $b-$metric space is generalized from strong $b-$metric space. A new kind of fractal structures are explored via the IFS of Kannan contraction maps in the controlled strong b-metric space as a general case.

5. Conclusion

In this research article, the Hausdorff controlled strong $b-$metric space (HCSbMS) is newly defined and it is proved that HCSbMS is complete. We developed controlled strong $b-$Kannan iterated function system, a new class of iterated function system on controlled strong $b-$metric space based on the Kannan contraction. Using controlled strong $b-$Kannan iterated function system, we constructed the controlled strong $b-$Kannan fractals on controlled strong $b-$metric space. This study is expected to lead us in a new direction for constructing the controlled strong $b-$fractal interpolation function and controlled strong $b-$multifractals.

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