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Boundedness of the Higher Order Commutators of Marcinkiewicz Integral Operator on Grand Variable Herz-Morrey Spaces

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Abstract. In this paper, the authors prove the boundedness of higher order commutators of Marcinkiewicz integral operator under some proper assumptions on grand variable Herz-Morrey spaces. Then we obtain the estimates for the Marcinkiewicz fractional operator of variable order in grand variable Herz spaces.

1. Introduction

In recent years, function spaces with variable exponents have gained increasing attention in research due to their applications in a variety of fields for instance see [12]. Over time, there has been significant theoretical work on variable Lebesgue, Orclicz, Sobolev and Lorentz spaces. This work has helped to establish the properties and characteristics of these spaces, and has contributed to our understanding of their potential applications. Some important references in this area include articles published by researchers such as [1–3, 6, 7]. In harmonic analysis, Herz space are notable for their norm which incorporates both local and global information. In [4] authors defined the idea of variable Herz spaces.

We consider the Riesz potential operator

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$$I^{\zeta(z_1)}g(z_1) = \int_{\mathbb{R}^n} \frac{g(z_2)}{|z_1 - z_2|^{n-\zeta(z_1)}} dz_2, \quad 0 < \zeta(z_1) < n. \tag{1.1}$$

In [8], Kokilashvili and Samko obtained the boundedness of Riesz potential operators $I^{\zeta(x)}$ of variable order on variable Lebesgue space.

More recently, in [13] variable parameters were used to define continual Herz spaces. Additionally, the boundedness of sublinear operators on these continual Herz spaces was demonstrated. Additionally, boundedness of other operators the Marcinkiewicz integral and Riesz potential operator can be confirmed in [9, 11]. For continual Herz-Morrey spaces see [10].

The grand variable Herz spaces are the generalization of variable Herz spaces and boundedness of some operators can be checked in works such as [17, 19, 25, 26, 29]. Moreover the grand variable Herz-Morrey spaces is the generalization of original grand variable Herz spaces and it was introduced in [18]. The study of higher-order commutators of fractional integral operator can provide important insights into the behavior of these operators in function spaces. For more results on variable exponent function spaces see [15, 16, 20–22, 22, 24, 27, 28, 31–34].

In fact in this article are aiming to prove the boundedness of higher order commutators of Marcinkiewicz integral operator $[\mathcal{U}, \mu_\Xi]^m$ under some proper assumptions on grand Herz-Morrey spaces with variable exponents. There are different sections of this paper. We have different sections in this research paper to organize and present the information clearly. Apart from introduction, the preliminaries section is important to establish the necessary background knowledge and definitions that will be used throughout the paper. Another section is for the boundedness of Marcinkiewicz fractional integral operator of variable order in grand variable Herz spaces. In the last section we will focus on boundedness of higher order commutators of Marcinkiewicz integral operator on grand variable Herz-Morrey spaces.

2. Preliminaries

Definition 2.1. *Let $\mathcal{A}$ be measurable subset in $\mathbb{R}^n$ and a measurable function $p(\cdot) : \mathcal{A} \to [1, \infty)$. Then*

(a) *Now Lebesgue space with variable exponent $L^{p(\cdot)}(\mathcal{A})$ is defined by*

$$L^{p(\cdot)}(\mathcal{A}) = \left\{ f \text{ is measurable} : \int_{\mathcal{A}} \left(\frac{|f(i)|}{\Gamma} \right)^{p(i)} di < \infty \text{ where } \Gamma \text{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(\mathcal{A})$ can be defined as,

$$\|f\|_{L^{p(\cdot)}(\mathcal{A})} = \inf \left\{ \Gamma > 0 : \int_{\mathcal{A}} \left(\frac{|f(i)|}{\Gamma} \right)^{p(i)} di \leq 1 \right\}.$$

(b) *The space $L^{p(\cdot)}_{\text{loc}}(\mathcal{A})$ can be defined as*

$$L^{p(\cdot)}_{\text{loc}}(\mathcal{A}) := \left\{ f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \mathcal{A} \right\}.$$

We use these notations in this paper:

(i) Suppose that $f \in L^1_{\text{loc}}(\mathcal{A})$ now the Hardy-Littlewood maximal operator $\mathfrak{M}$ can be given as

$$\mathfrak{M}f(i) := \sup_{0<r} \frac{1}{r^n} \int\limits_{B(i,r)} |f(i)| di \quad (i \in \mathcal{A}),$$

where $B(i,r) := \{x \in \mathcal{A} : |i - x| < r\}$.

(ii) Let $p(\cdot)$ be a measurable function then the set $\mathfrak{B}(\mathcal{A})$ is consists of those measurable functions satisfying the following properties:

$$p_- := \operatorname{ess}\inf_{x \in \mathcal{A}} p(x) > 1, \;\; p_+ := \operatorname{ess}\sup_{x \in \mathcal{A}} p(x) < \infty. \tag{2.1}$$

(iii) Let $p \in \mathfrak{B}(\mathcal{A})$ then $\mathfrak{B}^{\log} = \mathfrak{B}^{\log}(\mathcal{A})$ is the class of functions satisfying (2.1) and log-condition given by,

$$|\pi(x_1) - \pi(x_2)| \leq \frac{C(\pi)}{-\ln|x_1 - x_2|}, \;\; |x_1 - x_2| \leq \frac{1}{2}, \;\; x_1, x_2 \; \in \mathcal{A}. \tag{2.2}$$

(iv) For an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, $\mathfrak{B}_\infty(\mathcal{A})$ is the subset of $\mathfrak{B}(\mathcal{A})$ and values are in $[1, \infty)$ satisfying following condition

$$|\pi(x) - \pi_\infty| \leq \frac{C}{\ln(e + |x|)}, \tag{2.3}$$

where $\pi_\infty \in (1, \infty)$. $\mathfrak{B}_{0,\infty}(\mathcal{A})$ is the subset of $\mathfrak{B}(\mathcal{A})$ satisfying the following condition

$$|\pi(x) - \pi_0| \leq \frac{C}{\ln|x|}, |x| \leq \frac{1}{2}. \tag{2.4}$$

(v) For an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, then $\mathfrak{B}_\infty(\mathcal{A})$ and $\mathfrak{B}_{0,\infty}(\mathcal{A})$ are the subsets of $\mathfrak{B}(\mathcal{A})$.

(vi) Consider an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, then $\mathfrak{B}^{\log}_\infty(\mathcal{A})$ is the set of exponent $p \in \mathfrak{B}_\infty(\mathcal{A})$ satisfying condition (2.1). $\mathfrak{B}_\infty(\mathcal{A})$ is the subsets of exponents of $L^\infty(\mathcal{A})$ and its values are in $[1, \infty]$ satisfying both conditions (2.2) and (2.3),

(vii) $p(\cdot) \in \mathcal{A}$ then $\mathcal{B}(\mathcal{A})$ is the collection of $p(\cdot)$ where $\mathfrak{M}$ is bounded on $L^{p(\cdot)}(\mathcal{A})$.

(viii) $\mathcal{S}_t = \mathcal{S}(0, 2^t) = \{z_1 \in \mathbb{R}^n : |z_1| < 2^t\}$ for all $l \in \mathbb{Z}$. $F_t = \mathcal{S}_t \setminus \mathcal{S}_{t-1}$, $\chi_{F_t} = \chi_t$.

Lemma 2.1. *[13] Let $U > 1$ and $u \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{u(0)}} \leq \|\chi_{R_{s,U_s}}\|_{u(\cdot)} \leq t_0 s^{\frac{n}{p(0)}}, \textit{ for } 0 < s \leq 1 \tag{2.5}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{u_\infty}} \leq \|\chi_{R_{s,U_s}}\|_{u(\cdot)} \leq t_\infty s^{\frac{n}{u_\infty}}, \textit{ for } s \geq 1, \tag{2.6}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ and depending on U but independent of s.

Lemma 2.2. *[1] Let $\mathcal{A} \subseteq \mathbb{R}^n$, and $1 \leq p_-(\mathcal{A}) \leq p_+(\mathcal{A}) \leq \infty$. Then*

$$\|gf\|_{L^{r(\cdot)}(\mathcal{A})} \leq \|g\|_{L^{p(\cdot)}(\mathcal{A})} \|f\|_{q(\cdot)(\mathcal{A})}$$

holds, where $g \in L^{p(\cdot)}(\mathcal{A})$, $f \in q(\cdot)(\mathcal{A})$ and $\frac{1}{r(i)} = \frac{1}{p(i)} + \frac{1}{q(i)}$ for every $i \in \mathcal{A}$.

Definition 2.2 (BMO space). *Let u is a locally integrable function then a BMO function is consist of those functions whose mean oscillation given by $\frac{1}{|Q|}\int_Q |u(i)-u_Q|di$ is bounded. A Mathematically,*

$$\|u\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |u(i)-u_Q|di < \infty.$$

Lemma 2.3. *[5] Let $w, l \in \mathbb{Z}$ with $l < w$, and $\mathcal{U} \in BMO$, then*

$$\frac{1}{C}\|\mathcal{U}\|^k_{BMO} \le \sup_{\mathcal{S}:ball} \frac{1}{\|\chi_{\mathcal{S}}\|_{p(\cdot)}}\|(\mathcal{U}-\mathcal{U}_{\mathcal{S}})^k\chi_{\mathcal{S}}\|_{p(\cdot)} \tag{2.7}$$

$$\le C\|\mathcal{U}\|^k_{BMO}, \tag{2.8}$$

$$\|(\mathcal{U}-\mathcal{U}_{\mathcal{S}_t})^k\chi_{\mathcal{S}_w}\|_{p(\cdot)} \le C(w-t)^k\|\mathcal{U}\|^k_{BMO}\|\chi_{\mathcal{S}_w}\|_{p(\cdot)}. \tag{2.9}$$

Lemma 2.4. *[5] Consider a ball $\mathcal{S}$ in $\mathbb{R}^n$ and $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$; then we have*

$$|\mathcal{S}|^{-1}\|\chi_{\mathcal{S}}\|_{p(\cdot)}\|\chi_{\mathcal{S}}\|_{p'(\cdot)} \le C. \tag{2.10}$$

Definition 2.3. *Let $\gamma(\cdot) \in L^\infty(\mathbb{R}^n)$, $\varepsilon \in [1,\infty)$, $s : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$, $0 \le \lambda < \infty$. We define the homogeneous grand variable Herz-Morrey spaces can be defined by the norm:*

$$M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} = \sup_{\varpi>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varpi^\theta \sum_{o=-\infty}^{k_0} \|2^{o\gamma(\cdot)}g\chi_o\|^{\varepsilon(1+\varpi)}_{L^{s(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}.$$

For $\lambda = 0$, grand Herz-Morrey spaces becomes grand Herz spaces.

Non-homogeneous grand variable Herz-Morrey spaces can be defined in the similar way. For more details we refer to [14–16, 18, 28]

Let $\mathfrak{S}^{n-1}$ is denoting the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. $\Xi \in L^r(\mathfrak{S}^{n-1})$ is a function of degree zero which is a homogeneous such that

$$\int_{\mathfrak{S}^{n-1}} \Xi(z_2')d\Xi(z_2') = 0, \tag{2.11}$$

where $z_2' = z_2/|z_2|$ and y is not zero. The Marcinkiewicz fractional operator is introduced with Littlewood-Paley g-function as:

$$\mu_\Xi(g)(z_1) = \left(\int_0^\infty |F_{\Xi,s}(g)(z_1)|^2\frac{ds}{s^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Xi,s}(f)(z_1) = \int_{|z_1-z_2|\le s} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}}g(z_2)dz_2.$$

Let us take a locally integrable function $\mathcal{U}$ on $\mathbb{R}^n$, now define as higher order commutators of Marcinkiewicz integral $[\mathcal{U}, \mu_\Xi]^m$ by using μ_Ξ and $\mathcal{U}$ we have

$$[\mathcal{U}, \mu_\Xi]^m(g)(z_1) = \left(\int_0^\infty \left| \int_{|z_1-z_2|\leq s} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}} [\mathcal{U}(z_1) - \mathcal{U}(z_2)]^m g(z_2) dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

The variable Marcinkiewicz fractional integral are given by:

$$\mu_\Xi(g)(z_1) = \left(\int_0^\infty \left| \int_{|z_1-z_2|\leq s} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-\zeta(z_1)-1}} g(z_2) dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

Lemma 2.5 ([35]). *If $a > 0$, $s \in [1, \infty]$, $0 < d \leq s$ and $-m + (m-1)ds < u < \infty$, then*

$$\left(\int_{|z_2|\leq a|z_1|} |z_2|^u |\Xi(z_1 - z_2)|^d dz_2 \right)^{1/d} \leq |z_1|^{(u+m)/d} \|\Xi\|_{L^s(\mathbb{S}^{m-1})}.$$

Lemma 2.6 ([35]). *Let $H \subset \mathbb{R}^n$ denotes an open set and F is a family. If for some q_0 with $0 < q_0 < \infty$ and every $\omega \in A_\infty$,*

$$\int_H g(z_1)^{q_0} \omega(z_1) dz_1 \leq C_0 \int_H g(z_1)^{q_0} \omega(z_1) dz_1, \qquad (f, g) \in F.$$

Lemma 2.7 ([35]). *Let $\Xi \in L^s(\mathfrak{S}^{n-1})$ with $s > 1$ and satisfying (2.1). If $\mathcal{U}(z_1) \in BMO$ and $\omega \in A_{q/s'}, s' < q < \infty$, then there exists a constant C, not depending on f, so that*

$$\int_{\mathbb{R}^n} |[\mathcal{U}, \mu_\Xi](f)(z_1)|^q \omega(z_1) dz_1 \leq C \int_{\mathbb{R}^n} |(f)(z_1)|^q \omega(z_1) dz_1.$$

3. Main results

Theorem 3.1. *Ift $0 < v \leq 1$, $\gamma(\cdot), q(\cdot) \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q^- \leq q^+ < \infty$, $m \in \mathbb{Z}$, $1 \leq \varepsilon < \infty$, $0 \leq \beta < \infty$ and $\mathcal{U} \in BMO$. Let $\Xi \in L^s(\mathfrak{S}^{n-1})$, and $s > q'^-$. C denote the constant. Let γ be such that :*

(i) $-\frac{n}{q(0)} - v - \frac{n}{s} < \gamma(0) < \frac{n}{q'(0)} - v - \frac{n}{s}$

(ii) $-\frac{n}{q_\infty} - v - \frac{n}{s} < \gamma_\infty < \frac{n}{q'_\infty} - v - \frac{n}{s}$,

then operator $[\mathcal{U}, \mu_\Xi]^m$ will be bounded on $M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)$.

Proof. If $g \in M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)$, and $g(x) = \sum_{i_0=-\infty}^{\infty} g(x)\chi_{i_0}(z_1) = \sum_{i_0=-\infty}^{\infty} g_{i_0}(z_1)$, for $k_0 > 0$ we have,

$$\begin{aligned} \|[\mathcal{U}, \mu_\Xi]^m g\|_{M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)} &= \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\varpi^\theta \sum_{o=-\infty}^{k_0} \|2^{o\gamma(\cdot)} \chi_o [\mathcal{U}, \mu_\Xi]^m g\|_{q(\cdot)}^{\varepsilon(1+\varpi)} \right)^{\frac{1}{\varepsilon(1+\varpi)}} \\ &\leq \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\varpi^\theta \sum_{o=-\infty}^{k_0} \left(\sum_{i_0=-\infty}^{\infty} \|2^{o\gamma(\cdot)} \chi_o [\mathcal{U}, \mu_\Xi]^m g(\chi_{i_0})\|_{q(\cdot)}^{\varepsilon(1+\varpi)} \right) \right)^{\frac{1}{\varepsilon(1+\varpi)}} \end{aligned}$$

$$\leq \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=-\infty}^{k_0}\left(\sum_{i_0=-\infty}^{o}\|2^{o\gamma(\cdot)}\chi_o[\mathcal{U},\mu_{\Xi}]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$+\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=-\infty}^{k_0}\left(\sum_{i_0=o+1}^{\infty}\|2^{o\gamma(\cdot)}\chi_o[\mathcal{U},\mu_{\Xi}]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$= E_1 + E_2.$$

For E_1, we get

$$E_1 \leq \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\varpi)}\left(\sum_{i_0=-\infty}^{o}\|\chi_o[\mathcal{U},\mu_{\Xi}]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$+\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\sum_{k=0}^{k_0}2^{o\gamma_{\infty}\varepsilon(1+\varpi)}\left(\sum_{i_0=-\infty}^{o}\|\chi_o[\mathcal{U},\mu_{\Xi}]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$= E_{11} + E_{12}.$$

Let $k\in\mathbb{Z}$ with $i_0 \leq o-2$ and a.e. $z_1 \in F_k$, $z_2 \in F_{i_0}$, it is easy to chech that $|z_1-z_2| \approx |z_1| \approx 2^k$,

$$|\mu_{\Xi}(g\chi_{i_0})(z_1)| \leq \left(\int_{o}^{|z_1|}\left|\int_{|z_1-z_2|\leq t}\frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}}[\mathcal{U}(z_1-\mathcal{U}(z_2))]^m g_{i_0}(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_{|z_1|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}}[\mathcal{U}(z_1-\mathcal{U}(z_2))]^m g_{i_0}(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}$$
$$=: I_{11} + I_{12}.$$

The mean value theorem yields

$$\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right| \leq \frac{|z_2|}{|z_1-z_2|^3}. \tag{3.1}$$

For I_{11}, we get

$$I_{11} \leq \int_{\mathbb{R}^n}\frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1}}|\mathcal{U}(z_1-\mathcal{U}(z_2)|^m|g_{i_0}(z_2)|\left(\int_{|z_1-z_2|}^{|z_1|}\frac{dt}{t^3}\right)^{1/2}dz_2$$
$$\leq \int_{\mathbb{R}^n}\frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1}}|\mathcal{U}(z_1-\mathcal{U}(z_2)|^m|g_{i_0}(z_2)|\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right|^{1/2}dz_2$$
$$\leq \int_{\mathbb{R}^n}\frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1}}|\mathcal{U}(z_1-\mathcal{U}(z_2)|^m|g_{i_0}(z_2)|\left|\frac{|z_2|}{|z_1-z_2|^3}\right|^{1/2}dz_2$$

$$\leq \frac{2^{i_0/2}}{|z_1|^{n+1/2}} \int_{F_{i_0}} |\Xi(z_1 - z_2)||\mathcal{U}(z_1 - \mathcal{U}(z_2)|^m |g(z_2)| dz_2$$

$$\leq 2^{(i_0-o)/2} 2^{o(-n)} \Big\{ |\mathcal{U}(z_1) - \mathcal{U}_{\mathcal{S}_{i_0}}|^m \int_{F_{i_0}} |\Xi(z_1 - z_2)||g_{i_0}(z_2)| dz_2$$

$$+ \int_{F_{i_0}} |\mathcal{U}(z_2) - \mathcal{U}_{\mathcal{S}_{i_0}}|^m |\Xi(z_1 - z_2)||g_{i_0}(z_2)| dz_2$$

$$\leq 2^{(i_0-o)/2} 2^{o(-n)} \|g_{i_0}(z_2)\|_{q(\cdot)(\mathbb{R}^n)} \Big\{ |\mathcal{U}(z_1) - \mathcal{U}_{\mathcal{S}_{i_0}}|^m \|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}$$

$$+ \|(\mathcal{U}(\cdot) - \mathcal{U}_{\mathcal{S}_{i_0}})^m (\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} \Big\}.$$

Similarly, we can consider I_{12}, we have

$$I_{12} \leq \int_{\mathbb{R}^n} \frac{|\Xi(z - 1 - z_2)|}{|z_1 - z_2|^{n-1}} |\mathcal{U}(z_1 - \mathcal{U}(z_2)|^m |g_{i_0}(z_2)| \left(\int_{|z_1|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dz_2$$

$$\leq \int_{\mathbb{R}^n} \frac{|\Xi(z_1 - z_2)|}{|z_1 - z_2|^n} |\mathcal{U}(z_1) - \mathcal{U}(z_2)|^m |g_{i_0}(z_2)| dz_2$$

$$\leq |z_1|^{-n} \int_{F_{i_0}} |\Xi(z_1 - z_2)||\mathcal{U}(z_1) - \mathcal{U}(z_2)|^m |g(z_2)| dz_2$$

$$\leq 2^{-on} \|g_{i_0}(z_2)\|_{q(\cdot)} \Big\{ |\mathcal{U}(z_1) - \mathcal{U}_{\mathcal{S}_{i_0}}|^m \|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} + \|(\mathcal{U}(\cdot) - \mathcal{U}_{\mathcal{S}_{i_0}})^m (\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} \Big\}.$$

So we have,

$$|\mu_{\Xi}(g\chi_{i_0})(z_1)|$$
$$\leq 2^{-on} \|g_{i_0}(z_2)\|_{q(\cdot)} \Big\{ |\mathcal{U}(z_1) - \mathcal{U}_{\mathcal{S}_{i_0}}|^m \|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} + \|(\mathcal{U}(\cdot) - \mathcal{U}_{\mathcal{S}_{i_0}})^m (\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} \Big\}.$$

Let $\frac{1}{q'(x)} = \frac{1}{q(x)} + \frac{1}{s}$. Then (2.5) and Hölder's inequality yields

$$\|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)} \leq \|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_{i_0}(\cdot)\|_{q(\cdot)}$$

$$\leq 2^{-i_0 v} \left(\int_{2^{i_0-1} < |z_2| < 2^{i_0}} |\Xi(z_1 - z_2)|^s |z_2|^{sv} dz_2 \right)^{1/s} \|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}$$

$$\leq 2^{-i_0 v} 2^{o(\frac{n}{s} + v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}.$$

Applying Lemma (2.3) we get

$$\|(\mathcal{U}(\cdot) - \mathcal{U}_{\mathcal{S}_{i_0}})^m (\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}$$
$$\leq \|\Xi(z_1 - \cdot)\chi_{i_0}(\cdot)\|_{L^s(\mathbb{R}^n)} \|(\mathcal{U}(\cdot) - \mathcal{U}_{\mathcal{S}_{i_0}})^m \chi_{i_0}(\cdot)\|_{q(\cdot)}$$

$$\leq C\|\mathcal{U}\|_{BMO}^m \|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^s(\mathbb{R}^n)}$$
$$\leq C\|\mathcal{U}\|_{BMO}^m 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}.$$

Hence we have

$$\begin{aligned}
&\|[\mathcal{U}, \mu_\Xi(g_{i_0})]\chi_k\|_{q(\cdot)} \\
&\leq C2^{-on}\|g_{i_0}\|_{q(\cdot)} \Big\{ \|(\mathcal{U}(\cdot)-\mathcal{U}_{\mathcal{S}_{i_0}})^m \chi_o(\cdot)\|_{q(\cdot)} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \\
&+ \|\mathcal{U}\|_{BMO}^m 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \|\chi_o\|_{q(\cdot)} \Big\} \\
&\leq C2^{-on}\|g_{i_0}\|_{q(\cdot)} \Big\{ (o-i_0)^m \|\mathcal{U}\|_{BMO}^m \|\chi_{B_o}\|_{q(\cdot)} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \\
&+ \|\mathcal{U}\|_{BMO}^m 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \|\chi_{B_o}\|_{q(\cdot)} \Big\} \\
&\leq C2^{-on}\|g_{i_0}\|_{q(\cdot)} (o-i_0)^m \|\mathcal{U}\|_{BMO}^m \|\chi_{B_o}\|_{q(\cdot)} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \\
&\leq C(o-i_0)^m \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\mathcal{U}\|_{BMO}^m 2^{-on} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\chi_{B_o}\|_{q(\cdot)} \|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)} \|g_{i_0}\|_{q(\cdot)}.
\end{aligned}$$

Applying results to E_{11} we can get

$$\begin{aligned}
E_{11} \leq C \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg[\varpi^\theta \sum_{o=-\infty}^{-1} 2^{o\gamma(0)\varepsilon(1+\varpi)} \Bigg(\sum_{i_0=-\infty}^{o} (o-i_0)^m \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\mathcal{U}\|_{BMO}^m \\
\times 2^{(i_0-o)(n/q'(0)-v-\frac{n}{s})} \|g\chi_{i_0}\|_{q(\cdot)} \Bigg)^{\varepsilon(1+\varpi)} \Bigg]^{\frac{1}{\varepsilon(1+\varpi)}}
\end{aligned}$$

$$\leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg[\varpi^\theta \sum_{o=-\infty}^{-1} \Bigg(\sum_{i_0=-\infty}^{o} 2^{\gamma(0)i_0} \|g\chi_{i_0}\|_{q(\cdot)} 2^{\mathcal{U}(i_0-o)} (o-i_0)^m \Bigg)^{\varepsilon(1+\varpi)} \Bigg]^{\frac{1}{\varepsilon(1+\varpi)}}.$$

For $\frac{n}{q_1'(0)} - v - \frac{n}{s} - \gamma(0) = b > 0$. As we can see by the fact that $2^{-\varepsilon(1+\varpi)} < 2^{-\varepsilon}$. Again using well known Fubini's theorem and Hölder's inequality to get

$$\begin{aligned}
E_{11} \leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg[\varpi^\theta \sum_{o=-\infty}^{-1} \Bigg(\sum_{i_0=-\infty}^{o} 2^{\gamma(0)\varepsilon(1+\varpi)i_0} \|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)} 2^{b\varepsilon(1+\varpi)(i_0-o)/2} \\
\times \sum_{i_0=-\infty}^{o} 2^{b\varepsilon(1+\varpi)'(i_0-o)/2} (o-i_0)^{m\varepsilon(1+\varpi)'} \Bigg)^{\frac{\varepsilon(1+\varpi)}{\varepsilon(1+\varpi)'}} \Bigg]^{\frac{1}{\varepsilon(1+\varpi)}}
\end{aligned}$$

$$\leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg(\varpi^\theta \sum_{o=-\infty}^{-1} \sum_{i_0=-\infty}^{o} 2^{\gamma(0)\varepsilon(1+\varpi)i_0} \|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)} 2^{b\varepsilon(1+\varpi)(i_0-o)/2} \Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \Bigg(\varpi^\theta \sum_{i_0=-\infty}^{-1} 2^{\gamma(0)\varepsilon(1+\varpi)i_0} \|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)} \sum_{o=l+2}^{-1} 2^{b\varepsilon(1+\varpi)(i_0-o)/2} \Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{i_0=-\infty}^{-1}2^{\gamma(0)\varepsilon(1+\varpi)i_0}\|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}\sum_{o=l+2}^{-1}2^{b\varepsilon(1+\varpi)(i_0-o)/2}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{i_0=-\infty}^{-1}2^{\gamma(0)\varepsilon(1+\varpi)i_0}\|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{i_0=-\infty}^{k_0}\|2^{\gamma(\cdot)i_0}g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^{m}\|g\|_{M\dot{K}_{\beta,q(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}(\mathbb{R}^n)}.$$

Now for E_{12} using Minkowski's inequality we have

$$E_{12}\leq\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{k=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=-\infty}^{-1}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$+\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{k=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=0}^{o-2}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$=A_1+A_2.$$

Estimate of A_1 is easy to obtain. For A_1 we have

$$\|[\mathcal{U},\mu_\Xi]_\beta(g\chi_l)\chi_o\|_{q(\cdot)}$$
$$\leq C(o-i_0)^m\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|_{BMO}^m2^{-on}2^{-i_0v}2^{o(\frac{n}{s}+v)}\|\chi_{B_o}\|_{q(\cdot)}\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|g_{i_0}\|_{q(\cdot)}$$
$$\leq C(o-i_0)^m\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|_{BMO}^m2^{i_0(\frac{n}{q(0)}-v)}2^{o(\frac{n}{s}+v-\frac{n}{q'_\infty})}\|g_{i_0}\|_{q(\cdot)}.$$

Now by using the fact $-\frac{n}{q'_\infty}+\frac{n}{s}+v+\gamma_\infty<0$ we have,

$$A_1\leq\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{k=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=-\infty}^{-1}\|\chi_k[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|_{BMO}^m\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\left[\varpi^{\theta}\sum_{k=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=-\infty}^{-1}2^{i_0(\frac{n}{q(0)}-v)}2^{o(\frac{n}{s}+v-\frac{n}{q'_\infty})}(o-i_0)^m\|g\chi_{i_0}\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right]^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|_{BMO}^m\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\Bigg[\varpi^{\theta}\sum_{k=0}^{k_0}2^{o(\gamma_\infty+\frac{n}{s}+v-\frac{n}{q'_\infty}\varepsilon(1+\varpi)}(o-i_0)^{m\varepsilon(1+\varpi)}$$
$$\times\left(\sum_{i_0=-\infty}^{-1}2^{i_0(\frac{n}{q(0)}-v)}\|g\chi_{i_0}\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\Bigg]^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\left(\sum_{i_0=-\infty}^{-1} 2^{i_0(\frac{n}{q(0)}-v)}\|g\chi_{i_0}\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\leq C\|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\left(\sum_{i_0=-\infty}^{-1} 2^{i_0\gamma(0)}\|g\chi_{i_0}\|_{q(\cdot)}2^{i_0(\frac{n}{q(0)}-v-\gamma(0))}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}.$$

As we can check that $\frac{n}{q'(0)}-\frac{n}{s}-v-\gamma(0)>0$ and Hölder's inequality we have

$$\begin{aligned}\leq & C\|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\Bigg[\varpi^{\theta}\sum_{i_0=-\infty}^{-1} 2^{\gamma(0)i_0\varepsilon(1+\varpi)}\|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)} \\ & \times\left(\sum_{i_0=-\infty}^{-1} 2^{i_0(\frac{n}{q'(0)}-v-\gamma(0))\varepsilon(1+\varpi)'}\right)^{\frac{\varepsilon(1+\varpi)}{\varepsilon(1+\varpi)'}}\Bigg]^{\frac{1}{\varepsilon(1+\varpi)}} \\ \leq & C\|\mathcal{U}\|_{BMO}^m \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\varpi^{\theta}\left(\sum_{i_0=-\infty}^{k_0}\|2^{\gamma(\cdot)i_0}g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}\right)\right)^{\frac{1}{\varepsilon(1+\varpi)}} \\ \leq & C\|\mathcal{U}\|_{BMO}^m\|g\|_{M\dot{K}_{\beta,q(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}(\mathbb{R}^n)}.\end{aligned}$$

Now we will find the estimate for E_3. Let $k\in\mathbb{Z}$ with $i_0\geq k+2$ and a.e. $z_1\in F_k$, $z_2\in F_{i_0}$, it is easy to check that $|z_1-z_2|\approx|z_2|\approx 2^{i_0}$, we consider

$$\begin{aligned}|\mu_{\Xi}(g\chi_{i_0})(z_1)|\leq & \left(\int_o^{|z_2|}\left|\int_{|z_1-z_2|\leq t}\frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}}g_{i_0}(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2} \\ & +\left(\int_{|z_2|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1}}f_{i_0}(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2} \\ & =: I_{31}+I_{32}.\end{aligned}$$

Similarly using the estimate of I_{11}, we get

$$\begin{aligned}I_{31}\leq & 2^{(o-i_0)/2}2^{-i_0n}\|g_{i_0}\|_{q(\cdot)} \\ & \left\{|\mathcal{U}(z_1)-\mathcal{U}_{\mathcal{S}_{i_0}}|^m\|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}\|\mathcal{U}(\cdot)-(\mathcal{U}_{\mathcal{S}_{i_0}})^m(\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}\right\}.\end{aligned}$$

Similar to the arguments of I_{12}, we have

$$\begin{aligned}I_{32}\leq & 2^{-i_0n}\|g_{i_0}\|_{q(\cdot)} \\ & \left\{|\mathcal{U}(z_1)-\mathcal{U}_{\mathcal{S}_{i_0}}|^m\|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}+\|\mathcal{U}(\cdot)-(\mathcal{U}_{\mathcal{S}_{i_0}})^m(\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}\right\}.\end{aligned}$$

Hence we get

$$|[\mathcal{U},\mu_\Xi]-(f_{i_0})(z_1)| \le 2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}$$
$$\left\{|\mathcal{U}(z_1)-\mathcal{U}_{\mathcal{S}_{i_0}}|^m\|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}+\|\mathcal{U}(\cdot)-(\mathcal{U}_{\mathcal{S}_{i_0}})^m(\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{q'(\cdot)}\right\}.$$

Thus we have

$$\begin{aligned}
&\|[\mathcal{U},\mu_\Xi(g_{i_0})]\chi_k\|_{q(\cdot)}\\
&\le C2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}\left\{\|(\mathcal{U}(\cdot)-\mathcal{U}_{\mathcal{S}_{i_0}})^m\chi_o(\cdot)\|_{q(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\right.\\
&\left.+\|\mathcal{U}\|^m_{BMO}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|\chi_o\|_{q(\cdot)}\right\}\\
&\le C2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}\left\{(o-i_0)^m\|\mathcal{U}\|^m_{BMO}\|\chi_{B_o}\|_{q(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\right.\\
&\left.+\|\mathcal{U}\|^m_{BMO}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|\chi_{B_o}\|_{q(\cdot)}\right\}\\
&\le C2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}(o-i_0)^m\|\mathcal{U}\|^m_{BMO}\|\chi_{B_o}\|_{q(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\\
&\le C(o-i_0)^m\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\chi_{B_o}\|_{q(\cdot)}\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|g_{i_0}\|_{q(\cdot)}\\
&\le C(o-i_0)^m\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}2^{i_0 n/q_\infty}2^{on/q_\infty}\|g_{i_0}\|_{q(\cdot)}\\
&\le C(o-i_0)^m\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{(o-i_0)^m(\frac{n}{s}+v+\frac{n}{q_\infty})}\|g\chi_{i_0}\|_{q(\cdot)}.
\end{aligned}$$

Now splitting E_3 we have

$$\begin{aligned}
E_3 \le& \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^\theta\sum_{o=-\infty}^{k_0}\left(\sum_{i_0=k+2}^{\infty}\|2^{o\gamma(\cdot)}\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}\\
\le& \sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^\theta\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\varpi)}\left(\sum_{i_0=k+2}^{\infty}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}\\
&+\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^\theta\sum_{k=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=k+2}^{\infty}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}\\
=&E_{31}+E_{32}.
\end{aligned}$$

For E_{32} we have

$$\begin{aligned}
E_{32} \le& C\|\mathcal{U}\|^m_{BMO}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\varpi^\theta\sum_{o=0}^{k_0}2^{o\gamma_\infty\varepsilon(1+\varpi)}\left(\sum_{i_0=k+2}^{\infty}2^{(o-i_0)(\frac{n}{s}+v+\frac{n}{q_\infty})}(o-i_0)^m\|g\chi_{i_0}\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}
\end{aligned}$$

$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}$$
$$\times\left(\varpi^{\theta}\sum_{o=0}^{k_0}\left(\sum_{i_0=k+2}^{\infty}2^{i_0\gamma_\infty}\|g\chi_{i_0}\|_{q(\cdot)}2^{(o-i_0)(\frac{n}{s}+v+\frac{n}{q_\infty})}(o-i_0)^m\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

for $\frac{n}{q_{1\infty}}+a_\infty+v+\frac{n}{s}=d>0$. Again applying the Hölder's theorem and $2^{-\varepsilon(1+\varpi)}<2^{-\varepsilon}$ to obtain

$$E_{32}\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\Bigg[\varpi^{\theta}\sum_{o=0}^{k_0}\left(\sum_{i_0=k+2}^{\infty}2^{i_0\gamma_\infty\varepsilon(1+\varpi)}\|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}2^{d\varepsilon(1+\varpi)(o-i_0)/2}\right)$$
$$\times\left(\sum_{i_0=k+2}^{\infty}2^{d\varepsilon(1+\varpi)'(o-i_0)/2}(o-i_0)^{m(\varepsilon(1+\varpi))'}\right)^{\frac{\varepsilon(1+\varpi)}{\varepsilon(1+\varpi)'}}\Bigg]^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=0}^{k_0}\sum_{i_0=k+2}^{\infty}2^{l\gamma_\infty\varepsilon(1+\varpi)}\|g\chi_{i_0}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}2^{d\varepsilon(1+\varpi)(o-i_0)/2}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=0}^{k_0}\sum_{i_0=k+2}^{\infty}\sum_{j=-\infty}^{i_0}2^{j\gamma_\infty\varepsilon(1+\varpi)}\|g\chi_{j}\|_{q(\cdot)}^{\varepsilon(1+\varpi)}2^{d\varepsilon(1+\varpi)(o-i_0)/2}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|_{BMO}^{m}\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=0}^{k_0}\sum_{i_0=k+2}^{\infty}2^{d\varepsilon(1+\varpi)(o-i_0)/2}\right)^{\frac{1}{\varepsilon(1+\varpi)}}\|g\|_{M\dot{K}_{\beta,q(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}(\mathbb{R}^n)}$$
$$\leq C\|\mathcal{U}\|_{BMO}^{m}\|g\|_{M\dot{K}_{\beta,q(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}(\mathbb{R}^n)}.$$

Now for E_{31} using Minkowski's inequality we have

$$E_{31}\leq\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\varpi)}\left(\sum_{i_0=k+2}^{-1}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$+\sup_{\varpi>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\varpi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\varpi)}\left(\sum_{i_0=0}^{\infty}\|\chi_o[\mathcal{U},\mu_\Xi]^m(g\chi_{i_0})\|_{q(\cdot)}\right)^{\varepsilon(1+\varpi)}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$=B_1+B_2.$$

Estimate of B_1 is easy to obtain. For B_2 we have

$$\|[\mathcal{U},\mu_\Xi](g_{i_0})\chi_k\|_{q(\cdot)}$$
$$\leq C2^{-i_0n}\|g_{i_0}\|_{q(\cdot)}\Big\{\|(\mathcal{U}(\cdot)-\mathcal{U}_{\mathcal{S}_{i_0}})^m\chi_o(\cdot)\|_{q(\cdot)}2^{-i_0v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}$$
$$+\|\mathcal{U}\|_{BMO}^{m}2^{-i_0v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\mathfrak{S}^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|\chi_o\|_{q(\cdot)}\Big\}$$

$$\leq C2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}\Big\{(o-i_0)^m\|\mathcal{U}\|^m_{BMO}\|\chi_{B_o}\|_{q(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\Im^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}$$

$$+\|\mathcal{U}\|^m_{BMO}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\Im^{n-1})}\|\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|\chi_{B_o}\|_{q(\cdot)}\Big\}$$

$$\leq C2^{-i_0 n}\|g_{i_0}\|_{q(\cdot)}(o-i_0)^m\|\mathcal{U}\|^m_{BMO}\|\chi_{B_o}\|_{q(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\Xi\|_{L^s(\Im^{n-1})}\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}$$

$$\leq C(o-i_0)^m\|\Xi\|_{L^s(\Im^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\chi_{B_o}\|_{q(\cdot)}\|\chi_{\mathcal{S}_{i_0}}\|_{q(\cdot)}\|g_{i_0}\|_{q(\cdot)}$$

$$\leq C(o-i_0)^m\|\Xi\|_{L^s(\Im^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|g\chi_{i_0}\|_{q(\cdot)}2^{i_0 n/q_\infty}2^{on/q(0)}$$

$$\leq C(o-i_0)^m\|\Xi\|_{L^s(\Im^{n-1})}\|\mathcal{U}\|^m_{BMO}2^{-i_0(\frac{n}{s}+v+\frac{n}{q_\infty})}2^{o(v+\frac{n}{q(0)}+\frac{n}{s})}\|g\chi_{i_0}\|_{q(\cdot)}.$$

Next we have

$$B_2 \leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\varpi)}$$
$$\times\Bigg(\sum_{i_0=0}^{\infty}(o-i_0)^m\|g_{i_0}\|_{L^{q(\cdot)}}2^{-l(\frac{n}{q_\infty}+v+\frac{n}{s})}2^{o(\frac{n}{q(0)}+v+\frac{n}{s})}\Bigg)^{\varepsilon(1+\varpi)}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}$$
$$\times\Bigg(\varpi^\theta\sum_{o=-\infty}^{-1}\Bigg(\sum_{i_0=0}^{\infty}2^{i_0\gamma(0)}(o-i_0)^m\|g_{i_0}\|_{L^{q(\cdot)}}2^{-l(\frac{n}{q_\infty}+v+\frac{n}{s}-\gamma(0))}2^{o(\frac{n}{q(0)}+v+\frac{n}{s}+\gamma(0))}\Bigg)^{\varepsilon(1+\varpi)}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}.$$

Where $\frac{n}{q_\infty}+v+\frac{n}{s}-\gamma(0)=\theta_1$ and $\frac{n}{q(0)}+v+\frac{n}{s}+\gamma(0)=\theta_2$.

$$B_2 \leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{o=-\infty}^{-1}\Bigg(\sum_{i_0=0}^{\infty}2^{i_0\gamma(0)}(o-i_0)^m\|g_{i_0}\|_{L^{q(\cdot)}}2^{-l\theta_1}2^{k\theta_2}\Bigg)^{\varepsilon(1+\varpi)}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{o=-\infty}^{-1}\sum_{i_0=0}^{\infty}2^{i_0\gamma(0)\varepsilon(1+\varpi)}\|g_{i_0}\|^{\varepsilon(1+\varpi)}_{L^{q(\cdot)}}\Big((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\Big)^{\varepsilon(1+\varpi)/2}$$
$$\times\Bigg(\sum_{i_0=0}^{\infty}\Big((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\Big)^{\varepsilon(1+\varpi)'/2}\Bigg)^{\frac{\varepsilon(1+\varpi)}{\varepsilon(1+\varpi)'}}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{o=-\infty}^{-1}\sum_{i_0=0}^{\infty}2^{i_0\gamma(0)\varepsilon(1+\varpi)}\|g_{i_0}\|^{\varepsilon(1+\varpi)}_{L^{q(\cdot)}}\Big((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\Big)^{\varepsilon(1+\varpi)/2}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{i_0=0}^{\infty}\sum_{o=-\infty}^{i_0}2^{i_0\gamma(0)\varepsilon(1+\varpi)}\|g_{i_0}\|^{\varepsilon(1+\varpi)}_{L^{q(\cdot)}}\Big((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\Big)^{\varepsilon(1+\varpi)/2}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$
$$\leq C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Bigg(\varpi^\theta\sum_{i_0=0}^{\infty}2^{i_0\gamma(0)\varepsilon(1+\varpi)}\|g_{i_0}\|^{\varepsilon(1+\varpi)}_{L^{q(\cdot)}}\sum_{o=-\infty}^{i_0}\Big((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\Big)^{\varepsilon(1+\varpi)/2}\Bigg)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\le C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\sup_{\varpi>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\left(\varpi^\theta\sum_{i_0=0}^{\infty}\sum_{i_0=-\infty}^{i_0}2^{l\gamma(0)\varepsilon(1+\varpi)}\|g_{i_0}\|^{\varepsilon(1+\varpi)}_{L^{q(\cdot)}}\sum_{o=-\infty}^{i_0}\left((o-i_0)^m2^{-l\theta_1}2^{k\theta_2}\right)^{\varepsilon(1+\varpi)/2}\right)^{\frac{1}{\varepsilon(1+\varpi)}}$$

$$\le C\|\mathcal{U}\|^m_{BMO)(\mathbb{R}^n)}\|g\|_{M\dot{K}^{a(\cdot),\varepsilon),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1, E_2 and E_3 yields
$\|[\mathcal{U},\mu_\Xi]^m(g)\|_{M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)} \le \|\mathcal{U}\|^m_{BMO}\|g\|_{M\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}$,
which ends the proof. □

Theorem 3.2. *Let $0 < v \le 1$, $\gamma(\cdot), q(\cdot) \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q^- \le q^+ < \infty$, $1 \le u < \infty$ and $b \in BMO$. Let Ξ be homogenous of degree zero and $\Xi \in L^s(\mathfrak{S}^{n-1})$, $s > q'^-$. C denote the constant. Let γ be such that :*

(i) $-\frac{n}{q_1(0)} - v - \frac{n}{s} < \gamma(0) < \frac{n}{q_1'(0)} - v - \frac{n}{s}$
(ii) $-\frac{n}{q_{1\infty}} - v - \frac{n}{s} < \gamma_\infty < \frac{n}{q_{1\infty}'} - v - \frac{n}{s}$,

$$\|(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g)\|_{\dot{K}^{a(\cdot),\varepsilon),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|g\|_{\dot{K}^{a(\cdot),\varepsilon),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $g \in \dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{q_2(\cdot)}$, and $g(z_1) = \sum_{i_0=-\infty}^{\infty} g(z_1)\chi_{i_0}(z_1) = \sum_{i_0=-\infty}^{\infty} g_{i_0}(z_1)$, we have,

$$\|(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g)\|_{\dot{K}^{a(\cdot),\varepsilon),\theta}_{q_2(\cdot)}(\mathbb{R}^n)}$$
$$= \sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{\infty}\|2^{o\gamma(\cdot)}\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g)\|^{\varepsilon(1+\psi)}_{q_2(\cdot)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\le \sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{\infty}\left(\sum_{i_0=-\infty}^{\infty}\|2^{o\gamma(\cdot)}\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g)(\chi_{i_0})\|^{\varepsilon(1+\psi)}_{q_2(\cdot)}\right)\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\le \sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{\infty}\left(\sum_{i_0=-\infty}^{o}\|2^{o\gamma(\cdot)}\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$+\sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{\infty}\left(\sum_{i_0=o+1}^{\infty}\|2^{o\gamma(\cdot)}\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$=: E_1 + E_2.$$

For E_1, splitting E_1 by using Minkowski's inequality we have

$$E_1 \le \sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{o}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$+\sup_{\psi>0}\left(\psi^\theta\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{o}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$=: E_{11} + E_{12}.$$

For estimating E_{11}, Let $k \in \mathbb{Z}$ with $i_0 \leq k$ and a.e. $z_1 \in F_k$, $z_2 \in F_{i_0}$, and $|z_1 - z_2| \approx |z_1| \approx 2^k$, we get

$$
\begin{aligned}
|\mu_{\Xi}(g\chi_{i_0})(z_1)| \leq & \left(\int_{o}^{|z_1|} \left| \int_{|z_1-z_2|\leq t} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}} g\chi_{i_0}(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& + \left(\int_{|z_1|}^{\infty} \left| \int_{|z_1-z_2|\leq t} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}} g\chi_{i_0}(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& =: I_{11} + I_{12}.
\end{aligned}
$$

Again using the same arguments we get

$$
\left| \frac{1}{|z_1-z_2|^2} - \frac{1}{|z_1|^2} \right| \leq \frac{|z_2|}{|z_1-z_2|^3}. \tag{3.2}
$$

For I_{11}, 3.1, Minkowski inequaltiy and Hölder's inequality yields

$$
\begin{aligned}
I_{11} \leq & \int_{\mathbb{R}^n} \frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}} |g\chi_{i_0}(z_2)| \left(\int_{|z_1-z_2|}^{|z_1|} \frac{dt}{t^3} \right)^{1/2} dz_2 \\
\leq & \int_{\mathbb{R}^n} \frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}} |g\chi_{i_0}(z_2)| \left| \frac{1}{|z_1-z_2|^2} - \frac{1}{|z_1|^2} \right|^{1/2} dz_2 \\
\leq & \int_{\mathbb{R}^n} \frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}} |g\chi_{i_0}(z_2)| \left| \frac{|z_2|}{|z_1-z_2|^3} \right|^{1/2} dz_2 \\
\leq & \frac{2^{i_0/2}}{|z_1|^{n+\frac{1}{2}}.|z_1|^{-\zeta(z_1)}} \int_{F_{i_0}} |\Xi(z_1-z_2)||g(z_2)| dz_2 \\
\leq & 2^{(i_0-o)/2} 2^{-on} |z_1|^{\zeta(z_1)} \|g\chi_{i_0}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^{q_1'(\cdot)}}.
\end{aligned}
$$

For I_{12}, we get

$$
\begin{aligned}
I_{12} \leq & \int_{\mathbb{R}^n} \frac{|\Xi(z-1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}} |g\chi_{i_0}(z_2)| \left(\int_{|z_1|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dz_2 \\
\leq & \int_{\mathbb{R}^n} \frac{|\Xi(z_1-z_2)|}{|z_1-z_2|^{n-1-\zeta(z_1)}} |g\chi_{i_0}(z_2)| dz_2 \\
\leq & |z_1|^{-n} |z_1|^{\zeta(z_1)} \int_{F_{i_0}} |\Xi(z_1-z_2)||g(z_2)| dz_2 \\
\leq & 2^{-on} |z_1|^{\zeta(z_1)} \|g\chi_{i_0}(z_2)\|_{q_1(\cdot)} \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^{q_1'(\cdot)}}.
\end{aligned}
$$

Let $\frac{1}{q_1'(x)} = \frac{1}{q_1(x)} + \frac{1}{s}$. Then Lemma (2.5) yields

$$
\|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^{q_1'(\cdot)}} \leq \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_{i_0}(\cdot)\|_{q_1(\cdot)}
$$

$$\leq 2^{-i_0 v}\left(\int\limits_{2^{i_0-1}<|z_2|<2^{i_0}} |\Xi(z_1-z_2)|^s |z_2|^{sv} dz_2\right)^{1/s} \|\chi_{D_{i_0}}\|_{q_1(\cdot)}$$
$$\leq 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\Xi\|_{L^s(\mathfrak{S}^{n-1})} \|\|\chi_{D_{i_0}}\|_{q_1(\cdot)}.$$

It is known, see e.g. [36] that

$$\begin{aligned} I^{\zeta(\cdot)}(\chi_{B_o})(z_1) &\geq I^{\zeta(\cdot)}(\chi_{B_o})(z_1).(\chi_{B_o})(z_1) \\ &= \int_{B_o} \frac{1}{|z_1-z_2|^{\zeta(z_1)-n}} dy.\chi_{B_o}(z_1) \\ &\geq C|z_1|^{\zeta(z_1)}.\chi_{B_o}(z_1) \\ &\geq C|z_1|^{\zeta(z_1)}.\chi_k(z_1). \end{aligned}$$

Consequently, using the weighted Sobolev-type estimate for the fractional operator [8], we have

$$\|\chi_o|z_1|^{\zeta(z_1)}(|z_1|+1)^{-\lambda(z_1)}\|_{q_2(\cdot)} \geq \|(|z_1|+1)^{-\lambda(z_1)} I^{\zeta(\cdot)}(\chi_{B_o})\|_{q_2(\cdot)} \geq \|\chi_{B_o}\|_{q_1(\cdot)}.$$

Using these estimates we get

$$E_{11} \leq C\sup_{\psi>0}\left[\psi^\theta \sum_{o=-\infty}^{-1} 2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{o} 2^{(i_0-o)(n/q_1'(0)-v-\frac{n}{s})}\|g\chi_{i_0}\|_{q_1(\cdot)}\right)^{\varepsilon(1+\psi)}\right]^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left[\psi^\theta \sum_{o=-\infty}^{-1}\left(\sum_{i_0=-\infty}^{o} 2^{\gamma(0)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)} 2^{b(i_0-o)}\right)^{\varepsilon(1+\psi)}\right]^{\frac{1}{\varepsilon(1+\psi)}}.$$

Let $b = \frac{n}{q_1'(0)} - v - \frac{n}{s} - \gamma(0) > 0$, applying Hölders inequality, $2^{-\varepsilon(1+\psi)} < 2^{-\varepsilon}$ and Fubini's theorem for series and we get,

$$\begin{aligned} E_{11} \leq & C\sup_{\psi>0}\left[\psi^\theta \sum_{o=-\infty}^{-1}\left(\sum_{i_0=-\infty}^{o} 2^{\gamma(0)\varepsilon(1+\psi)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} 2^{b\varepsilon(1+\psi)(i_0-o)/2}\right.\right. \\ & \left.\left.\times \sum_{i_0=-\infty}^{o} 2^{b\varepsilon(1+\psi)'(i_0-o)/2}\right)^{\frac{\varepsilon(1+\psi)}{\varepsilon(1+\psi)'}}\right]^{\frac{1}{\varepsilon(1+\psi)}} \\ \leq & C\sup_{\psi>0}\left(\psi^\theta \sum_{o=-\infty}^{-1}\sum_{i_0=-\infty}^{o} 2^{\gamma(0)\varepsilon(1+\psi)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} 2^{b\varepsilon(1+\psi)(i_0-o)/2}\right)^{\frac{1}{\varepsilon(1+\psi)}} \\ \leq & C\sup_{\psi>0}\left(\psi^\theta \sum_{i_0=-\infty}^{-1} 2^{\gamma(0)\varepsilon(1+\psi)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} \sum_{o=l}^{-1} 2^{b\varepsilon(1+\psi)(i_0-o)/2}\right)^{\frac{1}{\varepsilon(1+\psi)}} \\ \leq & C\sup_{\psi>0}\left(\psi^\theta \sum_{i_0=-\infty}^{-1} 2^{\gamma(0)\varepsilon(1+\psi)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} \sum_{o=l}^{-1} 2^{bp(i_0-o)/2}\right)^{\frac{1}{\varepsilon(1+\psi)}} \end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{i_0=-\infty}^{-1}2^{\gamma(0)\varepsilon(1+\psi)i_0}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
&\le C\sup_{\psi>0}\left(\psi^{\theta}\sum_{i_0=-\infty}^{\infty}\|2^{\gamma(\cdot)i_0}g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
&\le C\|g\|_{\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{q_1(\cdot)}}.
\end{aligned}
$$

Now for E_{12}, we get

$$
\begin{aligned}
E_{12} \le &\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{-1}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
&+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=0}^{o}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
=:& A_1+A_2.
\end{aligned}
$$

Estimate of A_2 is easy to obtain. Next we find the estimate of A_1, we get

$$
\begin{aligned}
&\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\\
\le& C2^{-on}2^{-i_0v}2^{o(\frac{n}{s}+v)}\|\chi_{B_o}\|_{q_1(\cdot)}\|\chi_l\|_{q_1(\cdot)}\|g\chi_{i_0}\|_{q_1(\cdot)}\\
\le& C2^{i_0(\frac{n}{q_1(0)}-v)}2^{o(\frac{n}{s}+v-\frac{n}{q'_{1\infty}})}\|g\chi_{i_0}\|_{q_1(\cdot)}.
\end{aligned}
$$

Let $-\frac{n}{q'_{1\infty}}+\frac{n}{s}+v+\gamma_\infty<0$ then we get

$$
\begin{aligned}
A_1 \le &\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{-1}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\le& C\sup_{\psi>0}\left[\psi^{\theta}\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=-\infty}^{-1}2^{i_0(\frac{n}{q_1(0)}-v)}2^{o(\frac{n}{s}+v-\frac{n}{q'_{1\infty}})}\|g\chi_{i_0}\|_{q(\cdot)}\right)^{\varepsilon(1+\psi)}\right]^{\frac{1}{\varepsilon(1+\psi)}}\\
\le& C\sup_{\psi>0}\left[\psi^{\theta}\sum_{k=0}^{\infty}2^{o(\gamma_\infty+\frac{n}{s}+v-\frac{n}{q'_{1\infty}}\varepsilon(1+\psi)}\times\left(\sum_{i_0=-\infty}^{-1}2^{i_0(\frac{n}{q_1(0)}-v)}\|g\chi_{i_0}\|_{q_1(\cdot)}\right)^{\varepsilon(1+\psi)}\right]^{\frac{1}{\varepsilon(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{i_0=-\infty}^{-1}2^{i_0(\frac{n}{q_1(0)}-v)}\|g\chi_{i_0}\|_{q_1(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\le& C\sup_{\psi>0}\left(\psi^{\theta}\left(\sum_{i_0=-\infty}^{-1}2^{i_0\gamma(0)}\|g\chi_{i_0}\|_{q_1(\cdot)}2^{i_0(\frac{n}{q_1(0)}-v-\gamma(0))}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}.
\end{aligned}
$$

Now by applying Hölders inequality and using the fact that $\frac{n}{q_1'(0)} - \frac{n}{s} - v - \gamma(0) > 0$ we have

$$\leq C \sup_{\psi>0} \left[\psi^{\theta} \sum_{i_0=-\infty}^{-1} 2^{\gamma(0) i_0 \varepsilon(1+\psi)} \|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} \times \left(\sum_{i_0=-\infty}^{-1} 2^{i_0 (\frac{n}{q_1'(0)} - v - \gamma(0)) \varepsilon(1+\psi)'} \right)^{\frac{\varepsilon(1+\psi)}{\varepsilon(1+\psi)'}} \right]^{\frac{1}{\varepsilon(1+\psi)}}$$

$$\leq C \sup_{\psi>0} \left(\psi^{\theta} \left(\sum_{i_0=-\infty}^{\infty} \|2^{\gamma(\cdot) i_0} g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)} \right) \right)^{\frac{1}{\varepsilon(1+\psi)}}$$

$$\leq C \|g\|_{\dot{K}_{q_1(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}}.$$

For E_2, we have

$$|\mu_{\Xi}(g\chi_{i_0})(z_1)| \leq \left(\int_o^{|z_2|} \left| \int_{|z_1-z_2|\leq t} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}} g\chi_{i_0}(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2}$$

$$+ \left(\int_{|z_2|}^{\infty} \left| \int_{|z_1-z_2|\leq t} \frac{\Xi(z_1-z_2)}{|z_1-z_2|^{n-1-\zeta(z_1)}} g\chi_{i_0}(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2}$$

$$=: I_{31} + I_{32}.$$

Similarly for I_{11}, we get

$$I_{31} \leq 2^{(o-k)/2} 2^{-i_0 n} |z_1|^{\zeta(z_1)} \|g\chi_{i_0}\|_{q_1(\cdot)} \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^{q_1'(\cdot)}}.$$

For I_{12}, we get

$$I_{32} \leq 2^{-i_0 n} |z_1|^{\zeta(z_1)} \|g\chi_{i_0}\|_{q_1(\cdot)} \|\Xi(z_1-\cdot)\chi_{i_0}(\cdot)\|_{L^{q_1'(\cdot)}}.$$

Consequently, we get

$$\|\chi_o |z_1|^{\zeta(z_1)} (|z_1|+1)^{-\lambda(z_1)}\|_{q_2(\cdot)} \geq \|(|z_1|+1)^{-\lambda(z_1)} I^{\zeta(\cdot)}(\chi_{B_o})\|_{q_2(\cdot)} \geq \|\chi_{B_o}\|_{q_1(\cdot)}.$$

Hence we get

$$\begin{aligned}
&\|\chi_o (|z_1|+1)^{-\lambda(z_1)} \mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)} \\
&\leq C 2^{-i_0 n} \|g\chi_{i_0}\|_{q_1(\cdot)} \|\chi_{B_o}\|_{q_1(\cdot)} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\chi_l\|_{q_1(\cdot)} \\
&\leq C 2^{-i_0 n} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} \|\chi_{B_o}\|_{q_1(\cdot)} \|\chi_{D_{i_0}}\|_{q_1(\cdot)} \|g\chi_{i_0}\|_{q_1(\cdot)} \\
&\leq C 2^{-i_0 n} 2^{-i_0 v} 2^{o(\frac{n}{s}+v)} 2^{i_0 n/q_{1\infty}} 2^{on/q_\infty} \|g\chi_{i_0}\|_{q_1(\cdot)} \\
&\leq C 2^{(o-i_0)(\frac{n}{s}+v+\frac{n}{q_{1\infty}})} \|g\chi_{i_0}\|_{q_1(\cdot)}.
\end{aligned}$$

Now splitting E_2 we have

$$E_2 \leq \max \left\{ \sup_{\psi>0} \left(\psi^{\theta} \sum_{o=-\infty}^{\infty} 2 \left(\sum_{i_0=k+1}^{\infty} \|^{o\gamma(\cdot)} \chi_o (|z_1|+1)^{-\lambda(z_1)} \mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)} \right)^{\varepsilon(1+\psi)} \right)^{\frac{1}{\varepsilon(1+\psi)}}, \right.$$

$$\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=k+1}^{\infty}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$+\sup_{\psi>0}\left(\psi^{\theta}\sum_{k=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=k+1}^{\infty}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$=:E_{21}+E_{22}.$$

For E_{22} we have

$$E_{22}\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\left(\sum_{i_0=k+1}^{\infty}2^{(o-i_0)(\frac{n}{s}+v+\frac{n}{q_{1\infty}})}\|g\chi_{i_0}\|_{q_1(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=0}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\left(\sum_{i_0=k+1}^{\infty}2^{(o-i_0)(\frac{n}{s}+v+\frac{n}{q_{1\infty}})}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$

where $d=\frac{n}{q_\infty}+v+\frac{n}{s}>0$. Next we get

$$E_{22}\leq C\sup_{\psi>0}\left[\psi^{\theta}\sum_{o=0}^{\infty}\left(\sum_{i_0=k+1}^{\infty}2^{o\gamma_\infty\varepsilon(1+\psi)}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}2^{d\varepsilon(1+\psi)(o-i_0)/2}\right)\right.$$
$$\left.\times\left(\sum_{i_0=k+1}^{\infty}2^{d\varepsilon(1+\psi)'(o-i_0)/2}\right)^{\frac{\varepsilon(1+\psi)}{\varepsilon(1+\psi)'}}\right]^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=0}^{\infty}\sum_{i_0=k+1}^{\infty}2^{k\gamma_\infty\varepsilon(1+\psi)}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}2^{d\varepsilon(1+\psi)(o-i_0)/2}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\leq C\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=0}^{\infty}\|2^{o\gamma(\cdot)}g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$\leq C\|g\|_{\dot{K}^{\gamma(\cdot),\varepsilon),\theta}_{q_1(\cdot)}}.$$

Now for E_{21} using Minkowski's inequality we have

$$E_{21}\leq\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=k+1}^{-1}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$+\sup_{\psi>0}\left(\psi^{\theta}\sum_{o=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=0}^{\infty}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_{\Xi}(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}$$
$$=:B_1+B_2.$$

We will find the estimate of B_2 and for B_1 we find easily. So we have

$$\begin{aligned}
&\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\\
&\leq C2^{-i_0 n}\|g\chi_{i_0}\|_{q_1(\cdot)}\|\chi_{B_o}\|_{q_1(\cdot)}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\chi_l\|_{q_1(\cdot)}\\
&\leq C2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|\chi_{B_o}\|_{q_1(\cdot)}\|\chi_l\|_{q_1(\cdot)}\|g\chi_{i_0}\|_{q_1(\cdot)}\\
&\leq C2^{-i_0 n}2^{-i_0 v}2^{o(\frac{n}{s}+v)}\|g\chi_{i_0}\|_{q_1(\cdot)}2^{i_0 n/q_{1\infty}}2^{on/q_1(0)}\\
&\leq C2^{-i_0(\frac{n}{s}+v+\frac{n}{q_{1\infty}})}2^{o(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g\chi_{i_0}\|_{q_1(\cdot)}.
\end{aligned}$$

$$\begin{aligned}
B_2 \leq& \sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\left(\sum_{i_0=0}^{\infty}\|\chi_o(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g\chi_{i_0})\|_{q_2(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\leq& C\sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{o\gamma(0)\varepsilon(1+\psi)}\right.\\
&\qquad\left.\times\left(\sum_{i_0=0}^{\infty}2^{-i_0(\frac{n}{s}+v+\frac{n}{q_{1\infty}})}2^{o(v+\frac{n}{q_1(0)}+\frac{n}{s})}\|g\chi_{i_0}\|_{q_1(\cdot)}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\leq& C\sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{o(v+\frac{n}{q_1(0)}+\frac{n}{s}+\gamma(0))\varepsilon(1+\psi)}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right.\\
&\qquad\left.\times\left(\sum_{i_0=0}^{\infty}2^{-i_0(\frac{n}{s}+v+\frac{n}{q_{1\infty}})}\right)^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\leq& C\sup_{\psi>0}\left(\psi^\theta\sum_{k=-\infty}^{-1}2^{o(v+\frac{n}{q_1(0)}+\frac{n}{s}+\gamma(0))\varepsilon(1+\psi)}\|g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}.
\end{aligned}$$

Now by the virtue of Hölders inequality and we know that $\frac{n}{q_1(0)}+v+\frac{n}{s}+\gamma(0)>0$ yields

$$\begin{aligned}
B_2 \leq& C\sup_{\psi>0}\left(\psi^\theta\sum_{o=-\infty}^{\infty}\|2^{o\gamma(\cdot)}g\chi_{i_0}\|_{q_1(\cdot)}^{\varepsilon(1+\psi)}\right)^{\frac{1}{\varepsilon(1+\psi)}}\\
\leq& \|g\|_{\dot{K}_{q_1(\cdot)}^{\gamma(\cdot),\varepsilon),\theta}}.
\end{aligned}$$

Combining the estimates for E_1 and E_2 we get

$$\|(|z_1|+1)^{-\lambda(z_1)}\mu_\Xi(g)\|_{\dot{K}_{q_2(\cdot)}^{a(\cdot),\varepsilon),\theta}(\mathbb{R}^n)} \leq C\|g\|_{\dot{K}_{q_1(\cdot)}^{a(\cdot),\varepsilon),\theta}(\mathbb{R}^n)}.$$

which ends the proof. □

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