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# ANALYSIS OF DISCRETE MITTAG - LEFFLER FUNCTIONS

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Abstract. Discrete Mittag - Leffler functions play a major role in the development of the theory of discrete fractional calculus. In the present article, we analyze qualitative properties of discrete Mittag - Leffler functions and establish sufficient conditions for convergence, oscillation and summability of the infinite series associated with discrete Mittag - Leffler functions.

## 1. Introduction & Preliminaries

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of fractional orders. Many scientists have paid lot of attention due to its interesting applications in various fields of science and engineering, such as viscoelasticity, diffusion, neurology, control theory and statistics [24]. Like the exponential function in the theory of differential equations, Mittag - Leffler function plays an important role in the theory of fractional differential equations. The definition for one parameter Mittag - Leffler function was given by Gösta Mittag Leffler [22]. Later, Agarwal [1] defined the two parameter Mittag - Leffler function.

**Definition 1.** Let $t \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$. The one and two parameter Mittag - Leffler functions are defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \tag{1.1}$$

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{1.2}$$

The analogous theory for nabla discrete fractional calculus was initiated by Miller & Ross [21], Gray & Zhang [10] and Atici & Eloe [6], where basic approaches, definitions, and properties of fractional sums and differences were discussed. A series of papers continuing this research has appeared recently [6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 19, 23, 25, 26].

Throughout this article, we shall use the following notations, definitions and known results of nabla discrete fractional calculus [6, 25]. For any $a$, $b \in \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2, ..........\}$, $\mathbb{N}_{a,b} = \{a, a+1, a+2, ..........., b\}$ where $a < b$.

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**Definition 2.** For any $\alpha$, $t \in \mathbb{R}$, the $\alpha$ rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{......, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of rising factorial function.

**Lemma 1.** *Assume the following factorial functions are well defined.*

(1) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
(2) *If* $t \leq r$ *then* $t^{\overline{\alpha}} \leq r^{\overline{\alpha}}$.
(3) *If* $\alpha < t \leq r$ *then* $r^{\overline{-\alpha}} \leq t^{\overline{-\alpha}}$.

**Definition 3.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

(1) (Nabla Difference) The first order backward difference or nabla difference of $u$ is defined by

$$\nabla u(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the $N^{th}$ - order nabla difference of $u$ is defined recursively by

$$\nabla^N u(t) = \nabla(\nabla^{N-1} u(t)), \quad t \in \mathbb{N}_{a+N}.$$

In addition, we take $\nabla^0$ as the identity operator.

(2) (Fractional Nabla Sum) The $\alpha^{th}$ - order fractional nabla sum of $u$ is given by

$$\nabla_a^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a \tag{1.3}$$

where $\rho(s) = s-1$. Also, we define the trivial sum by $\nabla_a^{-0} u(t) = u(t)$ for $t \in \mathbb{N}_a$.

(3) (R - L Fractional Nabla Difference) The $\alpha^{th}$ - order Riemann - Liouville type fractional nabla difference of $u$ is given by

$$\nabla_a^{\alpha} u(t) = \nabla^N \Big[\nabla_a^{-(N-\alpha)} u(t)\Big], \quad t \in \mathbb{N}_{a+N}. \tag{1.4}$$

For $\alpha = 0$, we set $\nabla_a^0 u(t) = u(t),\ t \in \mathbb{N}_a$.

(4) (Caputo Fractional Nabla Difference) The $\alpha^{th}$ - order Caputo type fractional nabla difference of $u$ is given by

$$\nabla_{a*}^{\alpha} u(t) = \nabla_a^{-(N-\alpha)} \Big[\nabla^N u(t)\Big], \quad t \in \mathbb{N}_{a+N}. \tag{1.5}$$

For $\alpha = 0$, we set $\nabla_{a*}^0 u(t) = u(t),\ t \in \mathbb{N}_a$.

The unified definition for fractional sums and differences is as follows.

**Definition 4.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$. Then

(1) the $\alpha^{th}$ - order nabla fractional sum of $u$ is given by

$$\nabla_a^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

(2) the $\alpha^{th}$ - order R - L fractional difference of $u$ is given by

(1.6)
$$\nabla_a^{\alpha} u(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\alpha-1}} u(s), & \alpha \notin \mathbb{N}_1, \\ \nabla^N u(t), & \alpha = N \in \mathbb{N}_1, \end{cases} \quad \text{for } t \in \mathbb{N}_{a+N}.$$

**Theorem 2.** *(Power Rule) Let $\alpha > 0$ and $\mu > -1$. Then,*

(1) $\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}},\ t \in \mathbb{N}_a$.

(2) $\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}},\ t \in \mathbb{N}_{a+N}$.

**Definition 5.** A function $u$ is said to be slowly oscillating if $u(t) - u(s) \to 0$ for any $t, s \in \mathbb{N}_a$, whenever $s \to \infty, t > s, \frac{t}{s} \to 1$.

**Definition 6.** A function $u$ is said to be $T$ - periodic if $u(t+T) = u(t)$ for all $t \in \mathbb{N}_a$. The positive integer $T$ is called the period of the function $u$. Further $T$ is said to be the basic period if there does not exist a smaller period $T_1 \in \mathbb{Z}^+$ such that $T_1 < T$.

**Definition 7.** A continuous and bounded function $u$ is said to be $S$ - asymptotically periodic if there exists $T > 0$ such that $u(t+T) - u(t) \to 0$ as $t \to \infty$. In this case, we say that $T$ is an asymptotic period of $u$ and that $u$ is $S$ - asymptotically $T$ - periodic.

## 2. Qualitative Properties of Discrete Mittag - Leffler Functions

The definitions for one and two parameter discrete Mittag - Leffler functions are given by Atsushi Nagai [2] and Atici & Eloe [8] respectively.

**Definition 8.** Let $t \in \mathbb{N}_0$, $\lambda \in (-1, 1)$ and $\alpha, \beta \in \mathbb{R}^+$. The one and two parameter discrete Mittag - Leffler functions are defined by

(2.1)
$$F_{\alpha}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)},$$

(2.2)
$$F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + \beta)}.$$

**Lemma 3.** *We observe the following properties of (2.1) and (2.2).*

(1) $F_{\alpha,1}(\lambda, t^{\overline{\alpha}}) = F_{\alpha}(\lambda, t)$.
(2) $F_{1,1}(\lambda, t^{\overline{1}}) = F_1(\lambda, t) = (1-\lambda)^{-t}$.
(3) $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) \geq 0$.
(4) $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) \sim E_{\alpha,\beta}(\lambda t^{\alpha}) \quad (t \to \infty)$.
(5) $F_{\alpha}(\lambda, t) \geq \frac{[1+(\alpha-1)\lambda}{(1-\lambda)^2\Gamma(t)}, \quad t \in \mathbb{N}_2$.
(6) $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) \leq (1-\lambda)^{-1}, \quad 2 \leq t \leq \beta$.
(7) $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) \geq \frac{1}{(1-\lambda)\Gamma(t)}, \quad 2 \leq \beta \leq t$.

*Proof.* The proofs of (1), (2), (3) and (4) follow from (2.1) and (2.2). To prove (5), we consider (2.1). Clearly,

$$\frac{\Gamma(t+\alpha k)}{\Gamma(\alpha k + 2)} \geq 1, \quad t \in \mathbb{N}_2.$$

Hence

$$F_\alpha(\lambda,t)=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+1)}\geq\frac{1}{\Gamma(t)}\sum_{k=0}^{\infty}\lambda^k(\alpha k+1)=\frac{[1+(\alpha-1)\lambda}{(1-\lambda)^2\Gamma(t)}.$$

Now we consider (2.2) to prove (6) and (7). Clearly,

$$\frac{1}{\Gamma(t)}\leq 1 \text{ and } \frac{\Gamma(t+\alpha k)}{\Gamma(\alpha k+\beta)}\leq 1,\quad 2\leq t\leq\beta$$

and

$$\frac{\Gamma(t+\alpha k)}{\Gamma(\alpha k+\beta)}\geq 1,\quad 2\leq\beta\leq t.$$

Hence

$$F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)}\leq\sum_{k=0}^{\infty}\lambda^k=\frac{1}{1-\lambda}$$

and

$$F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)}\geq\frac{1}{\Gamma(t)}\sum_{k=0}^{\infty}\lambda^k=\frac{1}{(1-\lambda)\Gamma(t)}.$$

□

**Theorem 4.** *The two parameter discrete Mittag - Leffler function has the following properties.*

1. $F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})$ *is monotonically increasing on* $\mathbb{N}_0$.
2. $F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})$ *is slowly oscillating on* $\mathbb{N}_0$.
3. $F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})$ *is not a periodic function.*
4. $F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})$ *is* $S$ *- asymptotically periodic function on* $\mathbb{N}_{0,b}$.

*Proof.* Let $t,s\in\mathbb{N}_0$ such that $t>s$. Then $t-s=T\in\mathbb{Z}^+$. Consider

$$\begin{aligned}
&F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}})-F_{\alpha,\beta}(\lambda,s^{\overline{\alpha}})\\
&=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\left[t^{\overline{\alpha k}}-s^{\overline{\alpha k}}\right]\\
&=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\Big[\frac{\Gamma(t+\alpha k)}{\Gamma(t)}-\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big]\\
&=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\Big[\frac{\Gamma(s+T+\alpha k)}{\Gamma(s+T)}-\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big]\\
&=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big[\Big(\frac{s+T-1+k\alpha}{s+T-1}\Big)\Big(\frac{s+T-2+k\alpha}{s+T-2}\Big)...\Big(\frac{s+k\alpha}{s}\Big)-1\Big]
\end{aligned}$$

(2.3)

$$\begin{aligned}
&=\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big[\Big(1+\frac{k\alpha}{s+T-1}\Big)\Big(1+\frac{k\alpha}{s+T-2}\Big)...\Big(1+\frac{k\alpha}{s}\Big)-1\Big]\\
&>0.
\end{aligned}$$

Thus, we have

$$s<t\Rightarrow F_{\alpha,\beta}(\lambda,s^{\overline{\alpha}})<F_{\alpha,\beta}(\lambda,t^{\overline{\alpha}}).$$

Further, letting $s \to \infty$ in (2.3), we get

$$F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) - F_{\alpha,\beta}(\lambda, s^{\overline{\alpha}}) \to 0.$$

Hence we have (1) and (2). Let $T$ be any positive integer and consider

$$\begin{aligned} F_{\alpha,\beta}(\lambda, (t+T)^{\overline{\alpha}}) &= \sum_{k=0}^{\infty} \lambda^k \frac{(t+T)^{\overline{\alpha k}}}{\Gamma(\alpha k + \beta)} \\ &= \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(t+T+\alpha k)}{\Gamma(t+T)\Gamma(\alpha k + \beta)} \\ &\neq \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(t+\alpha k)}{\Gamma(t)\Gamma(\alpha k + \beta)} = F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}). \end{aligned}$$

Thus, $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}})$ is not a T - periodic function. Letting $s \to \infty$ in (2.3), we get

$$F_{\alpha,\beta}(\lambda, (s+T)^{\overline{\alpha}}) - F_{\alpha,\beta}(\lambda, s^{\overline{\alpha}}) \to 0. \tag{2.4}$$

Since $F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}})$ is continuous and bounded on $\mathbb{N}_{0,b}$, the proof of (4) is complete. □

**Lemma 5.** *Let $\alpha$, $\beta$ and $\gamma \in \mathbb{R}^+$. The following are valid.*

(1) $\nabla F_\alpha(\lambda, t) = \lambda t^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t+\alpha-1)^{\overline{\alpha}}), \quad t \in \mathbb{N}_0.$
(2) $\nabla \Big[ t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = t^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda, (t+\beta-1)^{\overline{\alpha}}), \quad t \in \mathbb{N}_0.$
(3) $\nabla_0^{-\beta} F_\alpha(\lambda, t) = t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda, (t+\beta)^{\overline{\alpha}}), \quad t \in \mathbb{N}_0.$
(4) $\nabla_0^{-\gamma} \Big[ t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = t^{\overline{\beta+\gamma}} F_{\alpha,\beta+\gamma+1}(\lambda, (t+\beta+\gamma)^{\overline{\alpha}}), \quad t \in \mathbb{N}_0.$
(5) $\nabla_{-1}^{-\gamma} \Big[ (t+1)^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = (t+1)^{\overline{\beta+\gamma-1}} F_{\alpha,\beta+\gamma}(\lambda, (t+\beta+\gamma)^{\overline{\alpha}}),$ $t \in \mathbb{N}_0.$
(6) $\nabla_{0*}^{\beta} F_\alpha(\lambda, t) = \lambda t^{\overline{\alpha-\beta}} F_{\alpha,1}(\lambda, (t+\alpha-\beta)^{\overline{\alpha}}), \quad 0 < \beta < 1, \quad t \in \mathbb{N}_1.$
(7) $\nabla_0^{\gamma} \Big[ t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = t^{\overline{\beta-\gamma}} F_{\alpha,\beta-\gamma+1}(\lambda, (t+\beta-\gamma)^{\overline{\alpha}}), \quad t \in \mathbb{N}_N.$
(8) $\nabla_{-1}^{\gamma} \Big[ (t+1)^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = (t+1)^{\overline{\beta-\gamma-1}} F_{\alpha,\beta-\gamma}(\lambda, (t+\beta-\gamma)^{\overline{\alpha}}),$ $\beta \neq \gamma, \quad t \in \mathbb{N}_{N-1}.$
(9) $\nabla_{-1}^{\beta} \Big[ (t+1)^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda, (t+\beta)^{\overline{\alpha}}) \Big] = \lambda (t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t+\alpha)^{\overline{\alpha}}), \quad t \in \mathbb{N}_{N-1}.$

*Proof.* Consider (1).

$$\begin{aligned}
\nabla F_\alpha(\lambda,t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \nabla t^{\overline{\alpha k}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \Big[\frac{\Gamma(t+\alpha k)}{\Gamma(t)} - \frac{\Gamma(t-1+\alpha k)}{\Gamma(t-1)}\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \frac{\Gamma(t-1+\alpha k)}{\Gamma(t-1)} \Big[\frac{t-1+\alpha k}{t-1} - 1\Big] \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k)} \frac{\Gamma(t-1+\alpha k)}{\Gamma(t)} \\
&= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \frac{\Gamma(t-1+\alpha k+\alpha)}{\Gamma(t)} \\
&= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} t^{\overline{\alpha k+\alpha-1}}
\end{aligned}$$

$$\begin{aligned}
&= \lambda t^{\overline{\alpha-1}} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} (t+\alpha-1)^{\overline{\alpha k}} \\
&= \lambda t^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda,(t+\alpha-1)^{\overline{\alpha}}).
\end{aligned}$$

Consider (2).

$$\begin{aligned}
\nabla\Big[t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda,(t+\beta)^{\overline{\alpha}})\Big] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \nabla[t^{\overline{\beta}}(t+\beta)^{\overline{\alpha k}}] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \nabla t^{\overline{\alpha k+\beta}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \Big[\frac{\Gamma(t+\alpha k+\beta)}{\Gamma(t)} - \frac{\Gamma(t-1+\alpha k+\beta)}{\Gamma(t-1)}\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \frac{\Gamma(t-1+\alpha k+\beta)}{\Gamma(t-1)} \Big[\frac{t-1+\alpha k+\beta}{t-1} - 1\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} \frac{\Gamma(t-1+\alpha k+\beta)}{\Gamma(t)} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} t^{\overline{\alpha k+\beta-1}} \\
&= t^{\overline{\beta-1}} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} (t+\beta-1)^{\overline{\alpha k}} \\
&= t^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda,(t+\beta-1)^{\overline{\alpha}}).
\end{aligned}$$

Consider (3).

$$\begin{aligned}
\nabla_0^{-\beta} F_\alpha(\lambda,t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \nabla_0^{-\beta} t^{\overline{\alpha k}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)} t^{\overline{\beta+\alpha k}} \quad \text{(using Power Rule)} \\
&= t^{\overline{\beta}} \sum_{k=0}^{\infty} \lambda^k \frac{(t+\beta)^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta+1)} \quad \text{(using Lemma 1(1))} \\
&= t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda,(t+\beta)^{\overline{\alpha}}).
\end{aligned}$$

Consider (4).

$$\begin{aligned}
\nabla_0^{-\gamma}\Big[t^{\overline{\beta}} F_{\alpha,\beta+1}(\lambda,(t+\beta)^{\overline{\alpha}})\Big] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \nabla_0^{-\gamma}[t^{\overline{\beta}}(t+\beta)^{\overline{\alpha k}}] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \nabla_0^{-\gamma} t^{\overline{\alpha k+\beta}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta+1)} \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+\beta+\gamma+1)} t^{\overline{\alpha k+\beta+\gamma}}
\end{aligned}$$

$$\begin{aligned}
&= t^{\overline{\beta+\gamma}} \sum_{k=0}^{\infty} \lambda^k \frac{(t+\beta+\gamma)^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta+\gamma+1)} \quad \text{(using Lemma 1(1))} \\
&= t^{\overline{\beta+\gamma}} F_{\alpha,\beta+\gamma+1}(\lambda,(t+\beta+\gamma)^{\overline{\alpha}}).
\end{aligned}$$

Consider (5).

$$\begin{aligned}
\nabla_{-1}^{-\gamma}\Big[(t+1)^{\overline{\beta-1}} F_{\alpha,\beta}(\lambda,(t+\beta)^{\overline{\alpha}})\Big] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} \nabla_{-1}^{-\gamma}[(t+1)^{\overline{\beta-1}}(t+\beta)^{\overline{\alpha k}}] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} \nabla_{-1}^{-\gamma}(t+1)^{\overline{\alpha k+\beta-1}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\beta)} \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\gamma)} (t+1)^{\overline{\alpha k+\beta+\gamma-1}} \\
&= (t+1)^{\overline{\beta+\gamma-1}} \sum_{k=0}^{\infty} \lambda^k \frac{(t+\beta+\gamma)^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta+\gamma)} \\
&= (t+1)^{\overline{\beta+\gamma-1}} F_{\alpha,\beta+\gamma}(\lambda,(t+\beta+\gamma)^{\overline{\alpha}}).
\end{aligned}$$

Consider (6).

$$\begin{aligned}
\nabla_{0*}^{\beta} F_\alpha(\lambda,t) &= \nabla_0^{-(1-\beta)}\Big[\nabla F_\alpha(\lambda,t)\Big] \quad \text{(using Definition 3(4))} \\
&= \nabla_0^{-(1-\beta)}\Big[\lambda t^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda,(t+\alpha-1)^{\overline{\alpha}})\Big] \quad \text{(using (1))} \\
&= \lambda t^{\overline{\alpha+1-\beta-1}} F_{\alpha,\beta+1-\beta}(\lambda,(t+\alpha+1-\beta-1)^{\overline{\alpha}}) \quad \text{(using (4))} \\
&= \lambda t^{\overline{\alpha-\beta}} F_{\alpha,1}(\lambda,(t+\alpha-\beta)^{\overline{\alpha}}).
\end{aligned}$$

(7) and (8) are obtained by replacing $\gamma$ by $-\gamma$ in (4) and (5) respectively. Consider (9).

$$\begin{aligned}
\nabla_{-1}^{\beta}\Big[(t+1)^{\overline{\beta-1}}F_{\alpha,\beta}(\lambda,(t+\beta)^{\overline{\alpha}})\Big] &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\nabla_{-1}^{\beta}[(t+1)^{\overline{\beta-1}}(t+\beta)^{\overline{\alpha k}}] \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\nabla_{-1}^{\beta}(t+1)^{\overline{\alpha k+\beta-1}} \\
&= \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k)}(t+1)^{\overline{\alpha k-1}} \\
&= \lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}(t+1)^{\overline{\alpha k+\alpha-1}} \\
&= \lambda(t+1)^{\overline{\alpha-1}}\sum_{k=0}^{\infty}\lambda^k\frac{(t+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)} \text{ (using Lemma 1(1))} \\
&= \lambda(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}}).
\end{aligned}$$

□

**Remark 1.** *From Lemma 5(6), we have*

$$\nabla_{0*}^{\alpha}F_{\alpha}(\lambda,t)=\lambda F_{\alpha}(\lambda,t), \quad 0<\alpha<1, \quad t\in\mathbb{N}_1, \tag{2.5}$$

*implies $F_\alpha(\lambda,t)$ is an eigenfunction of the operator $\nabla_{0*}^{\alpha}$. In other words, $F_\alpha(\lambda,t)$ is a nontrivial solution of the fractional nabla difference equation $\nabla_{0*}^{\alpha}u(t)=\lambda u(t), \quad t\in\mathbb{N}_1$.*

**Remark 2.** *From Lemma 5(9), we have*

$$\nabla_{-1}^{\alpha}\Big[(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})\Big]=\lambda(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}}), \quad t\in\mathbb{N}_1,$$

*implies $(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})$ is an eigenfunction of the operator $\nabla_{-1}^{\alpha}$. That is, $(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})$ is the solution of the Riemann - Liouville type fractional nabla difference equation $\nabla_{-1}^{\alpha}f(t)=\lambda f(t), \quad t\in\mathbb{N}_1$.*

Now, we prove that $(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})$ is also slowly oscillating on $\mathbb{N}_0$ and $S$ - asymptotically periodic on $\mathbb{N}_{0,b}$. For this purpose, let $t,s\in\mathbb{N}_0$ such that $t>s$. Then $t-s=T\in\mathbb{Z}^+$. Now consider

$$
\begin{aligned}
u(t)-u(s) &= (t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})-(s+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})\\
&= (t+1)^{\overline{\alpha-1}}\sum_{k=0}^{\infty}\lambda^k\frac{(t+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)}-(s+1)^{\overline{\alpha-1}}\sum_{k=0}^{\infty}\lambda^k\frac{(s+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)}\\
&= \sum_{k=0}^{\infty}\lambda^k\frac{(t+1)^{\overline{\alpha k+\alpha-1}}}{\Gamma(\alpha k+\alpha)}-\sum_{k=0}^{\infty}\lambda^k\frac{(s+1)^{\overline{\alpha k+\alpha-1}}}{\Gamma(\alpha k+\alpha)}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\Big[(t+1)^{\overline{\alpha k+\alpha-1}}-(s+1)^{\overline{\alpha k+\alpha-1}}\Big]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\Big[\frac{\Gamma(t+\alpha k+\alpha)}{\Gamma(t+1)}-\frac{\Gamma(s+\alpha k+\alpha)}{\Gamma(s+1)}\Big]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\Big[\frac{\Gamma(s+T+\alpha k+\alpha)}{\Gamma(s+T+1)}-\frac{\Gamma(s+\alpha k+\alpha)}{\Gamma(s+1)}\Big]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\frac{\Gamma(s+\alpha k+\alpha)}{\Gamma(s+1)}\Big[\Big(\frac{s+T-1+\alpha k+\alpha}{s+T}\Big)...\Big(\frac{s+\alpha k+\alpha}{s+1}\Big)-1\Big]\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\frac{\Gamma(s+\alpha k+\alpha)}{\Gamma(s+1)}\Big[\Big(1+\frac{\alpha k+\alpha-1}{s+T}\Big)...\Big(1+\frac{\alpha k+\alpha-1}{s+1}\Big)-1\Big].
\end{aligned}
\tag{2.6}
$$

Letting $s\to\infty$ in (2.6), we get $u(t)-u(s)\to 0$, i.e., $u(s+T)-u(s)\to 0$. Further, $(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})$ is continuous and bounded on $\mathbb{N}_{0,b}$. Hence the proof.

Finally, we show that $(t+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+\alpha)^{\overline{\alpha}})$ is also not a $T$ - periodic function. Let $T$ be any positive integer and consider

$$
\begin{aligned}
u(t+T) &= (t+T+1)^{\overline{\alpha-1}}F_{\alpha,\alpha}(\lambda,(t+T+\alpha)^{\overline{\alpha}})\\
&= (t+T+1)^{\overline{\alpha-1}}\sum_{k=0}^{\infty}\lambda^k\frac{(t+T+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)}\\
&= \sum_{k=0}^{\infty}\lambda^k\frac{(t+T+1)^{\overline{\alpha k+\alpha-1}}}{\Gamma(\alpha k+\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(t+T+\alpha k+\alpha)}{\Gamma(t+T+1)\Gamma(\alpha k+\alpha)} \\
&\neq \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(t+\alpha k+\alpha)}{\Gamma(t+1)\Gamma(\alpha k+\alpha)} \\
&= \sum_{k=0}^{\infty} \lambda^k \frac{(t+1)^{\overline{\alpha k+\alpha-1}}}{\Gamma(\alpha k+\alpha)} \\
&= (t+1)^{\overline{\alpha-1}} \sum_{k=0}^{\infty} \lambda^k \frac{(t+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)} \\
&= (t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t+\alpha)^{\overline{\alpha}}) = u(t).
\end{aligned}
$$

## 3. Convergence & Oscillation

In the present section we establish sufficient conditions on convergence and divergence of the infinite series

$$
\sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)} \tag{3.1}
$$

associated with discrete Mittag - Leffler function. The following theorem discusses the convergence of (3.1) using D'Alembert's ratio test.

**Theorem 6.** *The infinite series (3.1) converges absolutely for each $t \in \mathbb{N}_a$, $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ such that $|\lambda| < 1$.*

*Proof.* Consider (3.1). Here

$$
a_k = \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)} = \lambda^k \frac{\Gamma(t+\alpha k)}{\Gamma(t)\Gamma(\alpha k+\beta)} = \lambda^k \frac{1}{\Gamma(t)} \frac{\Gamma(\alpha k+t)}{\Gamma(\alpha k+\beta)}.
$$

Then

$$
\frac{a_{k+1}}{a_k} = \lambda \frac{\Gamma(\alpha k+t+\alpha)}{\Gamma(\alpha k+t)} \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\alpha)}.
$$

As $k \to \infty$,

$$
\left| \frac{a_{k+1}}{a_k} \right| = |\lambda| (\alpha k)^{(t+\alpha-t)} (\alpha k)^{(\beta-(\beta+\alpha))} = |\lambda|.
$$

Using D'Alembert's Ratio test, the proof is complete. ☐

**Remark 3.** *Since absolute convergence implies convergence, the infinite series (3.1) converges for $|\lambda| < 1$ and diverges for $|\lambda| \geq 1$.*

Divergent series are often classified further into properly divergent, oscillate finitely and oscillate infinitely series [18]. In the following theorems we discuss these properties for (3.1).

**Theorem 7.** *The infinite series (3.1) diverges properly for each $t \in \mathbb{N}_a$, $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ such that $t \geq \beta \geq 1$ and $\lambda \geq 1$.*

*Proof.* Consider the infinite series (3.1) with $\lambda \geq 1$. Here

$$
a_k = \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)} = \lambda^k \frac{\Gamma(t+\alpha k)}{\Gamma(t)\Gamma(\alpha k+\beta)}.
$$

Let $s_n$ be the $n$-th partial sum of the series (3.1). Then

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \lambda^k \frac{\Gamma(t+\alpha k)}{\Gamma(t)\Gamma(\alpha k+\beta)}.$$

Since $\lambda \geq 1$ and $\frac{\Gamma(t+\alpha k)}{\Gamma(\alpha k+\beta)} \geq 1$ for $t \geq \beta \geq 1$, we have

$$s_n \geq \sum_{k=0}^{n} \frac{1}{\Gamma(t)} = \frac{(n+1)}{\Gamma(t)} \text{ and hence } \lim_{n\to\infty} s_n = +\infty.$$

Thus, the infinite series (3.1) diverges properly for $t \geq \beta \geq 1$ and $\lambda \geq 1$. □

Let $\lambda \in \mathbb{R}^+$. Replacing $\lambda$ by $-\lambda$ in (3.1), we get an alternating series of the form

$$\sum_{k=1}^{\infty} (-1)^{k-1} \lambda^{k-1} \frac{t^{\overline{\alpha k-\alpha}}}{\Gamma(\alpha k-\alpha+\beta)}. \tag{3.2}$$

A general criterion for infinite oscillation and a different version of Leibnitz test for alternating series are given in Theorems 8 and 9 respectively.

**Theorem 8.** [4] *If $A = \sum_{k=1}^{\infty} (-1)^{k-1} a_k$, where $a_k > 0$ and $\frac{a_{k+1}}{a_k} \to \lambda > 1$ as $k \to \infty$, then $A$ oscillates infinitely.*

**Theorem 9.** [3] *Given an alternating series $A = \sum_{k=1}^{\infty} (-1)^{k-1} a_k$, $a_k > 0$, if $\frac{a_k}{a_{k+1}}$ can be expressed in the form*

$$\frac{a_k}{a_{k+1}} = 1 + \frac{\mu}{k} + O\Big(\frac{1}{k^p}\Big), \quad p > 1 \tag{3.3}$$

*then $A$ is convergent if $\mu > 0$, oscillatory $\mu \leq 0$.*

Using Theorems 8 and 9, we now discuss the oscillatory behaviour of (3.2).

**Theorem 10.** *The alternating series (3.2) oscillates infinitely for each $t \in \mathbb{N}_a$, $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ such that $\lambda > 1$.*

*Proof.* Consider (3.2). Here

$$a_k = \lambda^{k-1} \frac{t^{\overline{\alpha k-\alpha}}}{\Gamma(\alpha k-\alpha+\beta)} = \lambda^{k-1} \frac{\Gamma(t+\alpha k-\alpha)}{\Gamma(t)\Gamma(\alpha k-\alpha+\beta)} = \lambda^{k-1} \frac{1}{\Gamma(t)} \frac{\Gamma(\alpha k+t-\alpha)}{\Gamma(\alpha k+\beta-\alpha)} > 0.$$

Then

$$\frac{a_{k+1}}{a_k} = \lambda \frac{\Gamma(\alpha k+t)}{\Gamma(\alpha k+t-\alpha)} \frac{\Gamma(\alpha k+\beta-\alpha)}{\Gamma(\alpha k+\beta)}.$$

As $k \to \infty$,

$$\frac{a_{k+1}}{a_k} = \lambda (\alpha k)^{(t-(t-\alpha))} (\alpha k)^{((\beta-\alpha)-\beta)} = \lambda.$$

Using Theorem 8 the proof is complete. □

**Theorem 11.** *The alternating series (3.2) oscillates finitely for each $t \in \mathbb{N}_a$ and $\alpha, \beta \in \mathbb{R}^+$.*

*Proof.* Consider (3.2) with $\lambda = 1$. Here

$$a_k = \frac{t^{\overline{\alpha k-\alpha}}}{\Gamma(\alpha k-\alpha+\beta)} = \frac{\Gamma(t+\alpha k-\alpha)}{\Gamma(t)\Gamma(\alpha k-\alpha+\beta)} = \frac{1}{\Gamma(t)} \frac{\Gamma(\alpha k+t-\alpha)}{\Gamma(\alpha k+\beta-\alpha)} > 0.$$

Then
$$\frac{a_k}{a_{k+1}} = \frac{\Gamma(\alpha k + t - \alpha)}{\Gamma(\alpha k + t)} \frac{\Gamma(\alpha k + \beta)}{\Gamma(\alpha k + \beta - \alpha)}.$$
For large $k$,
$$\frac{a_k}{a_{k+1}} = (\alpha k)^{((t-\alpha)-t)}\Big[1 + O\Big(\frac{1}{\alpha k}\Big)\Big](\alpha k)^{(\beta-(\beta-\alpha))}\Big[1 + O\Big(\frac{1}{\alpha k}\Big)\Big] = 1 + O\Big(\frac{1}{k^2}\Big).$$
Using Theorem 9 the proof is complete. □

Combining all these results, we have

**Corollary 1.** *Let $t \in \mathbb{N}_a$, $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$. The infinite series (3.1)*
$$\begin{cases} converge, & for\ -1 < \lambda < 1; \\ diverge\ properly, & for\ \lambda \geq 1\ and\ t \geq \beta \geq 1; \\ oscillate\ finitely, & for\ \lambda = -1; \\ oscillate\ infinitely, & for\ \lambda < -1. \end{cases}$$

Since the radius of convergence of (3.1) is 1, we have the following result on the uniform convergence of (3.1).

**Corollary 2.** *Let $t \in \mathbb{N}_a$, $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$. For any $0 < r < 1$, the infinite series (3.1) converges uniformly for each $\lambda \in [-r, r]$.*

## 4. Preliminaries on Summability

The present section contains some basic definitions and results concerning summability theory [11, 20, 3] which will be useful in section 4.

The method of convergent series is simply a particular method of associating a definite number called *sum* denoted by $s$ with the series and using this number in place of the convergent series in calculations. But for the divergent series, this *sum* does not exist. The problem of divergent series is to associate a number with such a series called *Sum* denoted by $S$ so that it can be used in place of the divergent series in calculations. Any definite method by which we can associate a *Sum* with a given divergent series is called the method of summation. The methods of summation are designed primarily for the oscillating series.

We now discuss three important summability methods given by Abel, Borel and Le Roy [3, 11, 20]. Let $k \in \mathbb{N}_0$ and $a_k, \lambda \in \mathbb{C}$. Consider a divergent series
$$\sum_{k=0}^{\infty} a_k. \tag{4.1}$$

**Abel's Method:** [3, 11, 20] The series (4.1) is said to be Abel - summable (A - summable), if the series
$$\sum_{k=0}^{\infty} a_k \lambda^k \tag{4.2}$$
converges in the disk $D = \{\lambda : |\lambda| < 1\}$ and
$$\lim_{\lambda \to 1-} \sum_{k=0}^{\infty} a_k \lambda^k = S. \tag{4.3}$$

Then $S$ is called A-sum of the series (4.1) and is denoted by

$$\sum_{k=0}^{\infty} a_k = S \quad (A). \tag{4.4}$$

**Borel's Method:** [3, 11, 20] The series (4.1) is said to be Borel - summable (B - summable), if the series

$$e^{-\lambda} \sum a_k \frac{\lambda^k}{k!} \tag{4.5}$$

converges to $S$ and

$$\int_0^{\infty} e^{-\lambda} \sum_{k=0}^{\infty} a_k \frac{\lambda^k}{k!} d\lambda = \lim_{\lambda\to\infty} \int_0^{\lambda} e^{-\lambda} \sum_{k=0}^{\infty} a_k \frac{\lambda^k}{k!} d\lambda = S. \tag{4.6}$$

The complex number $S$ is called B-sum of the series (4.1) and is denoted by

$$\sum_{k=0}^{\infty} a_k = S \quad (B). \tag{4.7}$$

**Le Roy's Method:** [3, 11, 20] The series (4.1) is said to be Le Roy - summable (L - summable), if

$$\lim_{\alpha\to 1} \lim_{n\to\infty} \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha k+1)}{\Gamma(k+1)} = S, \quad 0 < \alpha < 1. \tag{4.8}$$

Here $S$ is called L-sum of the series (4.1) and is denoted by

$$\sum_{k=0}^{\infty} a_k = S \quad (L). \tag{4.9}$$

Finally we conclude this section with two important theorems on A and L summabilities.

**Theorem 12.** [3, 11, 20] *Every convergent series is summable (A, L) with Sum equal to sum i.e. Abel and Le Roy methods are regular.*

**Theorem 13.** [3, 11, 20] *A properly divergent series is not summable (A, L) with finite Sum.*

## 5. Summability of the Infinite Series (3.1)

In this section we discuss summability of (3.1) using the results obtained in section 3 and preliminaries described in section 4. The following corollary is a consequence of Theorems 12, 13 and Corollary 1.

**Corollary 3.** *For each $t \in \mathbb{N}_a$ and $\alpha, \beta \in \mathbb{R}^+$, the infinite series (3.1) is*

1. *(A, L)- summable for $-1 < \lambda < 1$.*
2. *not (A, L) - summable with finite Sum for $\lambda \geq 1$.*
3. *uniformly (A, L) - summable for each $\lambda \in [-r, r]$ such that $0 < r < 1$.*

**Theorem 14.** *For each $t \in \mathbb{N}_a$ and $\alpha, \beta \in \mathbb{R}^+$, the infinite series (3.1) is A - summable for $\lambda = -1$.*

*Proof.* From Corollary 1 we know that

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)} \tag{5.1}$$

oscillates finitely. Also,

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)}\lambda^k \tag{5.2}$$

converges absolutely in the disk $D = \{\lambda : |\lambda| < 1\}$ and

$$\lim_{\lambda\to 1-}\sum_{k=0}^{\infty}(-1)^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)}\lambda^k \tag{5.3}$$

exists and is finite. So, the infinite series (3.1) is A - summable with finite *Sum* for $\lambda = -1$. □

**Theorem 15.** *For each $t \in \mathbb{N}_a$, $0 < \alpha < 1$ and $\beta = 1$, the infinite series (3.1) is L - summable for $\lambda < 1$.*

*Proof.* Consider the alternating series

$$\sum_{k=0}^{\infty}(-1)^k(-\lambda)^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+\beta)} \tag{5.4}$$

with $\lambda \geq 1$. Clearly it oscillates finitely for $\lambda = 1$ and oscillate infinitely for $\lambda > 1$. Here

$$a_k = (-\lambda)^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k+1)}.$$

From (4.8), we have

$$S = \lim_{\alpha\to 1}\lim_{n\to\infty}\sum_{k=0}^{n} a_k\frac{\Gamma(\alpha k+1)}{\Gamma(k+1)} = \lim_{\alpha\to 1}\lim_{n\to\infty}\sum_{k=0}^{n}\binom{t+\alpha k-1}{k}(-\lambda)^k = \Big(\frac{1}{1+\lambda}\Big)^t$$

exists for each $\lambda > -1$. Thus (3.1) is L - summable for $\lambda < 1$. Hence the proof. □

## 6. Conclusion

Taking $\alpha = \beta = 1$ in (3.1), we get

$$\sum_{k=0}^{\infty}\lambda^k\frac{\Gamma(t+k)}{\Gamma(t)\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty}\binom{t+k-1}{k}\lambda^k = \sum_{k=0}^{\infty}\binom{-t}{k}\lambda^k. \tag{6.1}$$

We know that,

**Theorem 16.** *For each $t \in \mathbb{N}_a$, the infinite series (6.1)*

(1)
$$\begin{cases} \textit{converges to } (1-\lambda)^{-t}, & \textit{for } -1 < \lambda < 1;\\ \textit{diverges properly}, & \textit{for } \lambda \geq 1;\\ \textit{oscillates finitely}, & \textit{for } \lambda = -1;\\ \textit{oscillates infinitely}, & \textit{for } \lambda < -1.\end{cases}$$

(2) *is A - summable for $-1 \leq \lambda < 1$ and B - summable for $\lambda < 1$*

(3) *is not (A, B) - summable with finite Sum for $\lambda \geq 1$*

(4) *is uniformly (A, B) - summable for each $\lambda \in [-r, r]$ such that $0 < r < 1$.*

Theorem 16 gives a proper justification to Corollaries 1, 2, 3, Theorems 14 and 15 for $\alpha = \beta = 1$. Here we note that Le Roy's definition coincides with Borel's, whenever the later is convergent [3, 11, 20]. So one can replace B - Summability by L - Summability in (2), (3) and (4) of Theorem 16.

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