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Third Hankel Determinant and Zalcman Functional for Sakaguchi Type Starlike Functions Involving q-Derivative Operator Related with Sine Function

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Abstract. The purpose of this paper is to consider coefficient estimates for q-starlike function with respect to symmetric points associated with sine function $\mathcal{SS}_q^*(1+sin(z))$ consisting of analytic functions f normalized by $f(0)=f'(0)-1=0$ in the open unit disk $\mathcal{U}_d=\{z : z\in\mathbb{C} \quad \text{and} \quad |z|<1\}$ satisfying the condition $\frac{2[zD_qf(z)]}{f(z)-f(-z)} \prec 1+sin(z)=\psi(z),$ for all $z\in\mathcal{U}_d$ to derive certain coefficient estimates b_2, b_3 etc and Fekete-Szegö inequality for $f\in\mathcal{SS}_q^*(1+sin(z))$. Further to investigate the possible upper bound of third order Hankel determinant and also the Zalcman functional for $f\in\mathcal{SS}_q^*(1+sin(z))$.

1. Introduction

In geometric function theory, analytical classes are crucial for comprehending and characterizing the geometric features of functions, especially in complex analysis. The mathematical discipline of geometric function theory examines functions from a geometric perspective. Intervening analytical functions from a geometric perspective are the primary subject of the privileged mathematical field known as geometric function theory. Within the field of mathematical analysis, the study of analytical functions is very desirable. Coefficient estimates in Geometric Function Theory (GFT) are bounds on the coefficients of power series representations of analytic functions, especially within certain subclasses of univalent (one-to-one) functions. The geometric characteristics of these functions are shown by these estimates.Applications for coefficient estimates can be found in fluid dynamics, potential theory, and other branches of physics and mathematics. Recent days

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geometric function theory is advancing in image processsing techniques and has opened new avenues for interdisciplinary research in mathematical imaging.

Let $\mathcal{B}$ denote the class of functions in the open unit disc $\mathcal{U}_d = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. The class $\mathcal{B}$ has the following Taylor's series expansion

$$f(z) = z + b_2 z^2 + b_3 z^3 + \cdots . \tag{1.1}$$

Let $\mathcal{P}$ denote the class of functions defined by

$$h(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots \tag{1.2}$$

which are univalent and analytic in $\mathcal{U}_d$ is denoted by $\mathcal{S}$ and maps $\mathcal{U}_d$ onto the right half plane.
Let $g, h \in \mathcal{U}_d$. We say that $g \prec h$, if there exists a function ω analytic in $\mathcal{U}_d$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $g(z) = h(\omega(z))$. If the function h is univalent in $\mathcal{U}_d$ then $g \prec h$ if and only if $g(0) = h(0)$ and $g(\mathcal{U}_d) \subseteq h(\mathcal{U}_d)$.
The j^{th} Hankel determinant for $j \geq 1$ and $n \in N$ was defined by Pommerenke [24,26] as follows:

$$H_{j,k}(f) = \begin{vmatrix} b_k & b_{k+1} & b_{k+2} & \cdots & b_{k+j-1} \\ b_{k+1} & b_{k+2} & b_{k+3} & \cdots & b_{k+j} \\ b_{k+2} & b_{k+3} & b_{k+4} & \cdots & b_{k+j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k+j-1} & b_{k+j} & b_{k+j+1} & \cdots & b_{k+2j-2} \end{vmatrix}$$

where $b_1 = 1$. In [23] Inayath determined the rate of growth of $H_{j,k}$ as k$\rightarrow \infty$ for the function $f(z)$ given in (1.1) in $\mathcal{S}$ with bounded boundary. If $j = 2$ and $k = 1$ then $H_{2,1}(f) = \left|b_3 - b_2^2\right|$ and if j=2 and k=2 then $H_{2,2}(f) = \left|b_2 b_4 - b_3^2\right|$. Aditionally

$$\begin{aligned} H_{2,3}(f) &= \begin{vmatrix} b_3 & b_4 \\ b_4 & b_5 \end{vmatrix} \\ &= \left|b_3 b_5 - b_4^2\right|. \end{aligned}$$

The third order Hankel determinant $H_{3,1}(f)$ is given by,

$$H_3(1) = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} = b_3(b_2 b_4 - b_3^2) - b_4(b_4 - b_2 b_3) + b_5(b_3 - b_2^2). \tag{1.3}$$

We note that $H_{2,1}(f)$ is the well known Fekete-szego functional. In recent years many authors have studied the second order hankel determinant $H_{2,2}(f)$ and the third hankel determinant $H_{3,1}(f)$ for various classes of functions and they can be unified by considering a univalent function with a positive real part symmetric about the real axis with respect to starlike functions.
Very recently, Arif et al. [1], investigated upper bounds for third hankel determinant for the class of functions $S_q^*(\psi)$ associated with trignometric sine function. Stimulated by aforementioned work, we determine the upper bounds of the third Hankel for the class $S_q^*(\psi)$ of symmetric points associated with sine function by q-derivative operator.

Definition 1.1. *[5] The q-differential operator introduced by Jackson is defined as*

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ \ z \in \mathcal{U}_d. \tag{1.4}$$

In addition, the q-derivative at zero is $D_q f(0) = D_{q^{-1}} f(0)$ for $|q| > 1$. The q-derivative at zero is defined as $f'(0)$ if it exits. Equivalently (1.4) can be written as

$$D_q f(z) = 1 + \sum_{n=1}^{\infty} [n]_q b_n z^{n-1}, \ \ z \neq 0$$

where,

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}. \tag{1.5}$$

Sakaguchi [30] introduced a class of starlike functions with respect to symmetric points. Obviously the class of univalent functions and starlike with respect to symmetric points include the classes of convex functions and odd functions star like with respect to the origin (see [30]). Inspired by earlier works in [1,14,31–33], we now define a new class of $q-$starlike functions with respect to symmetric points associated with sine function $\mathcal{SS}_q^*(1 + sin(z))$ as in definition 1.2 to derive certain coefficient estimates b_2, b_3, b_4, b_5 and Fekete-Szegö inequality for $f \in \mathcal{SS}^*{}_q(1 + sin(z))$. Further we investigate the possible upper bound of third order Hankel determinant and the Zalcman functional for $f \in \mathcal{SS}_q^*(1 + sin(z))$.

Definition 1.2. *If a function* $f \in \mathcal{SS}_q^*(1 + sin(z))$ *then*

$$\frac{2[zD_q f(z)]}{f(z) - f(-z)} \prec 1 + sin(z) = \psi(z), \textit{ for all } z \in \mathcal{U}_d. \tag{1.6}$$

The lemmas listed below are needed to prove the desired results.

Lemma 1.1. *If* $h \in \mathcal{P}$ *then*

$$|d_n| \leq 2, \ \forall\, n \in \mathbb{N}, \tag{1.7}$$

$$|d_{i+j} - \mu d_i d_j| \leq 2, for \quad 0 \leq \mu \leq 1 \tag{1.8}$$

and for any complex number ξ*, we have*

$$|d_2 - \xi d_1^2| \leq 2max\{1, |2\xi - 1|\} \tag{1.9}$$

where the inequalities (1.7), (1.8) *are taken from [25] and* (1.9) *is obtained in [11].*

Lemma 1.2. *[1] If* $h \in \mathcal{P}$ *has power series* (1.2)*, then*

$$\left|\alpha d_1^3 - \beta d_1 d_2 + \gamma d_3\right| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|.$$

Lemma 1.3. *[28] Let m, n, l and r satisfy the inequalities* $0 < m < 1$, $0 < r < 1$ *and*

$$8r(1-r)\left[(mn-2l)^2+(m(r+m)-n)^2\right]+m(1-m)(n-2rm)^2 \leq 4m^2(1-m^2)r(1-r).$$

If $h \in \mathcal{P}$ *has power series* (1.2), *then*

$$\left|ld_1^4+rd_2^2+2md_1d_3-\frac{3}{2}nd_1^2d_2-d_4\right| \leq 2\ .$$

2. Coefficient estimates and Fekete-Szego inequality

In this section we evaluate the coefficient estimates for the class $\mathcal{SS}_q^*(1+sin(z))$. Further we evaluate Fekete-Szego functional for this class.

Theorem 2.1. *If* $f \in \mathcal{SS}_q^*(1+sin(z))$, *then*

$$|b_2| \leq \frac{1}{1+q},$$
$$|b_3| \leq \frac{1}{q+q^2},$$
$$|b_4| \leq \frac{1}{1+q+q^2+q^3},$$
$$|b_5| \leq \frac{3}{q+q^2+q^3+q^4}.$$

Proof. Since $f(z) \in \mathcal{SS}_q^*(1+sin(z))$, then from the principle of subordination we have,

$$\frac{2[zD_qf(z)]}{f(z)-f(-z)} = \psi(v(z)) = 1+sin(v(z)). \tag{2.1}$$

Since f(z) is of the form (1.1), we have

$$\frac{2[zD_qf(z)]}{f(z)-f(-z)} = 1+(1+q)a_2z+q(1+q)a_3z^2+[(1+q+q^2+q^3)a_4-(1+q)a_2a_3]z^3$$
$$+[q(1+q+q^2+q^3)a_5-q(1+q)a_3^2]z^4+\cdots$$

The function h(z)=$\dfrac{1+v(z)}{1-v(z)} = 1+d_1z+d_2z^2+d_3z^3+\cdots$ is analytic in $\mathcal{U}_d$ with d(0)=1 and maps $\mathcal{U}_d$ onto the right half of the ω-plane.
Computing $v(z)$ in terms of $h(z)$ we get,

$$v(z) = \frac{h(z)-1}{h(z)+1} = \frac{d_1z+d_2z^2+d_3z^3+\cdots}{2+d_1z+d_2z^2+d_3z^3+\cdots}\ .$$

Substituting for $v(z)$ in sine series we get,

$$1+sin(v(z)) = 1+v(z)-\frac{(v(z))^3}{3!}+\frac{(v(z))^5}{5!}-\frac{(v(z))^7}{7!}+\cdots\ .$$

This gives

$$\begin{aligned} 1+sin(v(z)) &= 1+\frac{1}{2}d_1z+\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right)z^2+\left(\frac{d_3}{2}-\frac{d_1d_2}{2}+\frac{5d_1^3}{48}\right)z^3 \\ &+ \left(\frac{-1}{32}d_1^4+\frac{5}{16}d_1^2d_2-\frac{1}{2}d_1d_3-\frac{1}{4}d_2^2+\frac{1}{2}d_4\right)z^4+\cdots . \end{aligned}$$

Substituting for $\frac{2[zD_qf(z)]}{f(z)-f(-z)}$ and $1+sin(v(z))$ in (2.1) and comparing the coefficients we get,

$$b_2 = \frac{d_1}{2(1+q)}, \tag{2.2}$$

$$b_3 = \frac{1}{(q+q^2)}\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right), \tag{2.3}$$

$$\begin{aligned} b_4 &= \frac{d_3}{2(1+q+q^2+q^3)}-\frac{d_1d_2(2(q+q^2)-1)}{4(1+q+q^2+q^3)(q+q^2)} \\ &+ \frac{d_1^3(5(1+q+q^2+q^3)-6)}{48(1+q+q^2+q^3)(q+q^2)}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} b_5 &= \frac{d_1^2d_2(5(q+q^2)-4)}{16(q+q^2)(q+q^2+q^3+q^4)}+\frac{d_2^2(1-(q+q^2))}{4(q+q^2)(q+q^2+q^3+q^4)} \\ &- \frac{d_1d_3}{2(q+q^2+q^3+q^4)}+\frac{d_4}{2(q+q^2+q^3+q^4)} . \end{aligned} \tag{2.5}$$

Now putting (1.7) in (2.2), we obtain

$$|b_2| \leq \frac{1}{1+q} . \tag{2.6}$$

Now using (1.9) with (2.3), we get

$$\begin{aligned} |b_3| &= \frac{1}{2}\left|d_2-\frac{d^2}{2}\right|\frac{1}{(q+q^2)} \\ &\leq max\left\{1,\left|2\left(\frac{1}{2}\right)-1\right|\right\}\frac{1}{(q+q^2)} \\ &\leq max\{1,0\}\frac{1}{(q+q^2)} . \end{aligned}$$

Therefore

$$|b_3| \leq \frac{1}{(q+q^2)} . \tag{2.7}$$

Using triangle inequality and lemma 1.2 in (2.4), gives

$$|b_4| \leq \frac{1}{1+q+q^2+q^3} . \tag{2.8}$$

By rearranging the equation (2.5), we get

$$|b_5| = \left|\frac{1}{2(q+q^2+q^3+q^4)}\left(d_4-\frac{d_2^2(q+q^2-1)}{2(q+q^2)}\right)-\frac{1}{2(q+q^2+q^3+q^4)}d_1\left(d_3-\frac{d_1d_2(5(q+q^2)-4)}{8(q+q^2)}\right)\right| .$$

Using (1.7) and (1.8), we get

$$|b_5| \quad \leq \frac{3}{(q+q^2+q^3+q^4)} \;. \tag{2.9}$$

□

Theorem 2.2. *If f(z) is of the form* (1.1) *and belongs to* $\mathcal{SS}_q^*(1+sin(z))$*, then for any complex number* μ

$$|b_3-\mu b_2^2| \leq \frac{1}{(q+q^2)} max\left\{1, \left|\frac{\mu(q+q^2)}{(1+q)^2}\right|\right\}.$$

Proof. Using (2.2) and (2.3) we get

$$\left|b_3-\mu b_2^2\right| = \left|\frac{d_2}{2(q+q^2)} - \frac{d_1^2}{4(q+q^2)} - \mu\left(\frac{d_1}{2(1+q)}\right)^2\right| \;.$$

This gives

$$\left|b_3-\mu b_2^2\right| = \frac{1}{2(q+q^2)}\left|d_2 - d_1^2\left(\frac{1}{2}+\frac{\mu(q+q^2)}{2(1+q)^2}\right)\right| \;.$$

Application of (1.9), leads us to

$$\left|b_3-\mu b_2^2\right| \leq \frac{1}{(q+q^2)} max\left\{1, \left|\frac{\mu(q+q^2)}{(1+q)^2}\right|\right\}.$$

□

Corollary 2.1. *If f(z)* $\in \mathcal{SS}_q^*(1+sin(z))$ *and* $\mu = 1$ *then,*

$$|b_3-b_2^2| \leq \frac{1}{q+q^2} \;. \tag{2.10}$$

3. Hankel determinants

Now we obtain other important results on the basis of which we will evaluate the third Hankel detrminant for this class.

Theorem 3.1. *If* $f \in \mathcal{SS}_q^*(1+sin(z))$*, then*

$$|b_2b_3-b_4| \leq \frac{1}{1+q+q^2+q^3} \;.$$

Proof. From equation (2.2), (2.3) and (2.4), we have

$$\begin{aligned}|b_2b_3-b_4| = \Bigg|&-\frac{d_1^3}{8(q+q^2)}\left(\frac{1}{1+q}+\frac{1}{(1+q+q^2+q^3)}\left(\frac{5(1+q+q^2)-6}{6}\right)\right) \\ &+\frac{d_1d_2}{4(q+q^2)}\left(\frac{1}{1+q}+\frac{(q+q^2-1)}{(1+q+q^2+q^3)}\right) - \frac{d_3}{2(1+q+q^2+q^3)}\Bigg|\end{aligned}$$

$$|b_2b_3 - b_4| = \left|\frac{d_1^3}{8(q+q^2)}\left(\frac{1}{1+q} + \frac{1}{(1+q+q^2+q^3)}\left(\frac{5(1+q+q^2)-6}{6}\right)\right)\right.$$
$$\left. - \frac{d_1 d_2}{4(q+q^2)}\left(\frac{1}{1+q} + \frac{(q+q^2-1)}{(1+q+q^2+q^3)}\right) + \frac{d_3}{2(1+q+q^2+q^3)}\right| .$$

By triangle inequality and Lemma 1.2, we get

$$|b_2b_3 - b_4| \leq \frac{1}{1+q+q^2+q^3} . \tag{3.1}$$

□

Theorem 3.2. *If* $f \in \mathcal{SS}_q^*(1+sin(z))$, *then* $|b_2b_4 - b_3^2| \leq \dfrac{8+17q+14q^2+10q^3+5q^4}{(1+q)(1+q+q^2+q^3)(q+q^2)^2}$.

Proof. Using (2.2), (2.3), (2.4) we get

$$\left|b_2b_4 - b_3^2\right| = \left|\frac{d_1 d_3}{4(1+q)(1+q+q^2+q^3)} + \frac{2+5q+3q^2}{8(1+q)(1+q+q^2+q^3))(q+q^2)^2} d_1^2 d_2\right.$$
$$\left. - \frac{d_1^4}{96(1+q)(1+q+q^2+q^3)(q+q^2)^2}\left(6+18q+13q^2+2q^3+q^4\right) - \frac{d_2^2}{4(q+q^2)^2}\right| .$$

Using (1.7), (1.8) and (1.9) we get

$$|b_2b_4 - b_3^2| \leq \frac{8+17q+14q^2+10q^3+5q^4}{(1+q)(1+q+q^2+q^3)(q+q^2)^2} . \tag{3.2}$$

□

Theorem 3.3. *Let f(z) is of the form* (1.1) *and if* $f \in \mathcal{SS}_q^*(1+sin(z))$, *then*
$$\left|H_{3,1}(f)\right| \leq \frac{q^4+2q^3+6q^2+9q+4}{(1+q+q^2+q^3)(1+q)(q+q^2)^3} + \frac{1}{(1+q+q^2+q^3)^2} + \frac{3}{(q+q^2+q^3+q^4)(q+q^2)} .$$

Proof. Third Hankel determinant from equation (1.3) can be written as,

$H_{3,1}(f) = \ b_3(b_2b_4 - b_3^2) - b_4(b_4 - b_2b_3) + b_5(b_3 - b_2^2)$, where a_1=1. This provides that

$$\left|H_{3,1}(f)\right| \leq \ |b_3|\left|(b_2b_4 - b_3^2)\right| + |b_4|\left|(b_4 - b_2b_3)\right| + |b_5|\left|(b_3 - b_2^2)\right| .$$

By implementing (2.7), (2.8), (2.9), (2.10), (3.1) and (3.2) we get our desired result.

□

4. Zalcman functional

In the field of geometric function theory, one of the classical conjectures proposed by Lawrence Zalcman in 1960 is that the coefficients of class $\mathcal{S}$ satisfy the inequality,

$$\left|b_n^2 - b_{2n-1}\right| \leq (n-1)^2.$$

The above form holds equality only for the famous Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ and its rotation. When $n = 2$, the equality holds for the famous Fekete-Szego inequality. In literature, many reseachers ([3], [4], [16]) have studied about Zalcman functional.

Theorem 4.1. *If $f \in \mathcal{SS}_q^*(1+sin(z))$, then*
$|b_3^2 - b_5| \leq \dfrac{1}{(q+q^2+q^3+q^4)}$.

Proof. In order to find Zalcman functional, we use the equations (2.3) and (2.5), then we get

$$\begin{aligned}\left|b_3^2 - b_5\right| = &\left|\frac{1}{(q+q^2)^2}\left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)^2 - \frac{d_1^2 d_2(5(q+q^2)-4)}{16(q+q^2+q^3+q^4)(q+q^2)} - \frac{d_2^2(1-(q+q^2))}{4(q+q^2+q^3+q^4)(q+q^2)}\right. \\ &\left. + \frac{d_1 d_3}{2(q+q^2+q^3+q^4)} - \frac{d_4}{2(q+q^2+q^3+q^4)}\right| \\ = &\left|\frac{d_1^4}{16(q+q^2)^2} - \frac{d_4}{2(q+q^2+q^3+q^4)} + \frac{d_1 d_3}{2(q+q^2+q^3+q^4)}\right. \\ &+ \frac{d_2^2}{4(q+q^2)^2(q+q^2+q^3+q^4)}\left(q^2+3q^3+2q^4\right) \\ &\left. - \frac{d_1^2 d_2}{16(q+q^2)^2(q+q^2+q^3+q^4)}\left(5q^2+14q^3+9q^4\right)\right| .\end{aligned}$$

Thus,

$$\begin{aligned}\left|b_3^2 - b_5\right| \leq \frac{1}{2(q+q^2+q^3+q^4)} &\left|\left(\frac{q+q^2+q^3+q^4}{8(q+q^2)^2}\right)d_1^4 - d_4 + 2\left(\frac{1}{2}\right)d_1 d_3\right. \\ &\left. + \frac{d_2^2}{2(q+q^2)^2}\left(q^2+3q^3+2q^4\right) - \frac{d_1^2 d_2}{18(q+q^2)^2}\left(5q^2+14q^3+9q^4\right)\right| .\end{aligned}$$

Using lemma 1.3, we get $\left|b_3^2 - b_5\right| \leq \dfrac{1}{q+q^2+q^3+q^4}$.

□

5. Concluding remarks

In the present paper, we have mainly obtained upper bounds of the second-order Hankel determinant of a new class of starlike functions connected with the sine function $SS_q^*(1+sin(z))$.

Also, we have discussed the related research of the coefficient problem, Fekete-Szegö inequality and Zalcman functional for $f \in \mathcal{SS}_q^*(1 + sin(z))$. For motivating further research in this subject-matter, we have chosen to draw the attention of the interested readers towards a considerably large number of related recent publications (see, for example, [14, 31–33]). In conclusion, with an opinion mostly to encourage and inspire further researches on subordinating with Van der Pol numbers (VPN) [21] and Gregory Coefficients [22] one can extend or generalize our results for $f \in SS_q^*(1 + sin(z))$ which is left as an exercise to interested readers.By increasing edge contrast, picture sharpening a basic image processing technique improves an image's clarity and detail. Conventional sharpening techniques can cause noise and distortions, especially in areas with low contrast. Recent advances in geometric function theory have improved image processing methods and created new opportunities for multidisciplinary mathematical imaging research. A masking framework one can use these bounds and the original coefficients as adaptive sharpening factors.

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