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Hyers-Ulam Stability of N-Dimensional Additive Functional Equation in Modular Spaces Using Fixed Point Method

Gowri Senthil[1,*], **S. Sudharsan**[2], **V. Banu Priya**[3], **S. Annadurai**[4], **G. Ganapathy**[5], **A. Vijayalakshmi**[6]

[1]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Tandalam, Chennai - 602 105, Tamil Nadu, India*

[2]*Department of Mathematics, C. Abdul Hakeem College of Engineering and Technology, Melvisharam-632509, Tamil Nadu, India*

[3]*Department of Mathematics, R.M.K. College of Engineering and Technology, Puduvoyal-601206, Tamil Nadu, India*

[4]*Department of Mathematics, St.Joseph's College of Engineering, OMR, Chennai - 600119, Tamil Nadu, India*

[5]*Department of Mathematics, R.M.D Engineering College, Kavaraipettai - 601206, Tamil Nadu, India*

[6]*Department of Mathematics, GRT Institute of Engineering and Technology, Tiruttani - 631209, Tamil Nadu, India*

[*]Corresponding author: gowrisenthil.sse@saveetha.com

Abstract. The Hyers–Ulam stability of functional equations is a subject of mathematical research that examines the approximate validity of these equations. This notion investigates if a function that nearly fulfills a specified functional equation must be near a precise solution of that equation. Numerous research have investigated this domain, examining the stability of diverse functional equations under varying situations. In this present work, we investigated Hyers-Ulam stability of a n-dimensional additive functional equation in modular spaces using the fixed point approach with the help of Fatou property.

1. Introduction

Stability in functional equations occurs when an inequality serves as a perturbation. In 1940, Ulam posed a question about the stability of functional equations [29], which Hyers responded

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to in [9]. Both Aoki [2] and Rassias [23] extended Hyers' theorem to additive mappings and linear mappings, respectively, by taking into account an unbounded Cauchy difference. Rassias considered a mapping $f : X \to Y$ satisfying the condition

$$\|f(x_1 + x_2) - f(x_1) - f(x_2)\|\epsilon \left(\|x_1\|^p + \|x_2\|^p\right)$$

for all $x_1, x_2 \in X$, where $\epsilon \geq 0$ and $0 \leq p < 1$. Afterwards, other mathematicians generalized and expanded this theorem for $p \neq 1$. In the last few decades, we have learned a lot about the stability of various functional equations [1, 6–8, 11, 12, 26, 27, 30, 31].

The modular theory of linear space was formulated by Nakano, with considerable developments contributed by Amemiya, Koshi, Shimogaki, Yamamuro, and others. Orlicz, Mazur, Musielak, Luxemburg, and Turpin [17,20,28], together with other researchers, have further and more comprehensively advanced these theories. Modular space theory is increasingly utilized, demonstrating its importance in various Orlicz spaces [22] and [16, 18], which have extensive applications [20]. The significance of applications lies in the complex structure of modular function spaces, which are also furnished with norm or metric concepts that are modularly equivalent to Banach Spaces.

In 1974, a comparable fixed point result was presented and proven by Ćirić in [3], relating to the original Banach fixed-point contracting theorem. Razani [24] have recently sought to expand their conceptual framework to include modular spaces. The results in metric spaces are comparable to those of ćirić; however, the Δ_2 condition remains unaddressed. The inquiry also pertained to whether the results of Ćirić could be established without imposing a stringent Δ_2-condition within the modular framework.

In 1950, Nakano introduced modular spaces in relation to order spaces [21]. The spaces were established based on Orlicz Spaces theory, which substitutes an abstract functional with advantageous qualities, namely an integrated nonlinear function that governs the evolution of space elements. Readers are advised to refere [4], [5], [10], [15], and [25] for further insights into fixed point theory in modular spaces, as detailed in those references. In the study cited in [25], Sadeghi conducted an investigation into the Hyers-Ulam stability of the generalised Jensen functional equation

$$f(rx + sy) = rg(x) + sh(x)$$

in modular spaces. In [13], Kim et al. explored the generalised Hyers-Ulam-Rassias stability of a nonic functional equation

$$\begin{aligned} f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) \\ -126f(x) + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!f(y) \end{aligned}$$

for mappings from linear spaces into modular spaces that satisfy the Δ_2-condition, utilising a fixed point theorem in modular spaces. Applying the fixed point theorem in probabilistic modular spaces, Zolfaghari et al. investigated the Hyers-Ulam stability of the general mixed additive and quadratic functional equation

$$f(x+ky)+f(x-ky)=f(x+y)+f(x-y)+\frac{2(k+1)}{k}f(ky)-2(k+1)f(y),$$

in [34].

Mohiuddine et al. presented a novel generalised quartic functional equation type and found its general solution in [19]. In addition, they looked into the stability outcomes when applying the Hyers technique in modular space without the Δ_b-condition, without the Fatou property, and without both of these conditions. In addition, they used a fixed-point method based on the concept of the Fatou property in modular spaces to study the stability results. The non-stability of a unique case is further proven by demonstrating a relevant counter example.

2. Preliminaries

We start by examining several critically significant concepts.

Definition 2.1. *[33] Let V be a vector space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). We called a generalized functional $f: V \to [0,\infty]$ is a modular if every scalars a_1, a_2 and for $u, v \in V$,*

(a) $f(u)=0 \Leftrightarrow u=0$,
(b) $f(a_1u)=f(u)$ *with* $|a_1|=1$,
(c) $f(a_1u+a_2v) \leq f(u)+f(v), \quad \forall\, a_1,a_2 \geq 0$ *and* $a_1+a_2=1$.
If (c) is substituted by
(c') $f(a_1u+a_2v) \leq a_1f(u)+a_2f(v)$ *for every scalars* $a_1,a_2 \geq 0$ *and* $a_1+a_2=1$, *then* f *is thus said to as convex modular.*

The f is a modular that defines an appropriate modular space, i.e. a V_f vector space provided with:

$$V_f=\{u \in V | f(cu) \to 0 \text{ as } c \to 0\}.$$

Definition 2.2. *[33] If V_f is a modular space and the sequence $\{v_n\}$ in V_f, then*

(i) $v_n \xrightarrow{f} v$ *if* $f(v_n-v) \to 0$ *as* $n \to \infty$.
(ii) $\{v_n\}$ *is known as f-Cauchy if* $f(v_l-v_n) \to 0$ *as l,n tends to* ∞.
(iii) *A subset $A \subseteq V_f$ is known as f-complete if and only if every f-Cauchy sequence is f-convergent in A.*

Definition 2.3. *[33] Let V_f be a modular space and a non-empty subset $A \subseteq V_f$. The mapping $\Upsilon: A \to A$ is referred to as a quasicontraction, if there is $k<1$ satisfies*

$$f\left(Jl-Jm\right) \leq k\max\{f(l-m), f(l-Jm), f(m-Jl), f(l-Jl), f(m-Jm)\},$$

for any $l,m \in A$.

Definition 2.4. *[33] Let V_f be a modular space, a non-empty subset $A \subseteq V_f$, and a function $\Upsilon: A \to A$, the Υ orbit around a point v is*

$$O(\Upsilon) := \{u, \Upsilon u, \Upsilon^2u, \cdots\},$$

the quantity

$$\Upsilon_f(\Upsilon) := \sup\{f(p-q)|p, q \in O(\Upsilon)\},$$

is then related to Υ *and is referred to as the orbital diameter of* Υ *at* v*. If* $\Upsilon_f(\Upsilon) < \infty$*, in particular one says that* Υ *has an orbit of* v *that is limited to* v*.*

Fatou property: The f-modular will have the Fatou property if and only if $f(v) \leq \lim_{m\to\infty} \inf f(v_m)$ whenever $\{v_m\} \xrightarrow{f} v$. A modular function is stated to fulfil the conditions Δ_2 if there is $k > 0$ which satisfies $f(2v) \leq kf(v)$, for every $v \in V_f$.

Theorem 2.1. *[33] Let a modular space* W_f *such that* f *fulfils the Fatou property and* $A \subseteq W_f$ *be a* f*-complete. If* $\Upsilon : A \to A$ *is a quasicontraction and* Υ *has a bounded orbit at* v_0*, then* $\{\Upsilon^n v_0\} \xrightarrow{f} \alpha$*, where* $\alpha \in A$*.*

In this work, we introduce new generalized additive functional equation

$$\zeta\Big(\sum_{1\leq j\leq n} jv_j\Big) = \sum_{1\leq j\leq n} j\zeta(v_j), \tag{2.1}$$

where $n \geq 2$, and investigated Hyers-Ulam stability of this additive functional equation in modular space by utilizing the fixed point theory with the help of Fatou property.

Theorem 2.2. *If* ζ *is an odd mapping between two real vector spaces* V *and* W*, which fulfils the equation* (2.1) *for all* $v_1, v_2, \cdots, v_n \in V$*, then the function* ζ *is additive.*

3. Hyers-Ulam Stability

Consider V as a linear space, W as a Banach space, W_f as a f-complete modular space, and f as a convex modular on W_f with the Fatou property which fulfils the Δ_2-condition with $0 < k \leq 2$.

For notational handiness, we can define the mapping $\zeta : V \to W_f$ as follows:

$$D\zeta(v_1, v_2, \cdots, v_n) = \zeta\Big(\sum_{1\leq j\leq n} jv_j\Big) - \sum_{1\leq j\leq n} j\zeta(v_j),$$

for all $v_1, v_2, \cdots, v_n \in V$.

Theorem 3.1. *Let a mapping* $\Gamma : V^n \to [0, +\infty)$ *such that*

$$\lim_{m\to\infty} \frac{1}{2^m}\Gamma(2^m v_1, 2^m v_2, \cdots, 2^m v_n) = 0, \tag{3.1}$$

and

$$\Gamma(2v_1, 2v_2, \cdots, 2v_n) \leq 2L\Gamma(v_1, v_2, \cdots, v_n), \tag{3.2}$$

for every $v_i \in V; i = 1, 2, \cdots, n$*, with* $0 < L < 1$*. If an odd mapping* $\zeta : V \to W_f$ *with* $\zeta(0) = 0$ *fulfils*

$$f(D\zeta(v_1, v_2, \cdots, v_n)) \leq \Gamma(v_1, v_2, \cdots, v_n), \tag{3.3}$$

for all $v_i \in V; i = 1, 2, \cdots, n$*, then there is only one additive solution* $Q_4 : V \to W_f$ *satisfies*

$$f(Q_4(v) - \zeta(v)) \leq \frac{1}{2(1-L)}\Gamma(0, v, 0, \cdots, 0), \tag{3.4}$$

for every $v \in V$.

Proof. Let us define the set

$$\xi = \{p : V \to W_f\}$$

and $\overline{f}$ is a function on ξ as

$$\overline{f}(p) =: \inf\{\alpha > 0 : f\,(p(v)) \leq \alpha\Gamma(0, v, 0, \cdots, 0),\ \forall\, v \in V\}.$$

Now, We need to demonstrate that the function $\overline{f}$ is a convex modular on ξ. Clearly, $\overline{f}$ holds the modular conditions (a) and (b). So that, it is enough to verify that the function $\overline{f}$ is convex, and so $(c^{'})$ is holds. For any given $\varepsilon > 0$, then there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ which are real constants such that

$$\overline{f}(p) \leq \alpha_1 \leq \overline{f}(p) + \varepsilon \text{ and } \overline{f}(q) \leq \alpha_2 \leq \overline{f}(q) + \varepsilon.$$

Also

$$f(p(v)) \leq \alpha_1\Gamma(0, v, 0, \cdots, 0),\ f(q(v)) \leq \alpha_2\Gamma(0, v, 0, \cdots, 0),\ v \in V.$$

Let us consider for any $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then we have

$$\begin{aligned} f\,(a_1 p(v) + a_2 q(v)) &\leq a_1 f(p(v)) + a_2 f(q(v)) \\ &\leq (\alpha_1 a_1 + \alpha_2 a_2)\,\Gamma(0, v, 0, \cdots, 0), \end{aligned}$$

so we get

$$\overline{f}(a_1 p + a_2 q) \leq a_1\overline{f}(p) + a_2\overline{f}(q) + (a_1 + a_2)\varepsilon.$$

From this, we conclude that the function $\overline{f}$ is convex modular on ξ. Next, we want to verify that $\xi_{\overline{f}}$ is $\overline{f}$-complete.

Suppose $\{p_n\}$ is a $\overline{f}$-Cauchy sequence in $\xi_{\overline{f}}$ and for every $\varepsilon > 0$. Then there is a non-negative integer $n_0 \in \mathbb{N}$ satisfies

$$\overline{f}\,(p_n - p_m) < \varepsilon,\ \forall\, n, m \geq n_0. \tag{3.5}$$

We have

$$f\,(p_n(v) - p_m(v)) \leq \varepsilon\Gamma(0, v, 0, \cdots, 0),\ v \in V,\ n, m \geq n_0. \tag{3.6}$$

Therefore, a f-Cauchy sequence $\{p_n(v)\}$ in W_f. As W_f is f-complete, thus $\{p_n(v)\}$ is convergent in W_f, for every $v \in V$. Now, let us define a mapping $p : V \to W_f$ by

$$p(v) := \lim_{n\to\infty} p_n(v),\ v \in V. \tag{3.7}$$

As f holds the Fatou property, using (3.6), it arrives that

$$f(p_n(v) - p(v)) \leq \lim\inf_{m\to\infty} f\,(p_n(v) - p_m(v)) \leq \varepsilon\Gamma(0, v, 0, \cdots, 0), \tag{3.8}$$

so

$$\overline{f}\,(p_n - p) \leq \varepsilon, \tag{3.9}$$

for all $n \geq n_0$. Thus, $\{p_n\}$ is $\overline{f}$-converges. Hence $\xi_{\overline{f}}$ is $\overline{f}$-complete.

Next, we want to prove that $\overline{f}$ holds the Fatou property. Suppose that a $\overline{f}$-convergent sequence $\{p_n\}$ converges to a point $p \in \xi_{\overline{f}}$.

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, let a real constant α_n such that

$$\overline{f}(p_n) \leq \alpha_n \leq \overline{f}(p_n) + \varepsilon. \tag{3.10}$$

So

$$f\left(p_n(v)\right) \leq \alpha_n \Gamma(0, v, 0, \cdots, 0), \ \forall v \in V. \tag{3.11}$$

We know that f holds the Fatou property, we get

$$\begin{aligned} f(p(v)) &\leq \lim_{n\to\infty} \inf f(p_n(v)) \\ &\leq \lim_{n\to\infty} \inf \alpha_n \Gamma(0, v, 0, \cdots, 0) \\ &\leq \left[\lim_{n\to\infty} \inf \overline{f}(p_n) + \varepsilon\right] \Gamma(0, v, 0, \cdots, 0). \end{aligned}$$

Thus, we obtain

$$\overline{f}(p) \leq \lim_{n\to\infty} \inf \overline{f}(p_n) + \varepsilon.$$

Hence, $\overline{f}$ also holds the Fatou property.

Let us define a mapping $\sigma : \xi_{\overline{f}} \to \xi_{\overline{f}}$ by

$$\sigma p(v) = \frac{1}{2} p(2v),$$

for all $v \in V$ and $p \in \xi_{\overline{f}}$. Let $p, q \in \xi_{\overline{f}}$ and an arbitrary constant $\alpha \in [0, 1]$ with $\overline{f}(p - q) < \alpha$. Utilizing the definition of $\overline{f}$, we obtain

$$f(p(v) - q(v)) \leq \alpha \Gamma(0, v, 0, \cdots, 0)$$

for every $v \in V$. By inequality (3.2) and the above inequality, we reach

$$\begin{aligned} f\left(\frac{p(2v)}{2} - \frac{q(2v)}{2}\right) &\leq \frac{1}{2} f\left(p(2v) - q(2v)\right) \\ &\leq \frac{1}{2} \alpha \Gamma\left(0, 2v, 0, \cdots, 0\right) \\ &\leq \alpha L \Gamma(0, v, 0, \cdots, 0), \end{aligned}$$

for every $v \in V$. Hence,

$$\overline{f}(\sigma p - \sigma q) \leq L \overline{f}(p - q), \ \forall \, p, q \in \xi_{\overline{f}}.$$

i.e., σ is a $\overline{f}$-contraction. We are now proving that σ has a ζ limited orbit. Replacing $(v_1, v_2, \cdots, v_n)$ by $(0, v, 0, \cdots, 0)$ in (3.3), we get

$$\begin{aligned} f\left(\zeta(2v) - 2\zeta(v)\right) &\leq \Gamma(0, v, 0, \cdots, 0), \\ \Rightarrow f\left(\frac{\zeta(2v)}{2} - \zeta(v)\right) &\leq \frac{1}{2} \Gamma(0, v, 0, \cdots, 0), \ v \in V. \end{aligned} \tag{3.12}$$

Replacing v with $2v$ in (3.12), we get

$$\begin{aligned} f\left(\frac{\zeta(2^2v)}{2}-\zeta(2v)\right) &\leq \frac{1}{2}\Gamma(0,2v,0,\cdots,0), \\ \Rightarrow f\left(\frac{\zeta(2^2v)}{2^2}-\frac{\zeta(2v)}{2}\right) &\leq \frac{1}{2^2}\Gamma(0,2v,0,\cdots,0),\ v\in V. \end{aligned} \tag{3.13}$$

By using (3.12) and (3.13), we get

$$\begin{aligned} f\left(\frac{\zeta(2^2v)}{2^2}-\zeta(v)\right) &\leq f\left(\frac{\zeta(2^2v)}{2^2}-\frac{\zeta(2v)}{2}+\frac{\zeta(2v)}{2}-\zeta(v)\right) \\ &\leq f\left(\frac{\zeta(2^2v)}{2^2}-\frac{\zeta(2v)}{2}\right)+f\left(\frac{\zeta(2v)}{2}-\zeta(v)\right) \\ &\leq \frac{1}{2^2}\Gamma(0,2v,0,\cdots,0)+\frac{1}{2}\Gamma(0,v,0,\cdots,0),\ v\in V. \end{aligned} \tag{3.14}$$

We can easily see this through induction

$$\begin{aligned} f\left(\frac{\zeta\left(2^nv\right)}{2^n}-\zeta(v)\right) &\leq \sum_{i=1}^{n}\frac{1}{2^i}\Gamma\left(0,2^{i-1}v,0,\cdots,0\right) \\ &\leq \frac{1}{L2}\Gamma(0,v,0,\cdots,0)\sum_{i=1}^{n}L^i \\ &\leq \frac{1}{2(1-L)}\Gamma(0,v,0,\cdots,0), \end{aligned} \tag{3.15}$$

for all $v\in V$. This results from (3.15) inequality

$$\begin{aligned} f\left(\frac{\zeta\left(2^nv\right)}{2^n}-\frac{\zeta\left(2^mv\right)}{2^m}\right) &\leq \frac{1}{2}f\left(2\frac{\zeta\left(2^nv\right)}{2^n}-2\zeta(v)\right)+\frac{1}{2}f\left(2\frac{\zeta\left(2^mv\right)}{2^m}-2\zeta(v)\right) \\ &\leq \frac{k}{2}f\left(\frac{\zeta\left(2^nv\right)}{2^n}-\zeta(v)\right)+\frac{k}{2}f\left(\frac{\zeta\left(2^mv\right)}{2^m}-\zeta(v)\right) \\ &\leq \frac{k}{2}\frac{1}{2(1-L)}\Gamma(0,v,0,\cdots,0)+\frac{k}{2}\frac{1}{2(1-L)}\Gamma(0,v,0,\cdots,0) \\ &\leq \frac{k}{2(1-L)}\Gamma(0,v,0,\cdots,0),\ v\in V, \end{aligned}$$

and $n,m\in\mathbb{N}$. We conclude that by defining $\overline{f}$,

$$\overline{f}\left(\sigma^n\zeta-\sigma^m\zeta\right)\leq\frac{k}{2(1-L)}.$$

This means that an orbit of σ at ζ is bounded. The sequence of $\{\sigma^n\zeta\}$ $\overline{f}$-converges into $Q_4\in\xi_{\overline{f}}$, according to Theorem 1.5 in [33]. Now, we have the $\overline{f}$-contractivity of σ, that

$$\overline{f}\left(\sigma^n\zeta-\sigma Q_4\right)\leq L\overline{f}\left(\sigma^{n-1}\zeta-Q_4\right).$$

If we pass $n \to \infty$ and apply $\overline{f}$ Fatou property, we get this

$$\begin{aligned}\overline{f}\left(\sigma Q_4 - Q_4\right) &\leq \liminf_{n\to\infty} \overline{f}\left(\sigma Q_4 - \sigma^n \zeta\right) \\ &\leq L \liminf_{n\to\infty} \overline{f}\left(Q_4 - \sigma^{n-1}\zeta\right) = 0.\end{aligned}$$

Thus, Q_4 is a fixed point of σ. Switching $(v_1, v_2, \cdots, v_n)$ by $\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right)$ in (3.3), we get

$$f\left(D\zeta\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right)\right) \leq \Gamma\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right),$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore

$$f\left(\frac{1}{2^l} D\zeta\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right)\right) \leq \frac{1}{2^l}\Gamma\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right). \tag{3.16}$$

Employing the limit $l \to \infty$, we get

$$DQ_4\left(v_1, v_2, \cdots, v_n\right) = 0,$$

for all $v_1, v_2, \cdots, v_n \in V$. It follows from Theorem 2.2, that Q_4 is additive. By using (3.15), we get (3.4).

Let $Q_4^{'} : V \to W_f$ be another additive mapping that meets inequality (3.4) to proved the unique character of Q_4. Then $Q4^{'}$ is a σ fixed point.

$$\overline{f}\left(Q_4 - Q_4^{'}\right) = \overline{f}\left(\sigma Q_4 - \sigma Q_4^{'}\right) \leq L\overline{f}\left(Q_4 - Q_4^{'}\right).$$

which implies that $\overline{f}\left(Q_4 - Q_4^{'}\right) = 0$. This proves that $Q_4 = Q_4^{'}$. Therefore, the function Q_4 is the unique solution. This completes the proof. □

Corollary 3.1. *Let a function $\Gamma : V^n \to [0, +\infty)$ such that*

$$\lim_{l\to\infty} \frac{1}{2^l}\Gamma\left(2^l v_1, 2^l v_2, \cdots, 2^l v_n\right) = 0,$$

and

$$\Gamma\left(2v_1, 2v_2, \cdots, 2v_n\right) \leq L2\Gamma\left(v_1, v_2, \cdots, v_n\right),$$

for all $v_1, v_2, \cdots, v_n \in V$ with $0 < L < 1$. If an odd mapping $\zeta : V \to W$ with $\zeta(0) = 0$ which fulfils

$$||D\zeta(v_1, v_2, \cdots, v_n)|| \leq \Gamma(v_1, v_2, \cdots, v_n), \tag{3.17}$$

for every $v_i \in V; i = 1, 2, \cdots, n$, then there is only one additive solution $Q_4 : V \to W$ satisfies

$$||Q_4(v) - \zeta(v)|| \leq \frac{1}{2(1-L)}\Gamma(0, v, 0, \cdots, 0),$$

for every $v \in V$.

Proof. Each normed space is known to be a $f(v) = ||v|$ modular space and to hold the Δ_2-condition with $k = 2$. □

Remark 3.1. *If we replacing $\Gamma(v_1, v_2, \cdots, v_n)$ by $\alpha\left(\sum_{i=1}^{n}\|v_i\|^p\right)$ and taking $L = 2^{p-1}$ in Corollary 3.1, we arrive the stability results for the sum of norms that*

$$\|Q_4(v) - \zeta(v)\| \leq \frac{\alpha\|v\|^p}{(2-2^p)},\ v \in V,$$

where α and p are constants with $p < 1$.

Remark 3.2. *If we replacing $\Gamma(v_1, v_2, \cdots, v_n)$ by $\alpha\left(\sum_{i=1}^{n}\|v_i\|^{np} + \prod_{i=1}^{n}\|v_i\|^p\right)$ and taking $L = 2^{np-1}$ in Corollary 3.1, we arrive the stability results for the sum of product of norms that*

$$\|Q_4(v) - \zeta(v)\| \leq \frac{\alpha\|v\|^{np}}{(2-2^{np})},\ v \in V,$$

where α and p are constants with $np < 1$.

Theorem 3.2. *Let a mapping $\Gamma : V^n \to [0, +\infty)$ satisfies*

$$\lim_{m\to\infty} 2^m \Gamma\left(\frac{v_1}{2^m}, \frac{v_2}{2^m}, \cdots, \frac{v_n}{2^m}\right) = 0 \tag{3.18}$$

and

$$\Gamma\left(\frac{v_1}{2}, \frac{v_2}{2}, \cdots, \frac{v_n}{2}\right) \leq \frac{L}{2}\Gamma(v_1, v_2, \cdots, v_n) \tag{3.19}$$

for all $v_1, v_2, \cdots, v_n \in V$ with $0 < L < 1$. If an odd mapping $\zeta : V \to W_f$ with $\zeta(0) = 0$ which fulfils (3.3)*, then there is only one additive solution $Q_4 : V \to W_f$ satisfies*

$$f(Q_4(v) - \zeta(v)) \leq \frac{L}{2(1-L)}\Gamma(0, v, 0, \cdots, 0), \tag{3.20}$$

for every $v \in V$.

Proof. Let us define a set

$$\xi = \{p : V \to W_f\}$$

and $\overline{f}$ be a function on ξ as,

$$\overline{f}(p) =: \inf\{\alpha > 0 : f(p(v)) \leq \alpha\Gamma(0, v, 0, \cdots, 0),\ \forall\, v \in V\}.$$

We have the same evidence as Theorem 3.1:

1. The function $\overline{f}$ is a convex modular on ξ.
2. $\xi_{\overline{f}}$ is $\overline{f}$-complete.
3. $\overline{f}$ holds the Fatou property.

We now think of the mapping $\sigma : \xi_{\overline{f}\to\xi_{\overline{f}}}$ defined by:

$$\sigma p(v) = 2p\left(\frac{v}{2}\right),\ v \in V,$$

and $p \in \xi_{\overline{f}}$. Let $p, q \in \xi_{\overline{f}}$ and an arbitrary constant $\alpha \in [0,1]$ with $\overline{f}(p-q) < \alpha$. Utilizing the definition of $\overline{f}$, we obtain

$$f(p(v) - q(v)) \leq \alpha\Gamma(0, v, 0, \cdots, 0),\ v \in V.$$

We get through assumption and above inequality, that

$$\begin{aligned} f\left(2p\left(\frac{v}{2}\right)-2q\left(\frac{v}{2}\right)\right) &\leq kf\left(p\left(\frac{v}{2}\right)-q\left(\frac{v}{2}\right)\right) \\ &\leq k\alpha\Gamma\left(0,\frac{v}{2},0,\cdots,0\right) \\ &\leq \alpha L\Gamma(0,v,0,\cdots,0),\ v\in V. \end{aligned}$$

Hence,

$$\overline{f}(\sigma p-\sigma q)\leq L\overline{f}(p-q),\ p,q\in\xi_{\overline{f}},$$

i.e., σ is a $\overline{f}$-contraction.

Next, we prove then that σ has a bounded orbit at ζ. Setting $(v_1,v_2,\cdots,v_n)$ by $(0,v,0,\cdots,0)$ in (3.3), we get

$$f\left(2\zeta(v)-\zeta(2v)\right)\leq\Gamma(0,v,0,\cdots,0), \tag{3.21}$$

for every $v\in V$. Replacing v with $\frac{v}{2}$ in (3.21), we get

$$f\left(2\zeta\left(\frac{v}{2}\right)-\zeta\left(v\right)\right)\leq\Gamma\left(0,\frac{v}{2},0,\cdots,0\right), \tag{3.22}$$

for every $v\in V$. Replacing v with $\frac{v}{2}$ in (3.22), we get

$$f\left(2\zeta\left(\frac{v}{2^2}\right)-\zeta\left(\frac{v}{2}\right)\right)\leq\Gamma\left(0,\frac{v}{2^2},0,\cdots,0\right), \tag{3.23}$$

for all $v\in V$. By using (3.21) , (3.22) and (3.23), we get

$$\begin{aligned} f\left(2^2\zeta\left(\frac{v}{2^2}\right)-\zeta(v)\right) &\leq f\left(2^2\zeta\left(\frac{v}{2^2}\right)-2\zeta\left(\frac{v}{2}\right)\right)+f\left(2\zeta\left(\frac{v}{2}\right)-\zeta(v)\right) \\ &\leq kf\left(2\zeta\left(\frac{v}{2^2}\right)-\zeta\left(\frac{v}{2}\right)\right)+f\left(2\zeta\left(\frac{v}{2}\right)-\zeta(v)\right) \\ &\leq 2\Gamma\left(0,\frac{v}{2^2},0,\cdots,0\right)+\Gamma\left(0,\frac{v}{2},0,\cdots,0\right), \end{aligned} \tag{3.24}$$

for every $v\in V$. We can easily see this through induction

$$\begin{aligned} f\left(2^n\zeta\left(\frac{v}{2^n}\right)-\zeta(v)\right) &\leq \frac{1}{2}\sum_{i=1}^{n}2^i\Gamma\left(0,\frac{v}{2^i},0,\cdots,0\right) \\ &\leq \frac{1}{2}\Gamma(0,v,0,\cdots,0)\sum_{i=1}^{n}L^i \\ &\leq \frac{L}{2(1-L)}\Gamma(0,v,0,\cdots,0), \end{aligned} \tag{3.25}$$

for every $v\in V$. It is the result of inequality (3.25) that

$$\begin{aligned} f\left(2^n\zeta\left(\frac{v}{2^n}\right)-2^m\zeta\left(\frac{v}{2^m}\right)\right) &\leq \frac{1}{2}f\left(2(2^n)\zeta\left(\frac{v}{2^n}\right)-2\zeta(v)\right)+\frac{1}{2}f\left(2(2^m)\zeta\left(\frac{v}{2^m}\right)-2\zeta(v)\right) \\ &\leq \frac{kL}{2(1-L)}\Gamma(0,v,0,\cdots,0),\ v\in V, \end{aligned}$$

and every $n, m \in \mathbb{N}$. We can conclude that by defining $\overline{f}$,

$$\overline{f}\left(\sigma^n \zeta - \sigma^m \zeta\right) \le \frac{kL}{2(1-L)}.$$

That means that σ orbit is limited to ζ. The sequence $\{\sigma^n \zeta\}$ $\overline{f}$-converges to $Q_4 \in \xi_{\overline{f}}$ from Theorem 1.5 in [33].

Now, by the $\overline{f}$-contractivity of σ, we have

$$\overline{f}\left(\sigma^n \zeta - \sigma Q_4\right) \le L\overline{f}\left(\sigma^{n-1}\zeta - Q_4\right).$$

Taking $n \to \infty$ and utilizing the Fatou property of $\overline{f}$, we arrive

$$\begin{aligned} \overline{f}\left(\sigma Q_4 - Q_4\right) &\le \liminf_{n\to\infty} \overline{f}\left(\sigma Q_4 - \sigma^n \zeta\right) \\ &\le L \liminf_{n\to\infty} \overline{f}\left(Q_4 - \sigma^{n-1}\zeta\right) = 0. \end{aligned}$$

Therefore, Q_4 is a fixed point of σ. Replacing $(v_1, v_2, \cdots, v_n)$ by $\left(\frac{v_1}{2^l}, \frac{v_2}{2^l}, \cdots, \frac{v_n}{2^l}\right)$ in (3.3), we get

$$f\left(D\zeta\left(2^{-l}v_1, 2^{-l}v_2, \cdots, 2^{-l}v_n\right)\right) \le \Gamma\left(2^{-l}v_1, 2^{-l}v_2, \cdots, 2^{-l}v_n\right),$$

for all $v_1, v_2, \cdots, v_n \in V$. Therefore,

$$f\left(2^l D\zeta\left(\frac{v_1}{2^l}, \frac{v_2}{2^l}, \cdots, \frac{v_n}{2^l}\right)\right) \le k^l \Gamma\left(\frac{v_1}{2^l}, \frac{v_2}{2^l}, \cdots, \frac{v_n}{2^l}\right). \tag{3.26}$$

Passing to the limit $l \to \infty$, we get

$$DQ_4\left(v_1, v_2, \cdots, v_n\right) = 0,$$

for all $v_1, v_2, \cdots, v_n \in V$. It follows from Theorem 2.2, that Q_4 is additive. By using (3.25), we get (3.20).

In order to prove that the uniqueness of Q_4, consider another additive solution $Q_4^{'} : V \to W_f$ to satisfy the inequality (3.4). Then $Q_4^{'}$ is a fixed point of σ.

$$\overline{f}\left(Q_4 - Q_4^{'}\right) = \overline{f}\left(\sigma Q_4 - \sigma Q_4^{'}\right) \le L\overline{f}\left(Q_4 - Q_4^{'}\right).$$

which implies that $\overline{f}\left(Q_4 - Q_4^{'}\right) = 0$ or $Q_4 = Q_4^{'}$. Hence the proof is now completed. □

Corollary 3.2. *Let a mapping $\Gamma : V^n \to [0, +\infty)$ satisfies*

$$\lim_{l\to\infty} 2^l \Gamma\left(\frac{v_1}{2^l}, \frac{v_2}{2^l}, \cdots, \frac{v_n}{2^l}\right) = 0,$$

and

$$\Gamma\left(\frac{v_1}{2}, \frac{v_2}{2}, \cdots, \frac{v_n}{2}\right) \le \frac{L}{2}\Gamma\left(v_1, v_2, \cdots, v_n\right),$$

for every $v_i \in V$; $i = 1, 2, \cdots, n$, with $0 < L < 1$. If an odd mapping $\zeta : V \to W$ with $\zeta(0) = 0$ which fulfils (3.17), *then there is only one additive solution $Q_4 : V \to W$ fulfils*

$$\|Q_4(v) - \zeta(v)\| \le \frac{L}{2(1-L)}\Gamma(0, v, 0, \cdots, 0),$$

for every $v \in V$.

Proof. Each normed space is known to be a $f(v) = \|v|$ modular space and to hold the Δ_2-condition with $k = 2$. □

Remark 3.3. *If we replacing* $\Gamma(v_1, v_2, \cdots, v_n)$ *by* $\alpha\left(\sum_{i=1}^{n} \|v_i\|^p\right)$ *and taking* $L = 2^{1-p}$ *in Corollary 3.2, we arrive the stability results for the sum of norms that*

$$\|Q_4(v) - \zeta(v)\| \leq \frac{\alpha\|v\|^p}{(2^p - 2)},\ v \in V,$$

where α *and* p *are constants with* $p > 1$*.*

Remark 3.4. *If we replacing* $\Gamma(v_1, v_2, \cdots, v_n)$ *by* $\alpha\left(\sum_{i=1}^{n} \|v_i\|^{np} + \prod_{i=1}^{n} \|v_i\|^p\right)$ *and taking* $L = 2^{1-np}$ *in Corollary 3.2, we arrive the stability results for the sum of product of norms that*

$$\|Q_4(v) - \zeta(v)\| \leq \frac{\alpha\|v\|^{np}}{(2^{np} - 2)},\ v \in V,$$

where α *and* p *are constants with* $np > 1$*.*

4. Conclusion

In this present work, we investigated Hyers-Ulam stability of a n-dimensional additive functional equation in modular spaces using the fixed point approach with the help of Fatou property.

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