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Nearly Lindelöfness in N^{th}-Topological Spaces

Jamal Oudetallah[1], Rehab Alharbi[2,*], Salsabiela Rawashdeh[3], Iqbal M. Batiha[4,5], Ala Amourah[6], Tala Sasa[7]

[1]*Department of Mathematics, University of Petra, Amman, 11196, Jordan*

[2]*Department of Mathematics, Jazan University, Jazan 2097, Saudi Arabia*

[3]*Department of Mathematics, Irbid National University, Irbid 21110, Jordan*

[4]*Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan*

[5]*Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE*

[6]*Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 3111, Oman*

[7]*Applied Science Research Center, Applied Science Private University, Amman, Jordan*

[*]*Corresponding author:* ralharbi@jazanu.edu.sa

Abstract. The concepts of approximately N^{th}-Lindelöf spaces, introduced in [15] and [16], represent weaker forms of single Lindelöf spaces and have since attracted significant attention. In this paper, we revisit these spaces through the lens of a particular type of cover, known as a normal cover. The primary objective is to explore the properties of a new generalization, termed nearly n^{th}-Lindelöf spaces.

1. Introduction

Topology, often described as "rubber-sheet geometry," is a fundamental branch of mathematics that studies the properties of space preserved under continuous deformations such as stretching or bending, but not tearing or gluing. It provides a flexible and abstract framework to formalize concepts like convergence, continuity, and compactness across diverse mathematical and applied disciplines. Over the years, numerous generalizations and refinements of topological concepts have emerged, especially in the study of multi-topological structures and bitopological settings, which allow for richer investigations of separation axioms, compactness variations, and open set structures [1–11]. Recent research has demonstrated that topological structures play a foundational role in the qualitative analysis of complex systems governed by differential and integro-differential

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equations. For instance, the behavior of solutions to Volterra-type equations, fractional-order models, and systems influenced by noise can often be better understood when analyzed within suitable topological frameworks [12–14]. The application of topological concepts—such as compactness, continuity, and convergence—provides theoretical guarantees for the existence, uniqueness, and stability of solutions, while advanced numerical algorithms tailored to these models help simulate such systems under topologically meaningful constraints.

Lindelöfness, originally studied by Katětov and L. Krajewski, is one of the most prominent concepts in general topology. Over time, numerous generalizations of Lindelöf spaces have been explored, including compactness, paracompactness, and local finiteness. An n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is said to be γ_i-expandable with respect to γ_j if it is γ_i-m-expandable for every cardinal m, where $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. A bitopological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called an n^{th} expandable (or P-expandable) space if it satisfies the P-T_2 separation axiom and is:

- γ_1-expandable with respect to γ_2 or γ_3,
- γ_2-expandable with respect to γ_1 or γ_3, and
- γ_3-expandable with respect to γ_1 or γ_2.

A subset A of $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called γ_i-regularly open with respect to γ_j if

$$\mathrm{Int}^{\gamma_j}(\mathrm{Cl}^{\gamma_j}(A)) = A,$$

where $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. Moreover, A is said to be a pairwise regularly open set in μ if it is both γ_1-regularly open with respect to γ_2 and γ_2-regularly open with respect to γ_1. Clearly, every pairwise regularly open set is also a pairwise open set. Similarly, a subset $B \subseteq \mu$ is called a pairwise regularly closed set if it is the complement of a pairwise regularly open set.

2. Literature Review

The study of n^{th}-topological spaces is built upon foundational work in single, bi-topological, and tri-topological spaces, which themselves are extensions of classical topological spaces. In this context, we consider a non-empty set μ endowed with n distinct topologies denoted by γ_i, for $i = 1, 2, \ldots, n$, forming what is termed an n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$.

The concept of n^{th}-topological spaces emerged around the year 2000 and has since been explored primarily in relation to separation axioms. In the present study, we extend this framework by applying it more comprehensively to various separation axioms. These axioms comprise a collection of conditions or properties used to classify topological spaces based on their ability to distinguish between distinct points or sets. Specifically, separation axioms define how points or sets may be separated using open sets. Their importance in topology stems from their role in elucidating the geometric structure of a space and the relationships among its points and subsets. These axioms differ in the extent to which they enable such separation.

The concept of nearly Lindelöf spaces in the classical topological setting (μ, γ) was first introduced in [17]. Subsequent studies [15,18,19] have further developed this notion. The current paper investigates the extension of this idea to the framework of n^{th}-topological spaces and presents new results related to n^{th}-nearly Lindelöf spaces.

The topologies γ_u, γ_{dis}, γ_{cof}, and γ_{coc} refer to the usual (standard) topology, discrete topology, cofinite topology, and co-countable topology, respectively. An n^{th}-topological space may be denoted as $\mu = (\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$, where each γ_i is a topology on μ, for all $i = 1, 2, \dots, n$. This framework connects directly to prior research in the area, wherein each topology satisfies the usual topological axioms. In particular, the work in [16] discusses key separation properties such as n^{th}-Hausdorff, n^{th}-regular, and n^{th}-normal spaces, leveraging a collection of classical results including the Tietze extension theorem.

3. Preliminaries

Definition 3.1 ([20]). *Let μ be a non-empty set, and let $\gamma \subseteq \mathcal{P}(\mu) = \{A \mid A \subseteq \mu\}$. Then γ is called a topology on μ if the following conditions are satisfied:*

(1) *$\mu \in \gamma$ and $\varnothing \in \gamma$;*
(2) *For all $W, L \in \gamma$, we have $W \cap L \in \gamma$;*
(3) *If $\mathcal{M} = \{W_\gamma \mid W_\gamma \in \gamma, \gamma \in \gamma\}$, then $\bigcup \mathcal{M} = \bigcup_{\gamma \in \gamma} W_\gamma \in \gamma$.*

Definition 3.2. *Let μ be a non-empty set, and let $\gamma_i \subseteq \mathcal{P}(\mu) = \{A \mid A \subseteq \mu\}$ for each $i = 1, 2, \dots, n$. Then the ordered tuple $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called an n^{th}-*topological space *if each γ_i is a topology on μ.*

Example 3.1. *Let $\mu = \{1, 2, 3\}$. Consider the following topologies:*

$$\gamma_1 = \{\varnothing, \mu, \{1\}\} \subseteq \mathcal{P}(\mu),$$
$$\gamma_2 = \{\varnothing, \mu, \{1\}, \{2\}, \{1, 2\}\} \subseteq \mathcal{P}(\mu),$$
$$\gamma_3 = \{\varnothing, \mu, \{2\}, \{3\}, \{2, 3\}\} \subseteq \mathcal{P}(\mu).$$

Each γ_i satisfies the conditions for a topology on μ for $i = 1, 2, 3$. Hence, the structure $(\mu, \gamma_1, \gamma_2, \gamma_3)$ forms a tri-topological space. *However, consider $\gamma = \{\varnothing, \{1\}\}$. This is **not** a topology on μ, as it does not contain the whole set μ, violating the first axiom of topology.*

Definition 3.3. *Let $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ be an n^{th}-topological space, and let $E \subseteq \mu$. Then:*

(1) *E is called an n^{th}-*open set *if $E \in \gamma_i$ for some $i = 1, 2, \dots, n$;*
(2) *E is called an n^{th}-*closed set *if $E^c \in \gamma_i$ for some $i = 1, 2, \dots, n$;*
(3) *E is called an n^{th}-*clopen set *if both E and E^c belong to γ_i for some $i = 1, 2, \dots, n$.*

Example 3.2. *Let $\mu = \{x, y, z\}$, and consider the topologies:*

$$\gamma_1 = \{\varnothing, \mu, \{x\}\}, \quad \gamma_2 = \{\varnothing, \mu, \{y\}\}, \quad \gamma_3 = \{\varnothing, \mu, \{z\}\}.$$

Then the sets $\varnothing, \mu, \{x\}, \{y\}, \{z\}$ are n^{th}-open sets in μ.
The sets $\varnothing, \mu, \{y, z\}, \{x, z\}, \{x, y\}$ are n^{th}-closed sets in μ.
The sets $\varnothing$ and μ are n^{th}-clopen sets in μ.

Definition 3.4. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be an n^{th}-topological space with $\mu \neq \varnothing$, and let $A \subseteq \mu$. A point $d \in \mu$ is called an n^{th}*-limit point *of A if for every n^{th}-open set u_d containing d, we have*

$$u_d \cap (A \setminus \{d\}) \neq \varnothing.$$

The set of all n^{th}-limit points of A is called the n^{th}-derived set *of A, denoted by*

$$A' = \{d \in \mu : d \text{ is an } n^{th}\text{-limit point of } A\}.$$

Remark 3.1. *The derived set operator in an n^{th}-topological space satisfies the following properties for all $W, M \subseteq \mu$:*

(1) *The derived set of $\varnothing$ is $\varnothing$, i.e., $\varnothing' = \varnothing$.*
(2) *If $W \subseteq M$, then $W' \subseteq M'$.*
(3) $(W \cup M)' = W' \cup M'$.
(4) $(W \cap M)' = W' \cap M'$.

Example 3.3. *Let $\mu = \{x, y, z, q\}$, with topologies:*

$$\gamma_1 = \{\varnothing, \mu, \{x\}, \{x, y\}\}, \quad \gamma_2 = \{\varnothing, \mu, \{x\}\}, \quad \gamma_3 = \{\varnothing, \mu, \{x\}, \{x, z\}\},$$

and consider the set $A = \{x, z\}$.
Let $y \in \mu$. The set $\{x, y\} \in \gamma_1$ is an n^{th}-open set containing y, and we observe:

$$\{x, y\} \cap (A \setminus \{y\}) = \{x\} \neq \varnothing.$$

Thus, y is an n^{th}-limit point of A. Similarly, x and z are also limit points. Therefore, the derived set is

$$A' = \{x, y, z\}.$$

Definition 3.5. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be an n^{th}-topological space with $\mu \neq \varnothing$, and let $W \subseteq \mu$. The n^{th}-closure of W, denoted $\overline{W}$, is defined by:*

$$\overline{W} = W \cup W'.$$

Remark 3.2. *The set $\overline{W}$ is called the n^{th}-closure of W, and it satisfies the following properties:*

(1) *$\overline{W}$ is an n^{th}-closed set.*
(2) *$\overline{W}^c$ is an n^{th}-open set.*
(3) *A point $w \in \overline{W}$ if and only if for every γ_i-open set u_w with $w \in u_w$, we have $u_w \cap W \neq \varnothing$.*
(4) *A point $w \notin \overline{W}$ if and only if for every γ_i-open set u_w with $w \in u_w$, we have $u_w \cap W = \varnothing$.*

Definition 3.6. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be an n^{th}-topological space with $\mu \neq \varnothing$, and let $W \subseteq \mu$. A point $d \in W$ is called an n^{th}-interior point of W if there exists at least one neighborhood $N(d, \varepsilon)$ such that $N(d, \varepsilon) \subseteq W$. The set of all n^{th}-interior points is called the n^{th}-interior set of W, and is denoted by:*

$$W^\circ \equiv \operatorname{Int}(W) = \left(\overline{W^c}\right)^c.$$

Remark 3.3. ***Properties of the n^{th}-interior:*** *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be an n^{th}-topological space and $X, Y \subseteq \mu$. Then:*

(1) $\varnothing^\circ = \varnothing$ *and* $\mu^\circ = \mu$.
(2) $(X \cap Y)^\circ = X^\circ \cap Y^\circ$, *and* $X^\circ \cup Y^\circ \subseteq (X \cup Y)^\circ$.
(3) X° *is a* γ_i*-open set for some* $i \in \{1, 2, \ldots, n\}$.
(4) $x \in X^\circ$ *if and only if there exists a* γ_i*-open set* U_x *such that* $x \in U_x \subseteq X$.

Definition 3.7. *Let* $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ *be an* n^{th}*-topological space with* $\mu \neq \varnothing$*, and let* $W \subseteq \mu$*. A point* $d \in \mu$ *is said to be an* n^{th}*-exterior point of* W *if there exists at least one neighborhood* $N(d, \varepsilon)$ *such that* $N(d, \varepsilon) \cap W = \varnothing$*. The set of all such points is called the* n^{th}*-exterior set of* W*, and is denoted by:*

$$\mathrm{Ex}(W) = \mathrm{Int}(W^c) = \left(\overline{W}\right)^c.$$

Remark 3.4. *Note that* $\mathrm{Ex}(W)$ *is an* n^{th}*-open set.*

Remark 3.5. ***Properties of the exterior set:*** *Let* $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ *be an* n^{th}*-topological space, and let* $W, M \subseteq \mu$*. Then:*

(1) *The exterior set of* $\varnothing$ *is* μ*, and the exterior set of* μ *is* $\varnothing$.
(2) *If* $W \subseteq M$*, then* $\mathrm{Ex}(M) \subseteq \mathrm{Ex}(W)$.
(3) $\mathrm{Ex}(W)$ *is a* γ_i*-open set for some* $i = 1, 2, \ldots, n$.
(4) *A point* $m \in \mathrm{Ex}(W)$ *if and only if there exists a* γ_i*-open set* U_m *such that* $m \in U_m \subseteq W^c$.

Proof of (3). Since $\mathrm{Ex}(W) = \mathrm{Int}(W^c)$, we can write:

$$\mathrm{Ex}(W) = \left(\overline{(W^c)^c}\right)^c = \left(\overline{W}\right)^c.$$

By the definition of the n^{th}-closure set, $\overline{W}$ is a γ_i-closed set. Therefore, its complement, $\left(\overline{W}\right)^c = \mathrm{Ex}(W)$, is a γ_i-open set for $i = 1, 2, \ldots, n$. □

Definition 3.8 (Boundary Set)**.** *Let* $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ *be an* n^{th}*-topological space with* $\mu \neq \varnothing$*, and let* $W \subseteq \mu$*. A point* $d \in \mu$ *is said to be an* n^{th}*-boundary point of* W *if every neighborhood* $N(d, \varepsilon)$ *of* d *satisfies:*

$$N(d, \varepsilon) \cap W \neq \varnothing \quad \textit{and} \quad N(d, \varepsilon) \cap W^c \neq \varnothing.$$

The set of all such points is called the n^{th}*-boundary set of* W*, denoted by:*

$$\mathrm{Bd}(W) = \overline{W} \setminus W^\circ = \overline{W} \cap \overline{W^c}.$$

Remark 3.6. $\mathrm{Bd}(W)$ *is an* n^{th}*-closed set.*

Remark 3.7 (Properties of the Boundary Set)**.** *Let* $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ *be an* n^{th}*-topological space and let* $X, Y \subseteq \mu$*. Then:*

(i) $\mathrm{Bd}(\varnothing) = \mathrm{Bd}(\mu) = \varnothing$.
(ii) $\mathrm{Bd}(X)$ *is a* γ_i*-closed set for each* $i = 1, 2, \ldots, n$.
(iii) *A point* $y \in \mathrm{Bd}(X)$ *if and only if for every* γ_i*-open set* u_y *with* $y \in u_y$*, it holds that:*

$$u_y \cap X \neq \varnothing \quad \textit{and} \quad u_y \cap X^c \neq \varnothing.$$

Proof of (iii). Let $y \in \mathrm{Bd}(X)$. Then by definition:

$$y \in \overline{X} \cap \overline{X^c}.$$

This implies that $y \in \overline{X}$ and $y \in \overline{X^c}$, so every γ_i-open neighborhood u_y of y intersects both X and X^c. Hence:

$$u_y \cap X \neq \emptyset \quad \text{and} \quad u_y \cap X^c \neq \emptyset.$$

□

Definition 3.9 ($T_\circ$-Space). *An n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a $T_\circ$-space if for all distinct points $c \neq d$ in μ, there exists a γ_i-open set u_c such that $c \in u_c$ and $d \notin u_c$, or a γ_i-open set v_d such that $d \in v_d$ and $c \notin v_d$, for some $i = 1, 2, \dots, n$.*

Theorem 3.1. *Let $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ be an n^{th}-topological space. The following statements are equivalent:*

(i) *μ is an n^{th}-T_0 space.*
(ii) *For all $m \neq n$ in μ, we have $m \notin \overline{\{n\}}$ or $n \notin \overline{\{m\}}$.*
(iii) *For all $m \neq n$ in μ, we have $\overline{\{m\}} \neq \overline{\{n\}}$.*

Proof. **(i) ⇒ (ii):** Let $m \neq n \in \mu$. Since μ is an n^{th}-T_0 space, there exists a γ_i-open set u_m such that either:

$$m \in u_m \quad \text{and} \quad n \notin u_m, \quad \text{or} \quad n \in v_n \quad \text{and} \quad m \notin v_n,$$

for some γ_i-open set v_n. Hence, either $u_m \cap \{n\} = \emptyset$ or $v_n \cap \{m\} = \emptyset$, implying that $m \notin \overline{\{n\}}$ or $n \notin \overline{\{m\}}$.

(ii) ⇒ (iii): Assume $m \neq n$. If $m \notin \overline{\{n\}}$ and $m \in \overline{\{m\}}$, then clearly $\overline{\{m\}} \neq \overline{\{n\}}$. Similarly, if $n \notin \overline{\{m\}}$ and $n \in \overline{\{n\}}$, the conclusion follows.

(iii) ⇒ (i): Assume $\overline{\{m\}} \neq \overline{\{n\}}$ for all $m \neq n$ in μ. Without loss of generality, suppose there exists a point $p \in \overline{\{m\}} \setminus \overline{\{n\}}$. Since $\overline{\{m\}}$ is γ_i-closed for some i, its complement $v_n = \mu \setminus \overline{\{m\}}$ is γ_i-open, and $n \in v_n$ while $m \notin v_n$. Hence, μ satisfies the T_0 condition for all $i = 1, 2, \dots, n$, proving that μ is an n^{th}-T_0 space. □

Definition 3.10. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a T_1*-space *if for all $c \neq d$ in μ, there exist γ_i-open sets u_c and v_d such that $c \in u_c$, $d \notin u_c$, $d \in v_d$, and $c \notin v_d$, for all $i = 1, 2, \dots, n$.*

Definition 3.11. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a T_2*-space *(Hausdorff space) if for all $c \neq d$ in μ, there exist γ_i-open sets u_c and v_d such that $c \in u_c$, $d \in v_d$, and $u_c \cap v_d = \emptyset$, for all $i = 1, 2, \dots, n$.*

Definition 3.12. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a $T_{2\frac{1}{2}}$*-space *if for all $c \neq d$ in μ, there exist γ_i-closed sets A_c and B_d such that $c \in A_c$, $d \in B_d$, and $A_c \cap B_d = \emptyset$, for all $i = 1, 2, \dots, n$.*

Definition 3.13. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is said to be n^{th}*-regular *if for every point $c \in \mu$ and every n^{th}-closed set $A \subset \mu$ with $c \notin A$, there exist γ_i-open sets u_c and v_A such that:*

$$c \in u_c, \quad A \subset v_A, \quad \textit{and} \quad u_c \cap v_A = \emptyset,$$

for some $i \in \{1, 2, \dots, n\}$.

Theorem 3.2. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. Then the space is n^{th}-regular if and only if for every point $m \in u_m$, where u_m is a γ_i-open set, there exists a γ_i-open set w_m such that*

$$m \in w_m \subset \overline{w_m} \subset u_m.$$

Proof. ($\Rightarrow$) Suppose the space is n^{th}-regular. Let $m \in u_m$, where u_m is a γ_i-open set. Then the complement u_m^c is γ_i-closed. By regularity, since $m \notin u_m^c$, there exist γ_i-open sets w_m and $v_{u_m^c}$ such that:

$$m \in w_m, \quad u_m^c \subset v_{u_m^c}, \quad \text{and} \quad w_m \cap v_{u_m^c} = \emptyset.$$

Hence, $w_m \subset v^c_{u_m^c}$, and taking closures,

$$\overline{w_m} \subset \overline{v^c_{u_m^c}} = v^c_{u_m^c}.$$

But $u_m^c \subset v_{u_m^c} \Rightarrow v^c_{u_m^c} \subset u_m$, so $\overline{w_m} \subset u_m$. Thus,

$$m \in w_m \subset \overline{w_m} \subset u_m.$$

($\Leftarrow$) Conversely, suppose that for every point $m \in u_m$, there exists a γ_i-open set w_m such that

$$m \in w_m \subset \overline{w_m} \subset u_m.$$

Let $m \notin M$, where M is a γ_i-closed set. Then $m \in M^c$, a γ_i-open set. By hypothesis, there exists γ_i-open w_m such that

$$m \in w_m \subset \overline{w_m} \subset M^c.$$

This implies $M \subset \overline{w_m}^c$, and both w_m and $\overline{w_m}^c$ are γ_i-open. To prove regularity, it suffices to show that

$$w_m \cap \overline{w_m}^c = \emptyset.$$

Suppose not. Then there exists $y \in w_m \cap \overline{w_m}^c$, i.e., $y \in w_m$ and $y \notin \overline{w_m}$. But every point of w_m is in $\overline{w_m}$, a contradiction. Hence,

$$w_m \cap \overline{w_m}^c = \emptyset.$$

Thus, w_m and $\overline{w_m}^c$ are disjoint γ_i-open sets separating m and M, proving that the space is n^{th}-regular. □

Definition 3.14. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called a $\mathbf{T_3}$-space if it is both a n^{th}-regular space and a $\mathbf{T_1}$-space.*

Definition 3.15. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called n^{th}-normal if for every pair of disjoint n^{th}-closed sets M and W in μ, i.e., $M \cap W = \emptyset$, there exist γ_i-open sets u_M and v_W such that*

$$M \subseteq u_M, \quad W \subseteq v_W, \quad \textit{and} \quad u_M \cap v_W = \emptyset,$$

for some $i = 1, 2, \ldots, n$.

Definition 3.16. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called a $\mathbf{T_4}$-space if it is both n^{th}-normal and a $\mathbf{T_1}$-space.*

Definition 3.17. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space and let $A \subseteq \mu$. The complement of a n^{th}-α-open set is called a n^{th}-α-closed set.*

4. N^{TH} Lindelöf Spaces

This section introduces the concept of n^{th} Lindelöf spaces. Furthermore, we address other related notions derived from the concept of n^{th} Lindelöf spaces and investigate related hypotheses.

Definition 4.1. *A n^{th}-open cover is a disjoint collection of n^{th}-open sets that collectively cover the space μ, and is induced by an arbitrary open cover.*

Definition 4.2. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. A cover U of μ is called a n^{th}-open cover of $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ if there exist open covers $U_1, U_2 \subseteq \gamma_i$ such that:*

$$U \subseteq U_1 \cup U_2, \quad U \cap U_1 \neq \emptyset, \quad \textit{and} \quad U \cap U_2 \neq \emptyset.$$

Definition 4.3. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called γ_i-Lindelöf with respect to γ_j if for every γ_i-open cover of μ, there exists a countable γ_j-subcover. This holds for some $i, j \in \{1, 2, \ldots, n\}$, where $i \neq j$.*

Definition 4.4. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a B-compact space, and $(\beta, \lambda_1, \lambda_2, \ldots, \lambda_m)$ be a B-Lindelöf space. Then the product topology*

$$(\mu \times \beta, \gamma_1 \times \lambda_1, \gamma_2 \times \lambda_2, \ldots, \gamma_n \times \lambda_m)$$

is also B-Lindelöf.

Definition 4.5 ([21]). *A tri-topological space $(\mu, \gamma_1, \gamma_2, \gamma_3)$ is called a* tri-Lindelöf space *if for every γ_1-open cover of μ, there exists a countable γ_2-subcover of μ, and conversely.*

Definition 4.6 ([18]). *Let $(\mu, \gamma_1, \gamma_2)$ be a bitopological space, and let $\tilde{U}$ be a cover of μ. Then $\tilde{U}$ is said to be $\gamma_1\gamma_2$-open if $\tilde{U} \subseteq \gamma_1 \cup \gamma_2$.*

Definition 4.7. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called* second countable *(or 2^{nd} countable) if μ has a countable base with respect to each γ_i for all $i = 1, 2, \ldots, n$.*

Definition 4.8. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is said to be* S-Lindelöf *if and only if it is both Lindelöf and n^{th}-Lindelöf.*

Theorem 4.1. *If a n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is second countable, then it is n^{th}-Lindelöf.*

Proof. Let μ be a n^{th} second countable space. Then μ has a countable base

$$\mathcal{B} = \{B_i\}_{i=1}^{\infty}$$

with respect to each γ_i for $i = 1, 2, \ldots, n$. Let

$$\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$$

be a γ_i-open cover of μ for some i. Since $\mathcal{B}$ is a base, for each $x \in \mu$, there exists some $B_j \in \mathcal{B}$ such that $x \in B_j \subseteq U_\alpha$ for some $\alpha \in \Lambda$. Collecting all such B_j's gives a countable subcollection of $\mathcal{U}$ that still covers μ, because the base is countable and every point is covered. Therefore, μ is n^{th}-Lindelöf. □

Remark 4.1. *Every n^{th} compact space is a n^{th} Lindelöf space, but the converse is not necessarily true.*

Proof. Let μ be a n^{th} compact space. Then every γ_i-open cover of μ has a finite γ_j-subcover for all $i \neq j$, $i, j = 1, 2, \ldots, n$. Since finite sets are countable, it follows that μ is n^{th} Lindelöf. However, the converse does not necessarily hold. □

Example 4.1. *Let γ_u be the usual topology on $\mathbb{R}$. The space $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \ldots, \gamma_{u_n})$ is n^{th} Lindelöf but not n^{th} compact. Since*

$$\mathcal{B} = \{(a, b) \mid a < b,\ a, b \in \mathbb{Q}\}$$

is a countable base for $\mathbb{R}$, we have that $\mathbb{R}$ is second countable. Hence, it is Lindelöf in each γ_{u_i}, and therefore the space is n^{th} Lindelöf. However, $\mathbb{R}$ is not compact under the usual topology, and thus not n^{th} compact.

Theorem 4.2. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. If μ is a hereditary Lindelöf space, then μ is S-Lindelöf.*

Proof. Assume $\mathcal{U} = \{u_\alpha : \alpha \in \Gamma\} \cup \{v_\beta : \beta \in \Delta\}$ is a γ_i-open cover of $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$, such that $u_\alpha \in \gamma_1$ for every $\alpha \in \Gamma$ and $v_\beta \in \gamma_2$ for every $\beta \in \Delta$. Since μ is hereditary Lindelöf and $U = \bigcup_{\alpha \in \Gamma} u_\alpha$ is a γ_1-open cover, there exists a countable subset $\Gamma_1 \subset \Gamma$ such that

$$U = \bigcup_{\alpha \in \Gamma_1} u_\alpha.$$

Similarly, since $V = \bigcup_{\beta \in \Delta} v_\beta$ is a γ_2-open cover, there exists a countable subset $\Delta_1 \subset \Delta$ such that

$$V = \bigcup_{\beta \in \Delta_1} v_\beta.$$

Therefore,

$$\{u_\alpha : \alpha \in \Gamma_1\} \cup \{v_\beta : \beta \in \Delta_1\}$$

is a countable subcover of $\mathcal{U}$ that covers μ. Hence, μ is S-Lindelöf. □

Corollary 4.1. *Every second countable space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is a n^{th}-topological space. Thus, μ is n^{th}-S-Lindelöf.*

Example 4.2. *Let μ be a set and γ_1 be the topology on μ generated by the basis β_1, where*

$$\beta_1 = \{[x, y) : x < y;\ x, y \in \mu\}.$$

Then, $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is not hereditary Lindelöf but it is S-Lindelöf.

Theorem 4.3. *A n^{th}-Lindelöf space is preserved under an onto n^{th}-continuous function.*

Proof. Let $i \neq j$ where $i, j \in \{1, 2, \dots, n\}$, and let

$$f : (\mu, \gamma_1, \gamma_2, \dots, \gamma_n) \to (Y, \sigma_1, \sigma_2, \dots, \sigma_n)$$

be a surjective n^{th}-continuous function. Suppose μ is a Lindelöf space. We aim to show that Y is also a Lindelöf space. Let $\mathcal{U} = \{u_\alpha : \alpha \in \Gamma\}$ be a σ_i-open cover of Y. Since f is continuous, the preimage $f^{-1}(u_\alpha)$ is γ_i-open in μ for each $\alpha \in \Gamma$. As f is surjective, the collection $\{f^{-1}(u_\alpha) : \alpha \in \Gamma\}$ forms an open cover of μ. Since μ is Lindelöf, there exists a countable subcover $\{f^{-1}(u_\alpha) : \alpha \in \Gamma_0\}$ with $\Gamma_0 \subset \Gamma$ and $|\Gamma_0| \leq \aleph_0$. Then,

$$\mu \subseteq \bigcup_{\alpha \in \Gamma_0} f^{-1}(u_\alpha).$$

Applying f on both sides, and using surjectivity, we get

$$Y = f(\mu) \subseteq f\left(\bigcup_{\alpha \in \Gamma_0} f^{-1}(u_\alpha)\right) \subseteq \bigcup_{\alpha \in \Gamma_0} u_\alpha.$$

Therefore, $\mathcal{U}$ has a countable σ_j-subcover of Y, and so Y is a Lindelöf space with respect to σ_j. □

Remark 4.2. *We know that a compact subset in a n^{th} Hausdorff space is closed. However, a Lindelöf subset in a n^{th} Hausdorff space need not be closed. For example, $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \dots, \gamma_{u_n})$ is a n^{th} Hausdorff space, and $(0, 1)$ is a Lindelöf subset of $\mathbb{R}$. However, $(0, 1)$ is not n^{th}-closed in $\mathbb{R}$.*

Definition 4.9. *A space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a n^{th}-space if and only if the countable intersection of open sets is open.*

Theorem 4.4. *Every n^{th} Lindelöf subset of a T_2-space is n^{th}-closed.*

Proof. Let $i \neq j$ with $i, j \in \{1, 2, \dots, n\}$, and let A be a Lindelöf subset of the n^{th} T_2-space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$. To show that A is n^{th}-closed, it suffices to show that $\mu \setminus A$ is n^{th}-open. Let $x \in \mu \setminus A$. Then $x \notin A$, and for each $a \in A$, we have $x \neq a$. Since μ is a T_2 space, there exist γ_i-open sets u_a and v_a such that $x \in u_a$, $a \in v_a$, and $u_a \cap v_a = \emptyset$. Thus, the collection $\mathcal{V} = \{v_a : a \in A\}$ is a γ_i-open cover of A. Since A is a n^{th} Lindelöf subset of μ, the cover $\mathcal{V}$ reduces to a countable γ_j-subcover $\{v_{a_\alpha} : \alpha \in \Gamma_0\}$. Therefore,

$$A \subseteq \bigcup_{\alpha \in \Gamma_0} v_{a_\alpha}.$$

For each v_{a_α}, there exists a corresponding u_{a_α} such that $x \in u_{a_\alpha}$ and $u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$. Let

$$v^* = \bigcup_{\alpha \in \Gamma_0} v_{a_\alpha}, \quad u^* = \bigcap_{\alpha \in \Gamma_0} u_{a_\alpha}.$$

Then $A \subseteq v^*$ and $x \in u^*$, with $u^* \cap v^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$ for all α. Hence, $u^* \cap A = \emptyset$, and $u^* \subseteq \mu \setminus A$. Since μ is a n^{th}-space, u^* is γ_i-open as a countable intersection of γ_i-open sets. Thus, for every $x \in \mu \setminus A$, there exists a γ_i-open neighborhood contained in $\mu \setminus A$, and therefore, $\mu \setminus A$ is n^{th}-open. This implies that A is n^{th}-closed. □

Theorem 4.5. *If A is a Lindelöf subset of a n^{th} Hausdorff space μ, then for each $x \notin A$, we can separate x and A into two disjoint open sets in μ.*

Proof. Let $i \neq j$ with $i, j \in \{1, 2, \ldots, n\}$. For each $a \in A$, we have $a \neq x$ since $x \notin A$. Since μ is a n^{th} Hausdorff space, there exist n^{th}-open sets $u_a(x)$ and $v(a)$ such that $x \in u_a(x)$, $a \in v(a)$, and $u_a(x) \cap v(a) = \emptyset$. Hence, the collection $\mathcal{V} = \{v(a) : a \in A\}$ forms a γ_i-open cover of A. Since A is a n^{th} Lindelöf subset of μ, the cover $\mathcal{V}$ reduces to a countable γ_j-subcover $\{v(a_\alpha) : \alpha \in \Gamma_0\}$, where

$$A \subseteq \bigcup_{\alpha \in \Gamma_0} v(a_\alpha) = V.$$

For each $v(a_\alpha)$, there exists a corresponding n^{th}-open set $u_{a_\alpha}(x)$ such that $x \in u_{a_\alpha}(x)$ and $u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset$. Define

$$u = \bigcap_{\alpha \in \Gamma_0} u_{a_\alpha}(x).$$

Since μ is a n^{th} space, u is n^{th}-open. Moreover, for all $\alpha \in \Gamma_0$ we have:

$$u \subseteq u_{a_\alpha}(x) \quad \text{and} \quad u \cap v(a_\alpha) \subseteq u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset.$$

Thus,

$$u \cap V = u \cap \bigcup_{\alpha \in \Gamma_0} v(a_\alpha) = \emptyset.$$

Therefore, $x \in u$ and $A \subseteq V$, with $u \cap V = \emptyset$. This implies that x and A can be separated by two disjoint n^{th}-open sets in μ. □

Theorem 4.6. *Let μ be a n^{th} Lindelöf space, and let Y be a n^{th} topological space. Then, the projection map $P : \mu \times \mu \to Y$ is a closed map.*

Proof. Let $i \neq j$, with $i, j \in \{1, 2, \ldots, n\}$, and let $y \in Y$. Suppose G is open in $\mu \times \mu$ such that $P^{-1}(y) \subseteq G$. We aim to show that there exists a n^{th}-open set v containing y in Y such that $P^{-1}(v) \subseteq G$. Since G is open in $\mu \times \mu$, for every $(x, y) \in G$ with $(x, y) \in \mu \times Y$, there exist basic P-open sets u_x in μ and v_x in Y such that $x \in u_x$, $y \in v_x$, and $(x, y) \in u_x \times v_x \subseteq G$. Hence, the collection $\mathcal{U} = \{u_x : x \in \mu\}$ forms a γ_i-open cover of μ. Since μ is a n^{th} Lindelöf space, there exists a countable subcover $\{u_{x_\alpha} : \alpha \in T_0\}$ with $|T_0| \leq \aleph_0$, such that

$$\mu \subseteq \bigcup_{\alpha \in T_0} u_{x_\alpha}.$$

For each $\alpha \in T_0$, the corresponding v_{x_α} contains y, and so we define:

$$v = \bigcap_{\alpha \in T_0} v_{x_\alpha}.$$

Since T_0 is countable and Y is a n^{th} space, v is n^{th}-open in Y. Now, we have

$$P^{-1}(v) \subseteq \mu \times v \subseteq \bigcup_{\alpha \in T_0} u_{x_\alpha} \times v_{x_\alpha} \subseteq G.$$

Therefore, $P^{-1}(v) \subseteq G$, and hence P is a closed map. □

Theorem 4.7. *The product of two n^{th} Lindelöf spaces, one of which is a P-γ_2-space, is n^{th} Lindelöf.*

Proof. Let μ and Y be two n^{th} Lindelöf spaces, and suppose Y is a P-γ_2-space. By the known result: if for all $y \in Y$, the fibers $f^{-1}(y)$ are n^{th} Lindelöf and Y is n^{th} Lindelöf, then μ is n^{th} Lindelöf, provided $f : \mu \to Y$ is a closed, continuous, and surjective function. Now consider the projection function $P : \mu \times Y \to Y$. This function is continuous and surjective. For each $y \in Y$, we have

$$P^{-1}(y) = \mu \times \{y\} \cong \mu.$$

Since μ is n^{th} Lindelöf, it follows that each fiber $P^{-1}(y)$ is n^{th} Lindelöf. Moreover, since P is continuous and closed, it is a perfect map. Therefore, by the aforementioned result, the product space $\mu \times Y$ is n^{th} Lindelöf. □

Definition 4.10. *Let $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. A subset $A \subset \mu$ is called a n^{th}-nearly open set if*

$$A = \left(\overline{A}\right)^{\circ},$$

where $\overline{A}$ denotes the n^{th}-closure of A, and $\left(\overline{A}\right)^{\circ}$ denotes the n^{th}-interior of $\overline{A}$.

Definition 4.11. *Let $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. A subset $A \subset \mu$ is called a n^{th} nearly open set if*

$$A = \left(\overline{A}\right)^{\circ} \text{ in } \gamma_1 \quad \text{and} \quad A = \left(\overline{A}\right)^{\circ} \text{ in } \gamma_2,$$

where $\overline{A}$ denotes the closure of A, and $\left(\overline{A}\right)^{\circ}$ denotes the interior of $\overline{A}$ in the respective topologies.

5. On n^{th}-Nearly Lindelöf Spaces

In this section, we explore n^{th} nearly Lindelöf spaces, which extend the notion of nearly Lindelöf spaces into the framework of n^{th}-topological spaces. We define n^{th} nearly open covers and examine their role in characterizing n^{th} nearly compact and Lindelöf spaces. Furthermore, the relationship between n^{th} nearly Lindelöf spaces and n^{th} continuous functions is established. The investigation focuses on the preservation of the n^{th} nearly Lindelöf property under specific mappings and its applications in bitopological structures. Examples and counterexamples are presented to illustrate the distinctions between n^{th} nearly compact and n^{th} nearly Lindelöf spaces, emphasizing the implications of these properties in general topology.

Definition 5.1. *A collection $\mathcal{U} = \{u_\alpha : \alpha \in \Gamma\}$ is called a* n^{th}-nearly open cover *of μ if:*

(1) *u_α is a n^{th} nearly open set for all $\alpha \in \Gamma$, and*
(2) $\bigcup_{\alpha \in \Gamma} u_\alpha = \mu$.

Definition 5.2. *Let $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space, and let $\mathcal{U} = \{u_\alpha : \alpha \in \Gamma\}$ be a n^{th} nearly open cover of μ. A subcollection $\mathcal{S}_\Gamma = \{u_{\alpha_\beta} : \beta \in \Lambda, \Lambda \subseteq \Gamma\}$ is called a* n^{th} nearly subcover *of μ if*

$$\bigcup_{\beta \in \Lambda} u_{\alpha_\beta} = \mu.$$

Definition 5.3 ([16]). *Let $\mu = (\mu, \gamma)$ be a topological space. We say that μ is a* nearly compact space *if every nearly open cover of μ has a finite nearly subcover.*

Definition 5.4. *Let $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. Then, μ is called a* n^{th} nearly compact space *if every n^{th} nearly open cover has a finite n^{th} nearly subcover.*

Definition 5.5 ([17]). *Let $\mu = (\mu, \gamma)$ be a topological space. Then, μ is called a* nearly Lindelöf space *if every nearly open cover of μ has a countable nearly subcover.*

Definition 5.6. *Let $\mu = (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. Then, μ is called a* n^{th} nearly Lindelöf space *if every n^{th} nearly open cover has a countable nearly subcover.*

Definition 5.7. *A n^{th}-topological space $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ is called* n^{th} S-nearly Lindelöf *if and only if it is both nearly Lindelöf and n^{th} nearly Lindelöf.*

Theorem 5.1. *Every n^{th} nearly compact space is a n^{th} nearly Lindelöf space, but the converse is not necessarily true.*

Proof. Let μ be a n^{th} nearly compact space. Then, every nearly γ_i-open cover of μ has a finite γ_j-subcover of μ. Thus, for each nearly γ_i-open cover of μ, there exists a countable γ_j-subcover of μ for all $i \neq j$, where $(i, j = 1, 2, \ldots, n)$. Hence, μ is n^{th} nearly Lindelöf. The converse, however, is not necessarily true. □

Example 5.1. *Consider the space $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \ldots, \gamma_{u_n})$, where γ_u is the usual topology on $\mathbb{R}$. Since the set $\mathcal{B} = \{(a, b) : a < b, a, b \in \mathbb{Q}\}$ is a countable base for $(\mathbb{R}, \gamma_u)$, the space is second countable. Every second countable space is Lindelöf, hence $(\mathbb{R}, \gamma_u)$ is Lindelöf. Therefore, $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \ldots, \gamma_{u_n})$ is n^{th} Lindelöf, and consequently n^{th} nearly Lindelöf. However, $(\mathbb{R}, \gamma_u)$ is not compact because the open cover $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ does not admit a finite subcover. Hence, $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \ldots, \gamma_{u_n})$ is not n^{th} compact and thus not n^{th} nearly compact.*

Theorem 5.2. *Let $(\mu, \gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n^{th}-topological space. If μ is a hereditary n^{th}-nearly Lindelöf space, then μ is n^{th} S-nearly Lindelöf.*

Theorem 5.3. *A n^{th} nearly Lindelöf space is preserved under an onto n^{th} continuous function.*

Proof. Let $i \neq j$, where $(i, j = 1, 2, \ldots, n)$. Let

$$F : (\mu, \gamma_1, \gamma_2, \ldots, \gamma_n) \to (Y, \sigma_1, \sigma_2, \ldots, \sigma_n)$$

be a surjective continuous function, and suppose that μ is a n^{th}-nearly Lindelöf space. We aim to show that Y is also a n^{th}-nearly Lindelöf space. Assume $\mathcal{U} = \{u_\alpha : \alpha \in \lambda\}$ is a nearly γ_i-open cover of Y. Then each u_α is open for all $\alpha \in \lambda$. Since F is continuous, the preimage $F^{-1}(u_\alpha)$ is open in μ for each $\alpha \in \lambda$. As F is surjective, the collection $\{F^{-1}(u_\alpha) : \alpha \in \lambda\}$ forms an open cover of μ. Because μ is a nearly Lindelöf space, there exists a countable subcover $\{F^{-1}(u_\alpha) : \alpha \in \Gamma\}$ with $\Gamma \subset \lambda$ and $|\Gamma| \leq \aleph_0$, such that

$$\mu \subseteq \bigcup_{\alpha \in \Gamma} F^{-1}(u_\alpha).$$

Since F is surjective, we have

$$Y = F(\mu) \subseteq F\left(\bigcup_{\alpha\in\Gamma} F^{-1}(u_\alpha)\right) \subseteq \bigcup_{\alpha\in\Gamma} u_\alpha.$$

Hence, $\mathcal{U}$ has a countable γ_j-subcover of Y, which proves that Y is a nearly Lindelöf space. □

Remark 5.1. *A compact subset in a n^{th} nearly T_2-space is closed, but a nearly Lindelöf subset in a n^{th} nearly T_2-space need not be closed. For example, $(\mathbb{R}, \gamma_{u_1}, \gamma_{u_2}, \dots, \gamma_{u_n})$ is a n^{th} nearly γ_2-space, and $(0,1)$ is a nearly Lindelöf subset of $\mathbb{R}$. However, $(0,1)$ is not n^{th} nearly closed in $\mathbb{R}$.*

Definition 5.8. *A space $(\mu, \gamma_1, \gamma_2, \dots, \gamma_n)$ is called a n^{th} nearly n^{th} space if and only if the countable intersection of nearly open sets is n^{th} nearly open.*

Theorem 5.4. *Every nearly Lindelöf subset of a nearly n^{th}-T_2 space is n^{th} nearly closed.*

Proof. Let $i \neq j$, where $(i, j = 1, 2, \dots, n)$, and let A be a nearly Lindelöf subset of a nearly n^{th}-T_2 space μ. To show that A is n^{th} nearly closed, it suffices to prove that $\mu \setminus A$ is n^{th} nearly open. Take $x \in \mu \setminus A$. Then $x \notin A$, and for each $a \in A$, we have $x \neq a$. Since μ is a n^{th} nearly T_2 space, there exist nearly open sets u_a and v_a in μ such that $x \in u_a$, $a \in v_a$, and $u_a \cap v_a = \emptyset$. Thus, $\mathcal{V} = \{v_a : a \in A\}$ forms a nearly γ_i-open cover of A. Since A is a n^{th} nearly Lindelöf subset of μ, this cover admits a countable subcover $\{v_{a_\alpha} : \alpha \in T_0\}$, where T_0 is countable, such that:

$$A \subseteq \bigcup_{\alpha\in T_0} v_{a_\alpha}.$$

For each v_{a_α}, there exists a corresponding nearly open set u_{a_α} such that $x \in u_{a_\alpha}$ and $u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$. Define

$$V^* = \bigcup_{\alpha\in T_0} v_{a_\alpha}, \quad U^* = \bigcap_{\alpha\in T_0} u_{a_\alpha}.$$

Then $A \subseteq V^*$, $x \in U^*$, and

$$U^* \cap V^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset \quad \text{for all } \alpha \in T_0.$$

Hence, $U^* \cap A \subseteq U^* \cap V^* = \emptyset$, so $U^* \subseteq \mu \setminus A$. Since U^* is the countable intersection of nearly open sets and μ is a n^{th} nearly space, U^* is n^{th} nearly open. Therefore, $\mu \setminus A$ is n^{th} nearly open, which proves that A is n^{th} nearly closed. □

Theorem 5.5. *Let A be a nearly Lindelöf subset of a n^{th} nearly T_2-space μ. Then, for each $x \notin A$, we can separate x and A into two disjoint nearly open sets in μ.*

Proof. Let $i \neq j$, where $(i, j = 1, 2, \dots, n)$. For each $a \in A$, since $x \notin A$, we have $x \neq a$. As μ is a n^{th} nearly T_2-space, there exist n^{th} nearly open sets $u_a(x)$ and $v(a)$ in μ such that:

$$x \in u_a(x), \quad a \in v(a), \quad \text{and} \quad u_a(x) \cap v(a) = \emptyset.$$

Thus, the collection $\mathcal{V} = \{v(a) : a \in A\}$ forms a nearly γ_i-open cover of A. Since A is a n^{th} nearly Lindelöf subset of μ, there exists a countable subcover $\{v(a_\alpha) : \alpha \in T_0\}$ such that:

$$A \subseteq \bigcup_{\alpha \in T_0} v(a_\alpha) = V.$$

For each $\alpha \in T_0$, there exists a corresponding n^{th} nearly open set $u_{a_\alpha}(x)$ satisfying:

$$x \in u_{a_\alpha}(x), \quad u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset.$$

Define

$$U = \bigcap_{\alpha \in T_0} u_{a_\alpha}(x).$$

Since μ is a n^{th} nearly space and T_0 is countable, U is n^{th} nearly open. Moreover,

$$U \cap v(a_\alpha) \subseteq u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset \quad \text{for all } \alpha \in T_0,$$

and hence,

$$U \cap V = \emptyset.$$

Therefore, $x \in U$, $A \subseteq V$, and $U \cap V = \emptyset$, which shows that x and A can be separated into two disjoint n^{th} nearly open sets in μ. □

Theorem 5.6. *Every pair of disjoint nearly Lindelöf subsets in a n^{th} nearly Hausdorff space can be separated by disjoint nearly open sets in μ.*

Proof. Let $i \neq j$, where $(i, j = 1, 2, \ldots, n)$. Assume A and B are disjoint nearly Lindelöf subsets of a n^{th} nearly Hausdorff space μ. For each $a \in A$, we have $a \notin B$ since $A \cap B = \emptyset$. By a previous theorem, there exist nearly γ_i-open sets u_a and v_a in μ such that:

$$a \in u_a, \quad B \subseteq v_a, \quad \text{and} \quad u_a \cap v_a = \emptyset.$$

Thus, the collection $\mathcal{U} = \{u_a : a \in A\}$ forms a nearly γ_i-open cover of A. Since A is nearly Lindelöf, there exists a countable subcover $\{u_{a_\alpha} : \alpha \in T_0\}$, where T_0 is countable. Define:

$$U = \bigcup_{\alpha \in T_0} u_{a_\alpha}.$$

Then $A \subseteq U$ and U is nearly open. For each $\alpha \in T_0$, there exists a corresponding nearly open set v_{a_α} such that:

$$B \subseteq v_{a_\alpha} \quad \text{and} \quad u_{a_\alpha} \cap v_{a_\alpha} = \emptyset.$$

Let

$$V = \bigcap_{\alpha \in T_0} v_{a_\alpha}.$$

Since T_0 is countable and μ is a n^{th} nearly space, V is nearly open. Moreover, $B \subseteq V$ and:

$$V \cap u_{a_\alpha} \subseteq v_{a_\alpha} \cap u_{a_\alpha} = \emptyset \quad \text{for all } \alpha \in T_0,$$

so:

$$V \cap U = \emptyset.$$

Therefore, U and V are disjoint nearly open sets in μ such that $A \subseteq U$ and $B \subseteq V$, completing the proof. □

Theorem 5.7. *Let μ be a n^{th}-nearly Lindelöf space and Y a n^{th} nearly space. Then, the projection $P : \mu \times \mu \to Y$ is n^{th}-closed.*

Proof. Let $i \neq j$, where $(i, j = 1, 2, \ldots, n)$. Let $y \in Y$ and suppose G is an open set in $\mu \times \mu$ such that $P^{-1}(y) \in G$. We aim to show that there exists a n^{th} nearly open set v containing y in Y such that $P^{-1}(v) \subseteq G$. Since G is open in $\mu \times \mu$, for each $(x, y) \in G$, where $(x, y) \in \mu \times Y$, there exist n^{th} nearly open basic sets u_x in μ and v_x in Y such that

$$x \in u_x, \quad y \in v_x, \quad \text{and} \quad (x, y) \in u_x \times v_x \subseteq G.$$

Then, the collection $\mathcal{U} = \{u_x : x \in \mu\}$ forms a γ_i-nearly open cover of μ. Since μ is a n^{th} nearly Lindelöf space, this cover admits a countable γ_j-subcover

$$\{u_{x_\alpha} : \alpha \in T_0\}, \quad \text{where } |T_0| \leq \aleph_0.$$

Thus, $\mu \subseteq \bigcup_{\alpha \in T_0} u_{x_\alpha}$. For each u_{x_α}, there exists a corresponding v_{x_α} such that $y \in v_{x_\alpha}$. Let

$$v = \bigcap_{\alpha \in T_0} v_{x_\alpha}.$$

Since T_0 is countable and Y is a n^{th} nearly space, v is nearly open in Y. It follows that

$$P^{-1}(v) \subseteq \mu \times v \subseteq \bigcup_{\alpha \in T_0} (u_{x_\alpha} \times v_{x_\alpha}) \subseteq G.$$

Hence, $P^{-1}(v) \subseteq G$, which shows that P is n^{th}-closed. □

Theorem 5.8. *Let $f : \mu \to Y$ be a n^{th}-closed, continuous, and surjective function. If for all $y \in Y$, the fibers $f^{-1}(y)$ are n^{th}-nearly Lindelöf, and Y itself is n^{th}-nearly Lindelöf, then μ is n^{th}-nearly Lindelöf.*

Proof. Let $i \neq j$, where $i, j = 1, 2, \ldots, n$. Let $\mathcal{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a nearly γ_i-open cover of μ. For each $y \in Y$, we consider the fiber $f^{-1}(y) \subseteq \mu$. Then $\mathcal{U}$ is a cover for $f^{-1}(y)$. Since $f^{-1}(y)$ is n^{th}-nearly Lindelöf, there exists a countable subfamily $\{u_{\alpha_y}\}$ from $\mathcal{U}$ such that

$$f^{-1}(y) \subseteq \bigcup_{\alpha_y \in T_y} u_{\alpha_y},$$

where T_y is a countable index set. Define

$$F_y = \mu \setminus \bigcup_{\alpha_y \in T_y} u_{\alpha_y}, \quad \text{so that} \quad f^{-1}(y) \cap F_y = \emptyset.$$

Therefore,

$$y \notin f(F_y) \quad \Rightarrow \quad y \in Y \setminus f(F_y) =: O_y.$$

Since F_y is closed and f is closed, $f(F_y)$ is closed in Y, making O_y open in Y. Thus, the collection $\mathcal{O} = \{O_y : y \in Y\}$ is a nearly γ_i-open cover of Y. Because Y is n^{th}-nearly Lindelöf, there exists a countable subcover $\{O_{y_r} : r \in T_0\}$ such that:

$$Y \subseteq \bigcup_{r \in T_0} O_{y_r}.$$

Then,

$$\mu = f^{-1}(Y) \subseteq \bigcup_{r \in T_0} f^{-1}(O_{y_r}) \subseteq \bigcup_{r \in T_0} \bigcup_{\alpha_{y_r} \in T_{y_r}} u_{\alpha_{y_r}}.$$

This shows that μ is covered by a countable subcollection of $\mathcal{U}$. Hence, μ is n^{th}-nearly Lindelöf. □

Theorem 5.9. *The product of two n^{th}-nearly Lindelöf spaces, one of which is a n^{th} nearly T_2 space, is n^{th}-nearly Lindelöf.*

Proof. Let μ and Y be two n^{th}-nearly Lindelöf spaces, and suppose Y is a n^{th} nearly T_2 space. Consider the projection function

$$P : \mu \times Y \to Y, \quad P(x, y) = y.$$

Then P is continuous and surjective. Since Y is a n^{th} nearly T_2 space, it follows that P is n^{th}-closed. For each $y \in Y$, we have:

$$P^{-1}(y) = \mu \times \{y\} \cong \mu,$$

which implies that the fiber $P^{-1}(y)$ is homeomorphic to μ, and hence is n^{th}-nearly Lindelöf. Since P is a continuous, closed, and surjective map with n^{th}-nearly Lindelöf fibers, and Y itself is n^{th}-nearly Lindelöf, it follows that the product space $\mu \times Y$ is also n^{th}-nearly Lindelöf. □

6. Conclusion

In this paper, we have investigated several properties and results concerning nearly Lindelöf spaces within the framework of general topology and n^{th}-topological spaces. By extending classical notions such as Lindelöfness and compactness to the context of nearly open sets and n^{th}-topological structures, we have developed a deeper understanding of the behavior and structure of these generalized spaces. Notable results include the preservation of the n^{th}-nearly Lindelöf property under continuous and surjective mappings, the separability of disjoint nearly Lindelöf subsets in n^{th}-nearly Hausdorff spaces, and the conditions under which product spaces retain the nearly Lindelöf property. Through the use of illustrative examples and counterexamples, we have clarified distinctions between nearly Lindelöf spaces, compact spaces, and nearly compact spaces. These contributions not only enrich the theoretical development of generalized topological spaces but also suggest potential applications in functional analysis and mathematical modeling. Future research directions may include the study of further generalizations, the interplay between nearly Lindelöfness and other topological properties, and the extension of these results to more complex structures such as fuzzy topologies and multi-topological frameworks.

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