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# Generalized Soft Union Bi-Ideals and Interior Ideals of Semigroups: Soft Union Bi-Interior Ideals of Semigroups

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ABSTRACT. Generalizing the ideals of an algebraic structure has shown to be both beneficial and interesting for mathematicians. In this context, the idea of the bi-interior ideal was introduced as a generalization of the bi-ideal and interior ideal of a semigroup. By introducing "soft union (S-uni) bi-interior ideals of semigroups", we apply this idea to semigroups and soft set theory in this study. Finding the relationships between S-uni bi-interior ideals and other specific kinds of S-uni ideals of a semigroup is the main aim of this study. Our results show that an S-uni bi-interior ideal is an S-uni subsemigroup of a special soft simple semigroup, and that the S-uni bi-interior ideal of semigroup is a generalization of the S-uni left (right/two-sided) ideal, bi-ideal, interior ideal, and quasi-ideal, however, the converses are not true with counterexamples. We demonstrate that the semigroup should be a special soft simple semigroup in order to satisfy the converses. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of regarding soft set operations and notions supporting our assertions with particular, illuminating examples.

## 1. Introduction

In many areas of mathematics, semigroups have a crucial role given that they serve as the abstract algebraic structure for "memoryless" systems that reset at each iteration. Originally studied formally in the early 1900s, semigroups are crucial models for linear time-invariant systems in practical mathematics. It is crucial to theoretical computer science to investigate finite

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semigroups since they are inseparable from finite automata. Furthermore, semigroups and Markov processes are connected in probability theory. Since the idea of ideals is fundamental to comprehending mathematical structures and their uses, many mathematicians have concentrated a significant portion of their study on generalizing ideals in algebraic structures. In particular, the generalization of ideals in algebraic structures is essential for further exploration of these structures. Numerous mathematicians have established significant results and characterizations of algebraic structures by utilizing the concept and properties of generalized ideals. For the theory of algebraic numbers, Dedekind established the concept of ideals, and Noether expanded it to include associative rings. The concept of an ideal is extended by the concept of a one-sided ideal of any algebraic structure, and the concepts of a one-sided ideal and a two-sided ideal remain central to ring theory.

Good and Hughes [1] proposed the notion of bi-ideals for semigroups in 1952. The concept of quasi-ideals was initially introduced by Steinfeld [2] for semigroups and subsequently for rings. Bi-ideals are generalizations of quasi-ideals, while quasi-ideals are generalizations of left and right ideals. Lajos [3] established the concept of the interior ideal, and Szasz [4,5] expanded on it. Interior ideals are generalizations of ideals. Several new various kinds of ideals of semigroup, which are generalizations of the existing ones was defined by Rao [6–9]. Additionally, Baupradist et al. [10] introduced the concept of essential ideals of semigroups.

In 1999, the "Soft Set Theory" was first presented by Molodtsov [11] in order to comprehend and provide appropriate solutions for problems involving uncertainty. Since then, a great deal of substantial study has been done on soft set notions and operations. Çağman and Enginoğlu [12] revised the notions and operations of soft sets. Sezgin [13] and Sezgin et al. [14] using soft sets in the application of semigroup theory, defined various soft union (S-uni) ideals of semigroups and thoroughly examined their fundamental properties. The soft forms of different algebraic structures have been studied in [15-41].

As a generalization of bi-ideals and interior ideals of semigroups, Rao [6] established the concept of bi-interior ideals of semigroups and examined their characteristics. The concept of bi-interior ideals has also been studied by Rao and Venkateswarlu [42] for Γ-semirings, and Rao [43] for Γ-semigroup. By introducing "S-uni bi-interior ideals of semigroups", we apply this idea to semigroups and soft set theory in this study. We obtain the relationships between S-uni bi-interior ideals and various kinds S-uni ideals of semigroups. Our results show every S-uni bi-interior ideal of a special soft simple semigroup is an S-uni subsemigroup and S-uni bi-interior ideal is a generalization of S-uni left ideal, right ideal, ideal, bi-ideal, interior ideal, and quasi-ideal, however, the converses are not true with counterexamples. We demonstrate that the semigroup should be a special soft simple semigroup in order to satisfy the converses. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of regarding soft set operations and notions supporting our assertions with particular, illuminating examples. There are four sections in this study. An introduction to the subject is given in Section 1. The basic concepts of semigroup and soft sets, together with relevant definitions and implications, are presented in Section 2. In section 3, the concept of S-uni bi-interior ideal of semigroups was introduced and using specific examples to examine their characteristics and how they relate to other kinds of S-uni ideals. Our findings are outlined in Section 4 along with some directions for future study.

## 2. Preliminaries

Throughout this paper, $S$ denotes a semigroup. $\emptyset \neq Ţ \subseteq S$ is called a subsemigroup of $S$ if $ŢŢ \subseteq Ţ$, is called a bi-ideal of $S$ if $ŢŢ \subseteq Ţ$ and $ŢSŢ \subseteq Ţ$, is called an interior ideal of $S$ if $ŢŢ \subseteq Ţ$ and $SŢS \subseteq Ţ$, and is called a bi-interior ideal of $S$ if $SŢS \cap ŢSŢ \subseteq Ţ$.

**Definition 2.1.** [11, 12] Let $E$ be the parameter set, $U$ be the universal set, $P(U)$ be the power set of $U$, and $Ꝁ \subseteq E$. The soft set $f_{Ꝁ}$ over $U$ is a function such that $f_{Ꝁ}: E \to P(U)$, where for all $\mathrm{v} \notin Ꝁ$, $f_{Ꝁ}(\mathrm{v}) = \emptyset$. That is,

$$f_{Ꝁ} = \{(\mathrm{v}, f_{Ꝁ}(\mathrm{v})): \mathrm{v} \in E, f_{Ꝁ}(\mathrm{v}) \in P(U)\}$$

The set of all soft sets over $U$ is designated by $S_E(U)$ throughout this paper.

**Definition 2.2.** [12] Let $f_{Ꞵ} \in S_E(U)$. If $f_{Ꞵ}(\upsilon) = \emptyset$ for all $\upsilon \in E$, then $f_{Ꞵ}$ is called a null soft set and denoted by $\emptyset_E$.

**Definition 2.3.** [12] Let $f_{Ꞵ}, f_{\mathcal{T}} \in S_E(U)$. If $f_{Ꞵ}(\omega) \subseteq f_{\mathcal{T}}(\omega)$, for all $\omega \in E$, then $f_{Ꞵ}$ is a soft subset of $f_{\mathcal{T}}$ and indicated by $f_{Ꞵ} \,\tilde{\subseteq}\, f_{\mathcal{T}}$. If $f_{Ꞵ}(\omega) = f_{\mathcal{T}}(\omega)$, for all $\omega \in E$, then $f_{Ꞵ}$ is called soft equal to $f_{\mathcal{T}}$ and denoted by $f_{Ꞵ} = f_{\mathcal{T}}$.

**Definition 2.4.** [12] Let $f_{Ꞻ}, f_{\mathrm{B}} \in S_E(U)$. The union (intersection) of $f_{Ꞻ}$ and $f_{\mathrm{B}}$ is the soft set $f_{Ꞻ} \,\tilde{\cup}\, f_{\mathrm{B}}$ $(f_{Ꞻ} \,\tilde{\cap}\, f_{\mathrm{B}})$ where $(f_{Ꞻ} \,\tilde{\cup}\, f_{\mathrm{B}})(\hbar) = f_{Ꞻ}(\hbar) \cup f_{\mathrm{B}}(\hbar)$ $\left((f_{Ꞻ} \,\tilde{\cap}\, f_{\mathrm{B}})(\hbar) = f_{Ꞻ}(\hbar) \cap f_{\mathrm{B}}(\hbar)\right)$, for all $\hbar \in E$, respectively.

**Definition 2.5.** [15] Let $f_{\mathcal{H}} \in S_E(U)$ and $\text{ɓ} \subseteq U$. The lower ɓ-inclusion of $f_{\mathcal{H}}$, denoted by $\mathcal{S}(f_{\mathcal{H}}; \text{ɓ})$, is defined as

$$\mathcal{S}(f_{\mathcal{H}}; \text{ɓ}) = \{x \in \mathcal{H} \mid f_{\mathcal{H}}(x) \subseteq \text{ɓ}\}$$

**Definition 2.6.** [13] Let $\hbar_S, \text{ɓ}_S \in S_S(U)$. S-uni product $\hbar_S * \text{ɓ}_S$ is defined by

$$(\hbar_S * \text{ɓ}_S)(\eta) = \begin{cases} \bigcap_{\eta = \text{ɯ}\text{ȡ}} \{\hbar_S(\text{ɯ}) \cup \text{ɓ}_S(\text{ȡ})\}, & if\ \exists \text{ɯ}, \text{ȡ} \in S\ such\ that\ \eta = \text{ɯ}\text{ȡ} \\ U, & otherwise \end{cases}$$

**Theorem 2.7.** [13] Let $d_S, \eta_S, \mu_S \in S_S(U)$. Then,

i. $(d_S * \eta_S) * \mu_S = d_S * (\eta_S * \mu_S)$

ii. $d_S * \eta_S \neq d_S * \eta_S$, generally.

iii. $d_S * (\eta_S \,\tilde{\cup}\, \mu_S) = (d_S * \eta_S) \,\tilde{\cup}\, (d_S * \mu_S)$ and $(d_S \,\tilde{\cup}\, \eta_S) * \mu_S = (d_S * \mu_S) \,\tilde{\cup}\, (\eta_S * \mu_S)$

iv. $d_S * (\eta_S \,\tilde{\cap}\, \mu_S) = (d_S * \eta_S) \,\tilde{\cap}\, (d_S * \mu_S)$ and $(d_S \,\tilde{\cap}\, \eta_S) * \mu_S = (d_S * \mu_S) \,\tilde{\cap}\, (\eta_S * \mu_S)$

v. If $d_S \tilde{\subseteq} \eta_S$, then $d_S * \mu_S \tilde{\subseteq} \eta_S * \mu_S$ and $\mu_S * d_S \tilde{\subseteq} \mu_S * \eta_S$

vi. If $\wp_S, k_S \in S_S(U)$ such that $\wp_S \tilde{\subseteq} d_S$ and $k_S \tilde{\subseteq} \eta_S$, then $\wp_S * k_S \tilde{\subseteq} d_S * \eta_S$.

**Definition 2.8.** [13, 14] Let $\vartheta_S \in S_S(U)$. Then, $\vartheta_S$ is called

*i.* an S-uni subsemigroup (ꝸꝸ) if $\vartheta_S(ab) \subseteq \vartheta_S(a) \cup \vartheta_S(b)$ for all $a, b \in S$,

*ii.* an S-uni left (right) ideal (Ł(Ꝟ)-ideal) if $\vartheta_S(\mathrm{n}v) \subseteq \vartheta_S(v)$ $\left(\vartheta_S(\mathrm{n}v) \subseteq \vartheta_S(\mathrm{n})\right)$ for all $\mathrm{n}, v \in S$, and is called an S-uni two-sided ideal (S-uni ideal) if it is both S-uni Ł-ideal and S-uni Ꝟ-ideal,

*iii.* an S-uni bi-ideal (Ƀ-ideal) if $\vartheta_S$ is an S-uni ꝸꝸ and $\vartheta_S(\mathrm{jn}\rho) \subseteq \vartheta_S(\mathrm{j}) \cup \vartheta_S(\rho)$ for all $\mathrm{j}, \mathrm{n}, \rho \in S$,

*iv.* an S-uni interior ideal ($I$-ideal) if $\vartheta_S(\mathrm{jn}\rho) \subseteq \vartheta_S(\mathrm{n})$ for all $\mathrm{j}, \mathrm{n}, \rho \in S$.

Note that in [13], the definition of "S-uni ꝸꝸ" is given as "S-uni semigroup of $S$ over $U$"; however in this paper, without loss of generality, we prefer to use "S-uni ꝸꝸ". Moreover, for the sake of brevity, the phrases such "of $S$ over $U$" is not used throughout this paper.

If $\vartheta_S(x) = \emptyset$ for all $x \in S$, then $\vartheta_S$ is an S-uni ꝸꝸ (Ł-ideal, Ꝟ-ideal, ideal, Ƀ-ideal, $I$-ideal). Such a kind of S-uni ꝸꝸ (Ł-ideal, Ꝟ-ideal, ideal, Ƀ-ideal, $I$-ideal) is denoted by $\tilde{\Theta}$. It is obvious that $\tilde{\Theta}(x) = \emptyset$ for all $x \in S$ [13, 14].

**Definition 2.9.** [14] A soft set $\mathfrak{h}_S$ is called an S-uni quasi-ideal (Ꝗ-ideal) if $\left(\tilde{\Theta} * \mathfrak{h}_S\right) \tilde{\cup} \left(\mathfrak{h}_S * \tilde{\Theta}\right) \tilde{\supseteq} \mathfrak{h}_S$.

**Theorem 2.10.** [13] Let $\vartheta_S \in S_S(U)$. Then,

**i)** $\tilde{\Theta} * \tilde{\Theta} \tilde{\supseteq} \tilde{\Theta}$

**ii)** $\tilde{\Theta} * \vartheta_S \tilde{\supseteq} \tilde{\Theta}$ and $\vartheta_S * \tilde{\Theta} \tilde{\supseteq} \tilde{\Theta}$

**iii)** $\vartheta_S \tilde{\cap} \tilde{\Theta} = \tilde{\Theta}$ and $\vartheta_S \tilde{\cup} \tilde{\Theta} = \vartheta_S$

**Theorem 2.11.** [13, 14] Let $\vartheta_S \in S_S(U)$. Then,

(1) $\vartheta_S$ is an S-uni ꝸꝸ iff $\vartheta_S * \vartheta_S \tilde{\supseteq} \vartheta_S$

(2) $\vartheta_S$ is an S-uni Ł(Ꝟ)-ideal iff $\tilde{\Theta} * \vartheta_S \tilde{\supseteq} \vartheta_S$ $\left(\vartheta_S * \tilde{\Theta} \tilde{\supseteq} \vartheta_S\right)$

(3) $\vartheta_S$ is an S-uni Ƀ-ideal iff $\vartheta_S * \vartheta_S \tilde{\supseteq} \vartheta_S$ and $\vartheta_S * \tilde{\Theta} * \vartheta_S \tilde{\supseteq} \vartheta_S$

(4) $\vartheta_S$ is an S-uni $I$-ideal iff $\tilde{\Theta} * \vartheta_S * \tilde{\Theta} \tilde{\supseteq} \vartheta_S$

**Theorem 2.12.** [14] Every S-uni Ꝗ-ideal is an S-uni Ƀ-ideal.

Now, the concept of a special soft left (right) simple semigroup will be introduced in order to characterize S-uni ideals.

**Definition 2.13.** Let $\mathrm{f}_S \in S_S(U)$. Then, $S$ is called a special soft left simple semigroup (with respect to $\mathrm{f}_S$) if $\tilde{\Theta} = \tilde{\Theta} * \mathrm{f}_S$, is called a special soft right simple semigroup (with respect to $\mathrm{f}_S$) if $\tilde{\Theta} = \mathrm{f}_S * \tilde{\Theta}$, is called a special soft simple semigroup (with respect to $\mathrm{f}_S$) if $\tilde{\Theta} = \tilde{\Theta} * \mathrm{f}_S = \mathrm{f}_S * \tilde{\Theta}$. If $S$ is a special soft (left/right) simple semigroup with respect to all soft sets over $U$, then it is called a special soft (left/right) simple semigroup.

Special soft (left/right) simple semigroup is used in this paper as "special soft (left/right) simple".

**Example 2.14.** Consider the semigroup $S = \{$Ⱥ, Ꝉ, ⫛$\}$ defined by Table 1:

**Table 1.** Cayley table of '⍟' binary operation.

| ⍟ | Ⱥ | Ꝉ | ⫛ |
|---|---|---|---|
| Ⱥ | ⫛ | Ꝉ | Ⱥ |
| Ꝉ | Ꝉ | Ꝉ | Ꝉ |
| ⫛ | Ⱥ | Ꝉ | ⫛ |

Let $\mathrm{f}_S$ be soft set over $U = D_2 = \{\langle x, y\rangle : x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$$\mathrm{f}_S = \{(\text{Ⱥ}, \{e\}), (\text{Ꝉ}, \{x, y\}), (\text{⫛}, \{yx\})\}$$

By considering

$$\widetilde{\Theta} = \{(\text{Ⱥ}, \emptyset), (\text{Ꝉ}, \emptyset), (\text{⫛}, \emptyset)\}$$

we obtain that $S$ is special soft left simple with respect to $\mathrm{f}_S$. In fact, since

$$\widetilde{\Theta} * \mathrm{f}_S = \{(\text{Ⱥ}, \emptyset), (\text{Ꝉ}, \emptyset), (\text{⫛}, \emptyset)\}$$

we obtain that $\widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S$. Similarly, $S$ is special soft right simple with respect to $\mathrm{f}_S$. In fact, since

$$\mathrm{f}_S * \widetilde{\Theta} = \{(\text{Ⱥ}, \emptyset), (\text{Ꝉ}, \emptyset), (\text{⫛}, \emptyset)\}$$

we obtain $\widetilde{\Theta} = \mathrm{f}_S * \widetilde{\Theta}$. Hence, $\widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S = \mathrm{f}_S * \widetilde{\Theta}$. Therefore, $S$ is special soft simple with respect to $\mathrm{f}_S$.

**Theorem 2.15.** Let $S$ be special soft simple. Then, the following conditions hold.

i. Every S-uni Ƀ-ideal is an S-uni Ꝗ-ideal (Here, $S$ is enough to be special soft left or right simple).

ii. Every S-uni Ꝗ-ideal is an S-uni ideal.

iii. Every S-uni Ƀ-ideal is an S-uni ideal.

iv. Every S-uni $I$-ideal is an S-uni ideal.

**Proof:** (i) The proof is presented only for special soft left simple semigroups, as the proof for special soft right simple semigroups can be shown similarly. Let $S$ be special soft left simple and $\mathrm{f}_S$ be an S-uni Ƀ-ideal. Then, $\widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S$ and $\mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S \,\widetilde{\supseteq}\, \mathrm{f}_S$. Thus,

$$\left(\widetilde{\Theta} * \mathrm{f}_S\right) \,\widetilde{\cup}\, \left(\mathrm{f}_S * \widetilde{\Theta}\right) \,\widetilde{\supseteq}\, \mathrm{f}_S * \widetilde{\Theta} = \mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S \,\widetilde{\supseteq}\, \mathrm{f}_S$$

is obvious. Hence, $\mathrm{f}_S$ is an S-uni Ꝗ-ideal.

(ii) Let $S$ be special soft simple and $\mathrm{f}_S$ be an S-uni Ꝗ-ideal. Then, $\widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S = \mathrm{f}_S * \widetilde{\Theta}$ and $\left(\widetilde{\Theta} * \mathrm{f}_S\right) \,\widetilde{\cup}\, \left(\mathrm{f}_S * \widetilde{\Theta}\right) \,\widetilde{\supseteq}\, \mathrm{f}_S$. Since

$$\widetilde{\Theta} * \mathrm{f}_S = \left(\widetilde{\Theta} * \mathrm{f}_S\right) \,\widetilde{\cup}\, \left(\widetilde{\Theta} * \mathrm{f}_S\right) = \left(\widetilde{\Theta} * \mathrm{f}_S\right) \,\widetilde{\cap}\, \left(\mathrm{f}_S * \widetilde{\Theta}\right) \,\widetilde{\supseteq}\, \mathrm{f}_S$$

$\mathrm{f}_S$ is an S-uni Ƚ-ideal. Similarly, since

$$\mathrm{f}_S * \widetilde{\Theta} = \left(\mathrm{f}_S * \widetilde{\Theta}\right) \,\widetilde{\cup}\, \left(\mathrm{f}_S * \widetilde{\Theta}\right) = \left(\widetilde{\Theta} * \mathrm{f}_S\right) \,\widetilde{\cap}\, \left(\mathrm{f}_S * \widetilde{\Theta}\right) \,\widetilde{\supseteq}\, \mathrm{f}_S$$

$\mathrm{f}_S$ is an S-uni Ꞧ-ideal. Hence, $\mathrm{f}_S$ is an S-uni ideal.

(iii) Let $S$ be special soft simple and $\mathrm{f}_S$ be an S-uni Ƀ-ideal. Then, by Theorem 2.15 (i) $\mathrm{f}_S$ is an S-uni Ꝗ-ideal. The rest of the proof is obvious from Theorem 2.15 (ii).
(iv) Let $S$ be special soft simple and $\mathrm{f}_S$ be an S-uni $I$-ideal. Then, $\widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S = \mathrm{f}_S * \widetilde{\Theta}$ and $\widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, \mathrm{f}_S$. Since

$$\widetilde{\Theta} * \mathrm{f}_S = \widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, \mathrm{f}_S$$

$\mathrm{f}_S$ is an S-uni Ł-ideal. Similarly, since

$$\mathrm{f}_S * \widetilde{\Theta} = \widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, \mathrm{f}_S$$

$\mathrm{f}_S$ is an S-uni Ꞧ-ideal. Hence, $\mathrm{f}_S$ is an S-uni ideal.

## 3. Soft Union Bi-interior Ideals of Semigroups

In this section, soft union (S-uni) bi-interior ideal of a semigroup is defined and its relations between other certain soft union (S-uni) ideals are obtained.

**Definition 3.1.** A soft set $f_S$ over $U$ is called a soft union (S-uni) bi-interior ideal of $S$ over $U$ if $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \,\widetilde{\cup}\, \left(f_S * \widetilde{\Theta} * f_S\right) \,\widetilde{\supseteq}\, f_S$.

For the sake of brevity, S-uni bi-interior ideal of $S$ over $U$ is abbreviated by S-uni ɃI-ideal.

**Example 3.2.** Let $S = \{\text{₡}, \text{ŧ}, \text{Ꜧ}, \text{ນ}\}$ be:

**Table 2.** Cayley table of '•' binary operation.

| • | ₡ | ŧ | Ꜧ | ນ |
|---|---|---|---|---|
| ₡ | ₡ | ŧ | ŧ | ນ |
| ŧ | ŧ | ŧ | ŧ | ນ |
| Ꜧ | ŧ | ŧ | ŧ | ນ |
| ນ | ນ | ນ | ນ | ນ |

Let $\omega_S$ and $u_S$ be soft sets over $U = \mathbb{Z}$ as follows:

$$\omega_S = \{(\text{₡}, \{1,3,4\}), (\text{ŧ}, \{1,3\}), (\text{Ꜧ}, \{1,2,3\}), (\text{ນ}, \{1\})\}$$
$$u_S = \{(\text{₡}, \{3,5\}), (\text{ŧ}, \{3,5,7,8\}), (\text{Ꜧ}, \{5,8\}), (\text{ນ}, \{3,5,7\})\}$$

Then, $\omega_S$ is an S-uni ɃI-ideal. In fact;

$\left[\left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \,\widetilde{\cup}\, \left(\omega_S * \widetilde{\Theta} * \omega_S\right)\right](\text{₡}) = \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right)(\text{₡}) \cup \left(\omega_S * \widetilde{\Theta} * \omega_S\right)(\text{₡}) = \left[\widetilde{\Theta}(\text{₡}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{₡})\right] \cap \left[\omega_S(\text{₡}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{₡})\right] = \omega_S(\text{₡}) \cup \omega_S(\text{₡}) = \omega_S(\text{₡}) \supseteq \omega_S(\text{₡})$

$\left[\left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \,\widetilde{\cup}\, \left(\omega_S * \widetilde{\Theta} * \omega_S\right)\right](\text{ŧ}) = \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right)(\text{ŧ}) \cup \left(\omega_S * \widetilde{\Theta} * \omega_S\right)(\text{ŧ}) = \Big[\left[\widetilde{\Theta}(\text{₡}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{ŧ})\right] \cap \left[\widetilde{\Theta}(\text{₡}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{Ꜧ})\right] \cap \left[\widetilde{\Theta}(\text{ŧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{₡})\right] \cap \left[\widetilde{\Theta}(\text{ŧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{ŧ})\right] \cap \left[\widetilde{\Theta}(\text{ŧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{Ꜧ})\right] \cap \left[\widetilde{\Theta}(\text{Ꜧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{₡})\right] \cap \left[\widetilde{\Theta}(\text{Ꜧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{ŧ})\right] \cap \left[\widetilde{\Theta}(\text{Ꜧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{Ꜧ})\right]\Big] \cup \Big[\left[\omega_S(\text{₡}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{ŧ})\right] \cap \left[\omega_S(\text{₡}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{Ꜧ})\right] \cap \left[\omega_S(\text{ŧ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{₡})\right] \cap \left[\omega_S(\text{ŧ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{ŧ})\right] \cap \big[\omega_S(\text{ŧ}) \cup$

$\left(\widetilde{\Theta} * \omega_S\right)(\text{Ħ})\big] \cap \left[\omega_S(\text{Ħ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{₡})\right] \cap \left[\omega_S(\text{Ħ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{ŧ})\right] \cap \left[\omega_S(\text{Ħ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{Ħ})\right]\Big] =$
$[\omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ})] \cup [\omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ})] = \omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ}) \supseteq \omega_S(\text{ŧ})$

$\left[\left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right)\right](\text{Ħ}) = U \supseteq \omega_S(\text{Ħ})$

$\left[\left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right)\right](\upsilon) = \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right)(\upsilon) \cup \left(\omega_S * \widetilde{\Theta} * \omega_S\right)(\upsilon) = \Big[\left[\widetilde{\Theta}(\text{₡}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\upsilon)\right] \cap \left[\widetilde{\Theta}(\text{ŧ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\upsilon)\right] \cap \left[\widetilde{\Theta}(\text{Ħ}) \cup \left(\omega_S * \widetilde{\Theta}\right)(\upsilon)\right] \cap \left[\widetilde{\Theta}(\upsilon) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{₡})\right] \cap \left[\widetilde{\Theta}(\upsilon) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{ŧ})\right] \cap \left[\widetilde{\Theta}(\upsilon) \cup \left(\omega_S * \widetilde{\Theta}\right)(\text{Ħ})\right] \cap \left[\widetilde{\Theta}(\upsilon) \cup \left(\omega_S * \widetilde{\Theta}\right)(\upsilon)\right]\Big] \cup \Big[\left[\omega_S(\text{₡}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\upsilon)\right] \cap \left[\omega_S(\text{ŧ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\upsilon)\right] \cap \left[\omega_S(\text{Ħ}) \cup \left(\widetilde{\Theta} * \omega_S\right)(\upsilon)\right] \cap \left[\omega_S(\upsilon) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{₡})\right] \cap \left[\omega_S(\upsilon) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{ŧ})\right] \cap \left[\omega_S(\upsilon) \cup \left(\widetilde{\Theta} * \omega_S\right)(\text{Ħ})\right] \cap \left[\omega_S(\upsilon) \cup \left(\widetilde{\Theta} * \omega_S\right)(\upsilon)\right]\Big] = [\omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ}) \cap \omega_S(\upsilon)] \cup [\omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ}) \cap \omega_S(\upsilon)] = \omega_S(\text{₡}) \cap \omega_S(\text{ŧ}) \cap \omega_S(\text{Ħ}) \cap \omega_S(\upsilon) \supseteq \omega_S(\upsilon)$

Thus, $\omega_S$ is an S-uni ɃI-ideal. However, since

$$\begin{aligned}\left[\left(\widetilde{\Theta} * u_S * \widetilde{\Theta}\right) \tilde{\cup} \left(u_S * \widetilde{\Theta} * u_S\right)\right](\upsilon) &= \left(\widetilde{\Theta} * u_S * \widetilde{\Theta}\right)(\upsilon) \cup \left(u_S * \widetilde{\Theta} * u_S\right)(\upsilon)\\ &= [u_S(\text{₡}) \cap u_S(\text{ŧ}) \cap u_S(\text{Ħ}) \cap u_S(\upsilon)] \cup [u_S(\text{₡}) \cap u_S(\text{ŧ}) \cap u_S(\text{Ħ}) \cap u_S(\upsilon)]\\ &= u_S(\text{₡}) \cap u_S(\text{ŧ}) \cap u_S(\text{Ħ}) \cap u_S(\upsilon) \not\supseteq u_S(\upsilon)\end{aligned}$$

$u_S$ is not an S-uni ɃI-ideal.

**Corollary 3.3.** $\widetilde{\Theta}$ is an S-uni ɃI-ideal.

Now, we continue with the relationships between S-uni ɃI-ideals and other types of S-uni ideals of $S$.

**Theorem 3.4.** Every S-uni ɃI-ideal is an S-uni SS of a special soft simple semigroup.

**Proof:** Let $f_S$ be an S-uni ɃI-ideal of a special soft simple $S$. Then, $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \tilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \tilde{\supseteq} f_S$ and $\widetilde{\Theta} = \widetilde{\Theta} * f_S = f_S * \widetilde{\Theta}$. Thus,

$$\begin{aligned} f_S * f_S = (f_S * f_S) \tilde{\cup} (f_S * f_S) &\tilde{\supseteq} \left(\widetilde{\Theta} * f_S\right) \tilde{\cup} \left(f_S * \widetilde{\Theta}\right)\\ &= \left(\widetilde{\Theta} * f_S * f_S\right) \tilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \tilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \tilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \tilde{\supseteq} f_S\end{aligned}$$

Hence, $f_S$ is an S-uni SS.

**Theorem 3.5.** Every S-uni Ł-ideal is an S-uni ɃI-ideal.

**Proof:** Let $f_S$ be an S-uni Ł-ideal. Then, $\widetilde{\Theta} * f_S \tilde{\supseteq} f_S$ and $f_S * f_S \tilde{\supseteq} f_S$. Thus,

$$\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \tilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \tilde{\supseteq} f_S * \widetilde{\Theta} * f_S \tilde{\supseteq} f_S * f_S \tilde{\supseteq} f_S$$

Hence, $f_S$ is an S-uni ɃI-ideal.

We present a counterexample to demonstrate that the converse of Theorem 3.5 is not valid:

**Example 3.6.** Consider the semigroup $S = \{\text{ठ}, \text{Ʉ}\}$ defined by the following table:

**Table 3.** Cayley table of '$\circledcirc$' binary operation.

| $\circledcirc$ | ठ | Ʉ |
|---|---|---|
| ठ | ठ | ठ |
| Ʉ | Ʉ | Ʉ |

Let $\varrho_S$ be soft set over $U = \mathbb{Z}^+$ as follows:

$$\varrho_S = \{(\text{ठ}, \{4\}), (\text{Ʉ}, \{1,1999\})\}$$

Then, $\varrho_S$ is an S-uni ɃI-ideal. In fact;

$$[(\widetilde{\Theta} * \varrho_S * \widetilde{\Theta}) \tilde{\cup} (\varrho_S * \widetilde{\Theta} * \varrho_S)](\text{ठ}) = (\widetilde{\Theta} * \varrho_S * \widetilde{\Theta})(\text{ठ}) \cup (\varrho_S * \widetilde{\Theta} * \varrho_S)(\text{ठ})$$
$$= (\varrho_S(\text{ठ}) \cap \varrho_S(\text{Ʉ})) \cup \varrho_S(\text{ठ}) = \varrho_S(\text{ठ}) \supseteq \varrho_S(\text{ठ})$$

$$[(\widetilde{\Theta} * \varrho_S * \widetilde{\Theta}) \tilde{\cup} (\varrho_S * \widetilde{\Theta} * \varrho_S)](\text{Ʉ}) = (\widetilde{\Theta} * \varrho_S * \widetilde{\Theta})(\text{Ʉ}) \cup (\varrho_S * \widetilde{\Theta} * \varrho_S)(\text{Ʉ}) = (\varrho_S(\text{ठ}) \cap \varrho_S(\text{Ʉ})) \cup \varrho_S(\text{Ʉ})$$
$$= \varrho_S(\text{Ʉ}) \supseteq \varrho_S(\text{Ʉ})$$

Thus, $\varrho_S$ is an S-uni ɃI-ideal. However, since

$$\varrho_S(\text{ठɄ}) = \varrho_S(\text{ठ}) \not\supseteq \varrho_S(\text{Ʉ})$$

$\varrho_S$ is not an S-uni Ł-ideal.

Theorem 3.7 illustrates that the converse of Theorem 3.5 valid for the special soft simple semigroups.

**Theorem 3.7.** Let $\omega_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $\omega_S$ is an S-uni Ł-ideal.
2. $\omega_S$ is an S-uni ɃI-ideal.

**Proof:** (1) implies (2) is obvious by Theorem 3.5. Assume that $\omega_S$ is an S-uni ɃI-ideal. By assumption, $\widetilde{\Theta} = \widetilde{\Theta} * \omega_S = \omega_S * \widetilde{\Theta}$. Thus,

$$\widetilde{\Theta} * \omega_S = (\widetilde{\Theta} * \omega_S) \tilde{\cup} (\widetilde{\Theta} * \omega_S) = (\widetilde{\Theta} * \omega_S * \omega_S) \tilde{\cup} (\omega_S * \widetilde{\Theta} * \omega_S)$$
$$\tilde{\supseteq} (\widetilde{\Theta} * \omega_S * \widetilde{\Theta}) \tilde{\cup} (\omega_S * \widetilde{\Theta} * \omega_S) \tilde{\supseteq} \omega_S$$

Hence, $\omega_S$ is an S-uni Ł-ideal.

**Theorem 3.8.** Every S-uni Ʀ-ideal is an S-uni ɃI-ideal.

**Proof:** Let $\mathrm{f}_S$ be an S-uni Ʀ-ideal. Then, $\mathrm{f}_S * \widetilde{\Theta} \tilde{\supseteq} \mathrm{f}_S$ and $\mathrm{f}_S * \mathrm{f}_S \tilde{\supseteq} \mathrm{f}_S$. Thus,

$$(\widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta}) \tilde{\cup} (\mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S) \tilde{\supseteq} \mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S \tilde{\supseteq} \mathrm{f}_S * \mathrm{f}_S \tilde{\supseteq} \mathrm{f}_S$$

Hence, $\mathrm{f}_S$ is an S-uni ɃI-ideal.

We present a counterexample to demonstrate that the converse of Theorem 3.8 is not valid:

**Example 3.9.** Let $S = \{\text{ɟ}, \text{ç}\}$ be:

**Table 4.** Cayley table of '⬡' binary operation.

| ⬡ | ɟ | ç |
|---|---|---|
| ɟ | ɟ | ç |
| ç | ɟ | ç |

Let $\varpi_S$ be soft set over $U = \mathbb{Z}^-$ as follows:

$$\varpi_S = \{(\text{ɟ}, \{-1\}), (\text{ç}, \{-2\})\}$$

Then, $\varpi_S$ is an S-uni ɃI-ideal. In fact;

$$\left[\left(\widetilde{\Theta} * \varpi_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\varpi_S * \widetilde{\Theta} * \varpi_S\right)\right](\text{ɟ}) = \left(\widetilde{\Theta} * \varpi_S * \widetilde{\Theta}\right)(\text{ɟ}) \cup \left(\varpi_S * \widetilde{\Theta} * \varpi_S\right)(\text{ɟ}) = \left(\varpi_S(\text{ɟ}) \cap \varpi_S(\text{ç})\right) \cup \varpi_S(\text{ɟ})$$
$$= \varpi_S(\text{ɟ}) \supseteq \varpi_S(\text{ɟ})$$
$$\left[\left(\widetilde{\Theta} * \varpi_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\varpi_S * \widetilde{\Theta} * \varpi_S\right)\right](\text{ç}) = \left(\widetilde{\Theta} * \varpi_S * \widetilde{\Theta}\right)(\text{ç}) \cup \left(\varpi_S * \widetilde{\Theta} * \varpi_S\right)(\text{ç}) = \left(\varpi_S(\text{ɟ}) \cap \varpi_S(\text{ç})\right) \cup \varpi_S(\text{ç})$$
$$= \varpi_S(\text{ç}) \supseteq \varpi_S(\text{ç})$$

Thus, $\varpi_S$ is an S-uni ɃI-ideal. However since,

$$\varpi_S(\text{ɟç}) = \varpi_S(\text{ç}) \nsupseteq \varpi_S(\text{ɟ})$$

$\varpi_S$ is not an S-uni Ɍ-ideal.

Theorem 3.10 illustrates that the converse of Theorem 3.8 is valid for the special soft simple semigroups.

**Theorem 3.10.** Let $\omega_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $\omega_S$ is an S-uni Ɍ-ideal.
2. $\omega_S$ is an S-uni ɃI-ideal.

**Proof:** (1) implies (2) is obvious by Theorem 3.8. Assume that $\omega_S$ is an S-uni ɃI-ideal. By assumption, $\widetilde{\Theta} = \widetilde{\Theta} * \omega_S = \omega_S * \widetilde{\Theta}$. Thus,

$$\omega_S * \widetilde{\Theta} = \left(\omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta}\right)$$
$$= \left(\omega_S * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \tilde{\supseteq} \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \tilde{\supseteq} \omega_S$$

Hence, $\omega_S$ is an S-uni Ɍ-ideal.

**Theorem 3.11.** Every S-uni ideal is an S-uni ɃI-ideal.

**Proof:** It follows by Theorem 3.5 and Theorem 3.8.

Note that the converse of Theorem 3.11 is not true follows from Example 3.6 and Example 3.9.

Theorem 3.12 illustrates that the converse of Theorem 3.11 valid for the special soft simple semigroups.

**Theorem 3.12.** Let $f_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $f_S$ is an S-uni ideal.
2. $f_S$ is an S-uni ɃI-ideal.

**Proof:** It follows by Theorem 3.7 and Theorem 3.10.

**Theorem 3.13.** Every S-uni Ƀ-ideal is an S-uni ɃI-ideal.

**Proof:** Let $f_S$ be an S-uni Ƀ-ideal. Then, $f_S * \widetilde{\Theta} * f_S \,\widetilde{\supseteq}\, f_S$. Thus,

$$\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} f_S * \widetilde{\Theta} * f_S \widetilde{\supseteq} f_S$$

Hence, $f_S$ is an S-uni ɃI-ideal.

We present a counterexample to demonstrate that the converse of Theorem 3.13 is not valid:

**Example 3.14.** Let $S = \{\text{ꝑ}, \sigma, \upsilon, r\}$ be:

**Table 5.** Cayley table of '◎' binary operation.

| ◎ | ꝑ | $\sigma$ | $\upsilon$ | $r$ |
|---|---|---|---|---|
| ꝑ | ꝑ | ꝑ | ꝑ | ꝑ |
| $\sigma$ | ꝑ | ꝑ | ꝑ | ꝑ |
| $\upsilon$ | ꝑ | ꝑ | $\sigma$ | ꝑ |
| $r$ | ꝑ | ꝑ | $\sigma$ | $\sigma$ |

Suppose that $\text{ɓ}_S$ is soft set over $U = S_3$ as follows:

$$\text{ɓ}_S = \{(\text{ꝑ}, \{(1)\}), (\sigma, \{(1), (123)\}), (\upsilon, \{(1), (132)\}), (r, \{(1), (12)\})\}$$

Then, $\text{ɓ}_S$ is an S-uni ɃI-ideal. In fact;

$$\left[\left(\widetilde{\Theta} * \text{ɓ}_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\text{ɓ}_S * \widetilde{\Theta} * \text{ɓ}_S\right)\right](\text{ꝑ}) = \{(1)\} \supseteq \text{ɓ}_S(\text{ꝑ})$$

$$\left[\left(\widetilde{\Theta} * \text{ɓ}_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\text{ɓ}_S * \widetilde{\Theta} * \text{ɓ}_S\right)\right](\sigma) = U \supseteq \text{ɓ}_S(\sigma)$$

$$\left[\left(\widetilde{\Theta} * \text{ɓ}_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\text{ɓ}_S * \widetilde{\Theta} * \text{ɓ}_S\right)\right](\upsilon) = U \supseteq \text{ɓ}_S(\upsilon)$$

$$\left[\left(\widetilde{\Theta} * \text{ɓ}_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\text{ɓ}_S * \widetilde{\Theta} * \text{ɓ}_S\right)\right](r) = U \supseteq \text{ɓ}_S(r)$$

Thus, $\text{ɓ}_S$ is an S-uni ɃI-ideal. However, since

$$\text{ɓ}_S(\upsilon\upsilon) = \text{ɓ}_S(\sigma) \nsubseteq \text{ɓ}_S(\upsilon) \cup \text{ɓ}_S(\upsilon)$$

$\text{ɓ}_S$ is not an S-uni SS. Hence, $\text{ɓ}_S$ is not an S-uni Ƀ-ideal.

Theorem 3.15 illustrates that the converse of Theorem 3.13 is valid for the special soft simple semigroups.

**Theorem 3.15.** Let $\omega_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $\omega_S$ is an S-uni Ƀ-ideal.
2. $\omega_S$ is an S-uni ɃI-ideal.

**Proof:** (1) implies (2) is obvious by Theorem 3.13. Assume that $\omega_S$ is an S-uni ɃI-ideal. Then, by Theorem 3.4, $\omega_S$ is an S-uni SS and by assumption, $\widetilde{\Theta} = \widetilde{\Theta} * \omega_S = \omega_S * \widetilde{\Theta}$. Thus,

$$\begin{aligned}\omega_S * \widetilde{\Theta} * \omega_S &= \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right)\\ &= \left(\widetilde{\Theta} * \omega_S * \omega_S\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \widetilde{\supseteq} \omega_S\end{aligned}$$

Hence, $\omega_S$ is an S-uni Ƀ-ideal.

**Theorem 3.16.** Every S-uni $I$-ideal is an S-uni ꞖI-ideal.

**Proof:** Let $f_S$ be an S-uni $I$-ideal. Then, $\widetilde{\Theta} * f_S * \widetilde{\Theta} \supseteq f_S$. Thus,

$$\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \tilde{\cap} \left(f_S * \widetilde{\Theta} * f_S\right) \tilde{\supseteq} \widetilde{\Theta} * f_S * \widetilde{\Theta} \tilde{\supseteq} f_S$$

Hence, $f_S$ is an S-uni ꞖI-ideal.

We present a counterexample to demonstrate that the converse of Theorem 3.16 is not valid:

**Example 3.17.** Let the soft set $\varpi_S$ in Example 3.9. As seen in Example 3.9, $\varpi_S$ is an S-uni ꞖI-ideal. Since,

$$\varpi_S(\varsigma ɟ \varsigma) = \varpi_S(\varsigma) \nsubseteq \varpi_S(ɟ)$$

$\varpi_S$ is not an S-uni $I$-ideal.

Theorem 3.18 illustrates that the converse of Theorem 3.16 valid for the special soft simple semigroups.

**Theorem 3.18.** Let $\omega_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $\omega_S$ is an S-uni $I$-ideal.
2. $\omega_S$ is an S-uni ꞖI-ideal.

**Proof:** (1) implies (2) is obvious by Theorem 3.16. Assume that $\omega_S$ is an S-uni ꞖI-ideal. By assumption, $\widetilde{\Theta} = \widetilde{\Theta} * \omega_S = \omega_S * \widetilde{\Theta}$. Thus,

$$\begin{aligned}\widetilde{\Theta} * \omega_S * \widetilde{\Theta} &= \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right)\\ &= \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \widetilde{\Theta}\right) \tilde{\supseteq} \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta}\right)\\ &= \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \tilde{\supseteq} \omega_S\end{aligned}$$

Hence, $\omega_S$ is an S-uni $I$-ideal.

**Theorem 3.19.** Every S-uni Ꝗ-ideal is an S-uni ꞖI-ideal.

**Proof:** Let $f_S$ be an S-uni Ꝗ-ideal. Then, $f_S$ is an S-uni Ꞗ-ideal by Theorem 2.12. The rest is obvious by Theorem 3.13.

We present a counterexample to demonstrate that the converse of Theorem 3.19 is not valid:

**Example 3.20.** Let the soft set $\mathfrak{G}_S$ in Example 3.14. As seen in Example 3.14, $\mathfrak{G}_S$ is an S-uni ꞖI-ideal, however, it is not an S-uni Ꞗ-ideal. Since $\mathfrak{G}_S$ is not an S-uni Ꞗ-ideal, $\mathfrak{G}_S$ is not an S-uni Ꝗ-ideal.

Theorem 3.21 illustrates that the converse of Theorem 3.19 is valid for the special soft simple semigroups.

**Theorem 3.21.** Let $\omega_S \in S_S(U)$ and $S$ be special soft simple. Then, the following conditions are equivalent:

1. $\omega_S$ is an S-uni Ꝗ-ideal.
2. $\omega_S$ is an S-uni ꞖI-ideal.

**Proof:** (1) implies (2) is obvious by Theorem 3.19. Assume that $\omega_S$ is an S-uni ꞖI-ideal. By assumption, $\widetilde{\Theta} = \widetilde{\Theta} * \omega_S = \omega_S * \widetilde{\Theta}$. Thus,

$$\left(\widetilde{\Theta} * \omega_S\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta}\right) = \left(\widetilde{\Theta} * \omega_S * \omega_S\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * \omega_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\omega_S * \widetilde{\Theta} * \omega_S\right) \widetilde{\supseteq} \omega_S$$

Hence, $\omega_S$ is an S-uni Ꝗ-ideal.

**Theorem 3.22.** Let $\omega_S, \eta_S \in S_S(U)$. If $\omega_S$ or $\eta_S$ is an S-uni Ł-ideal, then $\omega_S * \eta_S$ is an S-uni ꞖI-ideal.

**Proof:** Let $\omega_S$ be an S-uni Ł-ideal. Then, $\widetilde{\Theta} * \omega_S \widetilde{\supseteq} \omega_S$ and $\omega_S * \omega_S \widetilde{\supseteq} \omega_S$. Thus,

$$\left(\widetilde{\Theta} * \left(\omega_S * \eta_S\right) * \widetilde{\Theta}\right) \widetilde{\cup} \left(\left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)\right) \widetilde{\supseteq} \left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)$$

$$= \omega_S * \eta_S * \left(\widetilde{\Theta} * \omega_S\right) * \eta_S \widetilde{\supseteq} \omega_S * \eta_S * \omega_S * \eta_S \widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \omega_S\right) * \eta_S \widetilde{\supseteq} \omega_S * \omega_S * \eta_S \widetilde{\supseteq} \omega_S * \eta_S$$

Hence, $\omega_S * \eta_S$ is an S-uni ꞖI-ideal. Also, the proof can be given similarly for $\eta_S$. Let $\eta_S$ be an S-uni Ł-ideal. Then, $\widetilde{\Theta} * \eta_S \widetilde{\supseteq} \eta_S$ and $\eta_S * \eta_S \widetilde{\supseteq} \eta_S$. Thus,

$$\left(\widetilde{\Theta} * \left(\omega_S * \eta_S\right) * \widetilde{\Theta}\right) \widetilde{\cup} \left(\left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)\right) \widetilde{\supseteq} \left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)$$

$$\widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \widetilde{\Theta}\right) * \widetilde{\Theta} * \eta_S \widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \widetilde{\Theta}\right) * \eta_S \widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \eta_S\right) \widetilde{\supseteq} \omega_S * \eta_S$$

Hence, $\omega_S * \eta_S$ is an S-uni ꞖI-ideal.

**Theorem 3.23.** Let $\omega_S, \eta_S \in S_S(U)$. If $\omega_S$ or $\eta_S$ is an S-uni Ꞧ-ideal, then $\omega_S * \eta_S$ is an S-uni ꞖI-ideal.

**Proof:** Let $\eta_S$ be an S-uni Ꞧ-ideal. Then, $\eta_S * \widetilde{\Theta} \widetilde{\supseteq} \eta_S$ and $\eta_S * \eta_S \widetilde{\supseteq} \eta_S$. Thus,

$$\left(\widetilde{\Theta} * \left(\omega_S * \eta_S\right) * \widetilde{\Theta}\right) \widetilde{\cup} \left(\left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)\right) \widetilde{\cup} \left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)$$

$$= \omega_S * \left(\eta_S * \widetilde{\Theta}\right) * \omega_S * \eta_S \widetilde{\supseteq} \omega_S * \eta_S * \omega_S * \eta_S \widetilde{\supseteq} \omega_S * \left(\eta_S * \widetilde{\Theta}\right) * \eta_S \widetilde{\supseteq} \omega_S * \eta_S * \eta_S \widetilde{\supseteq} \omega_S * \eta_S$$

Hence, $\omega_S * \eta_S$ is an S-uni ꞖI-ideal. Also, the proof can be given similarly for $\omega_S$. Let $\omega_S$ be an S-uni Ꞧ-ideal. Then, $\omega_S * \widetilde{\Theta} \widetilde{\supseteq} \omega_S$ and $\omega_S * \omega_S \widetilde{\supseteq} \omega_S$. Thus,

$$\left(\widetilde{\Theta} * \left(\omega_S * \eta_S\right) * \widetilde{\Theta}\right) \widetilde{\cup} \left(\left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)\right) \widetilde{\supseteq} \left(\omega_S * \eta_S\right) * \widetilde{\Theta} * \left(\omega_S * \eta_S\right)$$

$$\widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \widetilde{\Theta}\right) * \widetilde{\Theta} * \eta_S \widetilde{\supseteq} \omega_S * \left(\widetilde{\Theta} * \widetilde{\Theta}\right) * \eta_S \widetilde{\supseteq} \left(\omega_S * \widetilde{\Theta}\right) * \eta_S \widetilde{\supseteq} \omega_S * \eta_S$$

Hence, $\omega_S * \eta_S$ is an S-uni ꞖI-ideal.

**Theorem 3.24.** Let $f_S, \wp_S, h_S \in S_S(U)$. If $\varrho_S$ is an S-uni ideal, then $f_S * \wp_S * h_S$ is an S-uni ꞖI-ideal.

**Proof:** Let $\wp_S$ be an S-uni ideal. Then, $\widetilde{\Theta} * \wp_S \widetilde{\supseteq} \wp_S$, $\wp_S * \widetilde{\Theta} \widetilde{\supseteq} \wp_S$ and $\wp_S * \wp_S \widetilde{\supseteq} \wp_S$. Thus,

$$\left(\widetilde{\Theta} * (f_S * \wp_S * h_S) * \widetilde{\Theta}\right) \widetilde{\cup} \left((f_S * \wp_S * h_S) * \widetilde{\Theta} * (f_S * \wp_S * h_S)\right) \widetilde{\supseteq} (f_S * \wp_S * h_S) * \widetilde{\Theta} * (f_S * \wp_S * h_S)$$

$$\widetilde{\supseteq} f_S * \left(\wp_S * \widetilde{\Theta}\right) * \left(\widetilde{\Theta} * \widetilde{\Theta}\right) * \wp_S * h_S \widetilde{\supseteq} f_S * \left(\wp_S * \widetilde{\Theta}\right) * \wp_S * h_S \widetilde{\supseteq} f_S * \wp_S * \wp_S * h_S \widetilde{\supseteq} f_S * \wp_S * h_S$$

Hence, $f_S * \wp_S * h_S$ is an S-uni ꞖI-ideal.

**Theorem 3.25.** Let $f_S$ and $\eta_S$ be S-uni ꞖI-ideals. Then, $f_S \widetilde{\cup} \eta_S$ is an S-uni ꞖI-ideal.

**Proof:** Let $f_S$ and $\eta_S$ are S-uni ꞖI-ideals. Then, $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} f_S$ and $\left(\widetilde{\Theta} * \eta_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\eta_S * \widetilde{\Theta} * \eta_S\right) \widetilde{\supseteq} \eta_S$. Thus,

$$\left(\widetilde{\Theta} * (f_S \widetilde{\cup} \eta_S) * \widetilde{\Theta}\right) \widetilde{\cup} \left((f_S \widetilde{\cup} \eta_S) * \widetilde{\Theta} * (f_S \widetilde{\cup} \eta_S)\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} f_S$$

and

$$\left(\widetilde{\Theta} * (f_S \widetilde{\cup} \eta_S) * \widetilde{\Theta}\right) \widetilde{\cup} \left((f_S \widetilde{\cup} \eta_S) * \widetilde{\Theta} * (f_S \widetilde{\cup} \eta_S)\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * \eta_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\eta_S * \widetilde{\Theta} * \eta_S\right) \widetilde{\supseteq} \eta_S$$

Hence,

$$\left(\widetilde{\Theta} * (f_S \,\widetilde{\cup}\, \eta_S) * \widetilde{\Theta}\right) \widetilde{\cup} \left((f_S \,\widetilde{\cup}\, \eta_S) * \widetilde{\Theta} * (f_S \,\widetilde{\cup}\, \eta_S)\right) \widetilde{\supseteq} f_S \,\widetilde{\cup}\, \eta_S$$

Therefore, $f_S \,\widetilde{\cup}\, \eta_S$ is an S-uni ꞖI-ideal.

**Corollary 3.26.** The finite union of S-uni ꞖI-ideals is an S-uni ꞖI-ideal.

**Corollary 3.27.** The union of S-uni Ł-ideal (Ʀ-ideal/ideal/Ƀ-ideal/$I$-ideal/Ꝗ-ideal) and S-uni Ł-ideal (Ʀ-ideal/ideal/Ƀ-idel/$I$-ideal/Ꝗ-ideal) is an S-uni ꞖI-ideal.

**Theorem 3.28.** Let $\emptyset_S \neq f_S \in S_E(U)$. Then, every soft set containing $f_S$ which is the soft subset of $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right)$ is an S-uni ꞖI-ideal.

**Proof:** Let $\wp_S \widetilde{\supseteq} f_S$ soft subset $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right)$. Since,

$$\left(\widetilde{\Theta} * \wp_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(\wp_S * \widetilde{\Theta} * \wp_S\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cup} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} \wp_S$$

Hence, $\wp_S$ is an S-uni ꞖI-ideal.

**Theorem 3.29.** Let $\emptyset_S \neq f_S \in S_S(U)$. Then, every soft set containing $f_S$ which is the soft subset of $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cap} \left(f_S * \widetilde{\Theta} * f_S\right)$ is an S-uni ꞖI-ideal.

**Proof:** Let $\wp_S \widetilde{\supseteq} f_S$ soft subset $\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cap} \left(f_S * \widetilde{\Theta} * f_S\right)$. Since

$$\left(\widetilde{\Theta} * \wp_S * \widetilde{\Theta}\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cap} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} \wp_S$$

Thus, $\left(\widetilde{\Theta} * \wp_S * \widetilde{\Theta}\right) \widetilde{\supseteq} \wp_S$. Furthermore, since

$$\left(\wp_S * \widetilde{\Theta} * \wp_S\right) \widetilde{\supseteq} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} \left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right) \widetilde{\cap} \left(f_S * \widetilde{\Theta} * f_S\right) \widetilde{\supseteq} \wp_S$$

Thus, $\left(\wp_S * \widetilde{\Theta} * \wp_S\right) \widetilde{\supseteq} \wp_S$. Hence, $\left(\widetilde{\Theta} * \wp_S * \widetilde{\Theta}\right) \widetilde{\cap} \left(\wp_S * \widetilde{\Theta} * \wp_S\right) \widetilde{\supseteq} \wp_S$. Therefore, $\wp_S$ is an S-uni ꞖI-ideal.

**Proposition 3.30.** Let $f_S \in S_S(U)$, $ɓ \subseteq U$, $Im(f_S)$ be the image of $f_S$ such that $ɓ \in Im(f_S)$. If $f_S$ is an S-uni ꞖI-ideal, then $ȿ(f_S; ɓ)$ is a ꞖI-ideal.

**Proof:** Since $f_S(\upsilon) = ɓ$ for some $\upsilon \in S$, $\emptyset \neq ȿ(f_S; ɓ) \subseteq S$. Let $k \in [S \cdot ȿ(f_S; ɓ) \cdot S] \cap [ȿ(f_S; ɓ) \cdot S \cdot ȿ(f_S; ɓ)]$. Then, there exist $x, t, z \in ȿ(f_S; ɓ)$ and $a, b, c \in S$ such that $k = xat = bzc$. Thus, $f_S(x) \subseteq ɓ$, $f_S(t) \subseteq ɓ$ and $f_S(z) \subseteq ɓ$. Since $f_S$ is an S-uni ꞖI-ideal,

$$\begin{aligned}
\left(\widetilde{\Theta} * f_S * \widetilde{\Theta}\right)(k) &= \bigcap_{k=mn} \left\{\widetilde{\Theta}(m) \cup \left(f_S * \widetilde{\Theta}\right)(n)\right\} \\
&\subseteq \widetilde{\Theta}(b) \cup \left(f_S * \widetilde{\Theta}\right)(zc) \\
&= \emptyset \cup \left[\bigcap_{zc=sr} \left\{f_S(s) \cup \widetilde{\Theta}(r)\right\}\right] \\
&\subseteq f_S(z) \cup \widetilde{\Theta}(c) \\
&= f_S(z) \cup \emptyset \\
&= f_S(z) \\
&\subseteq ɓ
\end{aligned}$$

and

$$\left(f_S * \widetilde{\Theta} * f_S\right)(k) = \bigcap_{k=mn} \left\{f_S(m) \cup \left(\widetilde{\Theta} * f_S\right)(n)\right\}$$

$$\begin{aligned}
&\subseteq \mathrm{f}_S(x) \cup \left(\widetilde{\Theta} * \mathrm{f}_S\right)(at) \\
&= \mathrm{f}_S(x) \cup \left[\bigcap_{at=pq} \left\{\widetilde{\Theta}(p) \cap \mathrm{f}_S(q)\right\}\right] \\
&\subseteq \mathrm{f}_S(x) \cup \widetilde{\Theta}(a) \cup \mathrm{f}_S(t) \\
&= \mathrm{f}_S(x) \cup \emptyset \cup \mathrm{f}_S(t) \\
&= \mathrm{f}_S(x) \cup \mathrm{f}_S(t) \\
&\subseteq ɕ \cup ɕ \\
&= ɕ
\end{aligned}$$

Thus, $\left(\widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta}\right)(k) \cup \left(\mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S\right)(k) \subseteq ɕ$. Since $\mathrm{f}_S$ is an S-uni ɃI-ideal,

$$\mathrm{f}_S(k) \subseteq \left(\widetilde{\Theta} * \mathrm{f}_S * \widetilde{\Theta}\right)(k) \cup \left(\mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S\right)(k) \subseteq ɕ$$

Thus, $k \in$ Ꞩ$(f_S; ɕ)$. Therefore, $[S \cdot$ Ꞩ$(\mathrm{f}_S; ɕ) \cdot S] \cap [$Ꞩ$(\mathrm{f}_S; ɕ) \cdot S \cdot$ Ꞩ$(\mathrm{f}_S; ɕ)] \subseteq$ Ꞩ$(\mathrm{f}_S; ɕ)$. Hence, Ꞩ$(\mathrm{f}_S; ɕ)$ is a ɃI-ideal.

We illustrate Proposition 3.33 with Example 3.34.

**Example 3.31.** Let the soft set $\omega_S$ in Example 3.2. By considering the image set of $\omega_S$, that is,

$$Im(\omega_S) = \{\{1\}, \{1,3\}, \{1,2,3\}, \{1,3,4\}\}$$

we obtain the following:

$$Ꞩ(\omega_S; ɕ) \begin{cases} \{ʋ\}, & ɕ = \{1\} \\ \{tt, ʋ\}, & ɕ = \{1,3\} \\ \{tt, Ꜧ, ʋ\}, & ɕ = \{1,2,3\} \\ \{₵, tt, ʋ\}, & ɕ = \{1,3,4\} \end{cases}$$

Here, $\{₵, tt, Ꜧ, ʋ\}$, $\{₵, tt, Ꜧ\}$, $\{Ꜧ\}$ and $\{₵\}$ are all ɃI-ideals. In fact, since

$$(S \bullet \{₵, tt, ʋ\} \bullet S) \cap (\{₵, tt, ʋ\} \bullet S \bullet \{₵, tt, ʋ\}) = \{₵, tt, ʋ\} \subseteq \{₵, tt, ʋ\}$$
$$(S \bullet \{tt, Ꜧ, ʋ\} \bullet S) \cap (\{tt, Ꜧ, ʋ\} \bullet S \bullet \{tt, Ꜧ, ʋ\}) = \{tt, ʋ\} \subseteq \{tt, Ꜧ, ʋ\}$$
$$(S \bullet \{tt, ʋ\} \bullet S) \cap (\{tt, ʋ\} \bullet S \bullet \{tt, ʋ\}) = \{tt, ʋ\} \subseteq \{tt, ʋ\}$$
$$(S \bullet \{ʋ\} \bullet S) \cap (\{ʋ\} \bullet S \bullet \{ʋ\}) = \{ʋ\} \subseteq \{ʋ\}$$

each Ꞩ$(\omega_S; ɕ)$ is a ɃI-ideal.

Now, consider the soft set $u_S$ in Example 3.2. By taking into account

$$Im(u_S) = \{\{3,5\}, \{5,8\}, \{3,5,7\}, \{3,5,7,8\}\}$$

we obtain the following:

$$Ꞩ(u_S; ɕ) \begin{cases} \{₵\}, & ɕ = \{3,5\} \\ \{Ꜧ\}, & ɕ = \{5,8\} \\ \{₵, ʋ\}, & ɕ = \{3,5,7\} \\ \{₵, tt, Ꜧ, ʋ\}, & ɕ = \{3,5,7,8\} \end{cases}$$

Here, $\{Ꜧ\}$ is not a ɃI-ideal. In fact, since

$$(S \bullet \{Ꜧ\} \bullet S) \cap (\{Ꜧ\} \bullet S \bullet \{Ꜧ\}) = \{tt, ʋ\} \nsubseteq \{Ꜧ\}$$

one of the Ꞩ$(u_S; ɕ)$ is not a ɃI-ideal. It is seen that each of Ꞩ$(u_S; ɕ)$is not a ɃI-ideal. On the other hand, in Example 3.2 it was shown that $u_S$ is not an S-uni ɃI-ideal.

**Definition 3.32.** Let $f_S$ be an S-uni ꞖI-ideal. Then, the ꞖI-ideals $\mathcal{S}(f_S; \sigma)$ are called lower $\sigma$-ꞖI-ideals of $f_S$.

**Theorem 3.33.** Let $S$ be regular. Then, $f_S = (\widetilde{\Theta} * f_S * \widetilde{\Theta}) \tilde{\cup} (f_S * \widetilde{\Theta} * f_S)$ for every S-uni ꞖI-ideal $f_S$.

**Proof:** Let $S$ be regular, $f_S$ be an S-uni ꞖI-ideal and $x \in S$. Then, $(\widetilde{\Theta} * f_S * \widetilde{\Theta}) \tilde{\cup} (f_S * \widetilde{\Theta} * f_S) \tilde{\supseteq} f_S$, and there exists an element $y \in S$ such that $x = xyx$. Since

$$\begin{aligned}(\widetilde{\Theta} * f_S * \widetilde{\Theta})(x) &= \bigcap_{x=ab} \{(\widetilde{\Theta} * f_S)(a) \cup \widetilde{\Theta}(b)\} \\ &\subseteq (\widetilde{\Theta} * f_S)(x) \cup \widetilde{\Theta}(yx) \\ &= \left[\bigcap_{x=ab} \{\widetilde{\Theta}(a) \cup f_S(b)\}\right] \cup \emptyset \\ &\subseteq \widetilde{\Theta}(xy) \cup f_S(x) \\ &= \emptyset \cup f_S(x) \\ &= f_S(x)\end{aligned}$$

and

$$\begin{aligned}(f_S * \widetilde{\Theta} * f_S)(x) &= \bigcap_{x=ab} \{(f_S * \widetilde{\Theta})(a) \cup f_S(b)\} \\ &\subseteq (f_S * \widetilde{\Theta})(xy) \cup f_S(x) \\ &= \left[\bigcap_{xy=mn} \{f_S(m) \cup \widetilde{\Theta}(n)\}\right] \cup f_S(x) \\ &\subseteq [f_S(x) \cup \widetilde{\Theta}(yxy)] \cup f_S(x) \\ &= [f_S(x) \cup \emptyset] \cup f_S(x) \\ &= f_S(x)\end{aligned}$$

$(\widetilde{\Theta} * f_S * \widetilde{\Theta})(x) \cup (f_S * \widetilde{\Theta} * f_S)(x) \subseteq f_S(x)$ implying that $(\widetilde{\Theta} * f_S * \widetilde{\Theta}) \tilde{\cup} (f_S * \widetilde{\Theta} * f_S) \tilde{\subseteq} f_S$. Therefore, $f_S = (\widetilde{\Theta} * f_S * \widetilde{\Theta}) \tilde{\cup} (f_S * \widetilde{\Theta} * f_S)$.

**Theorem 3.34.** Let $S$ be regular, $f_S$ be an S-uni ꞖI-ideal, and $\acute{u}_S$ be an S-uni ideal. Then, $f_S \tilde{\cup} \acute{u}_S \tilde{\supseteq} (\acute{u}_S * f_S * \acute{u}_S) \tilde{\cup} (f_S * \acute{u}_S * f_S)$.

**Proof:** Let $S$ be regular, $f_S$ be an S-uni ꞖI-ideal, $\acute{u}_S$ be an S-uni ideal and $\mathrm{r} \in S$. Then, $(\widetilde{\Theta} * f_S * \widetilde{\Theta}) \tilde{\cup} (f_S * \widetilde{\Theta} * f_S) \tilde{\supseteq} f_S$, for $\acute{p}, q \in S$, $\acute{u}_S(\acute{p}q) \subseteq \acute{u}_S(\acute{p})$ and $\acute{u}_S(\acute{p}q) \subseteq \acute{u}_S(q)$, and there exists an element $y \in S$ such that $\mathrm{r} = \mathrm{r}y\mathrm{r}$. Since

$$\begin{aligned}(\acute{u}_S * f_S * \acute{u}_S)(\mathrm{r}) &= \bigcap_{\mathrm{r}=ab} \{\acute{u}_S(a) \cup (f_S * \acute{u}_S)(b)\} \\ &\subseteq \acute{u}_S(\mathrm{r}y) \cup (f_S * \acute{u}_S)(\mathrm{r}) \\ &= \acute{u}_S(\mathrm{r}y) \cup \left[\bigcap_{\mathrm{r}=ab} \{f_S(a) \cup \acute{u}_S(b)\}\right] \\ &\subseteq \acute{u}_S(\mathrm{r}y) \cup [f_S(\mathrm{r}) \cup \acute{u}_S(y\mathrm{r})]\end{aligned}$$

$$\subseteq ɰ_S(\mathrm{r}) \cup [f_S(\mathrm{r}) \cup ɰ_S(\mathrm{r})]$$
$$= f_S(\mathrm{r}) \cup ɰ_S(\mathrm{r})$$
$$= (f_S \tilde{\cup} ɰ_S)(\mathrm{r})$$

and

$$\begin{aligned}(f_S * ɰ_S * f_S)(\mathrm{r}) &= \bigcap_{\mathrm{r}=ab} \{(f_S * ɰ_S)(a) \cup f_S(b)\} \\ &\subseteq (f_S * ɰ_S)(\mathrm{r}y) \cup f_S(\mathrm{r}) \\ &= \left[\bigcap_{\mathrm{r}y=mn} \{f_S(m) \cup ɰ_S(n)\}\right] \cup f_S(\mathrm{r}) \\ &\subseteq [f_S(\mathrm{r}) \cup ɰ_S(y\mathrm{r}y)] \cup f_S(\mathrm{r}) \\ &\subseteq [f_S(\mathrm{r}) \cup ɰ_S(\mathrm{r})] \cup f_S(\mathrm{r}) \\ &= f_S(\mathrm{r}) \cup ɰ_S(\mathrm{r}) \\ &= (f_S \tilde{\cup} ɰ_S)(\mathrm{r})\end{aligned}$$

Thus, $(ɰ_S * f_S * ɰ_S)(\mathrm{r}) \cup (f_S * ɰ_S * f_S)(\mathrm{r}) \subseteq (f_S \tilde{\cup} ɰ_S)(\mathrm{r})$. Hence, $f_S \tilde{\cup} ɰ_S \tilde{\supseteq} (ɰ_S * f_S * ɰ_S) \tilde{\cup} (f_S * ɰ_S * f_S)$.

## 4. Conclusions

Rao [6] proposed bi-interior ideals of semigroups and examined the properties of bi-interior ideals of semigroups as a generalization of the bi-ideals and interior ideals of semigroups. In this paper, we appling the concept of bi-interior ideals of semigroups to semigroup theory and soft set theory, introduced "S-uni bi-interior ideals (abbreviated by "S-uni ɃI-ideals" throughout the text) of semigroups". The relations between different types of S-uni ideals of a semigroup and S-uni bi-interior ideals were established. We demonstrated that an S-uni left ideal, right ideal, ideal, bi-ideal, interior ideal, and quasi-ideal is an S-uni bi-interior ideal, however the opposite is not true with counterexamples. For the converses, we illustrated that the semigroup should be special soft simple. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of regarding soft set operations and notions supporting our assertions with particular, illuminating examples. In later studies, various semigroup types can be used to characterize S-uni bi-interior ideals.

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where A → B denotes that A is B but B may not always be A.

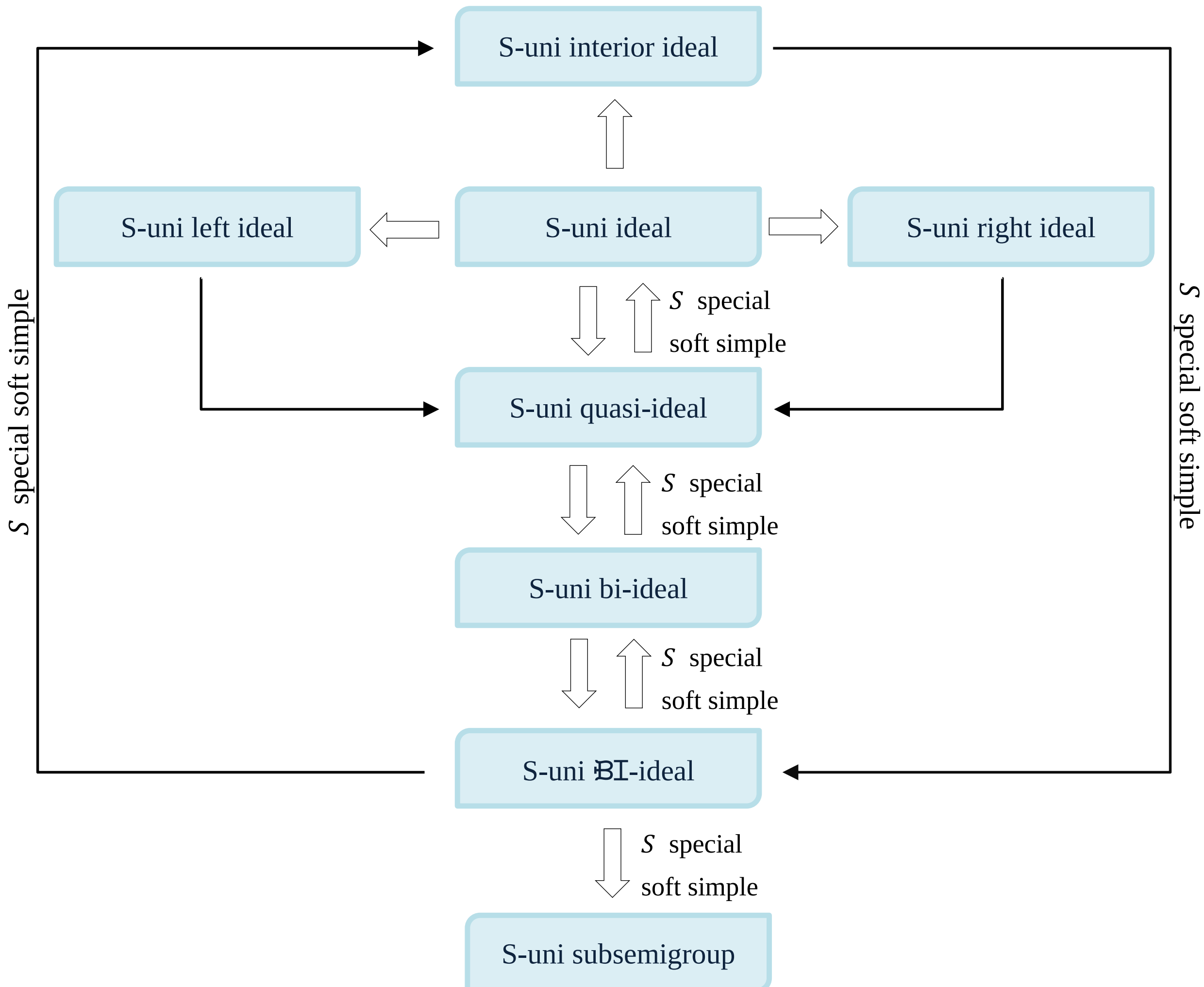


**Figure 1.** Diagram showing the relationships between the certain S-uni ideals

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