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New Fixed Point Theorems for $\theta-\phi-$Contraction on Quasi-Metric Spaces

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Abstract. In this paper, we introduce the concept of $\theta-$contraction and $\theta-\phi-$contraction in a generalized setting such as quasi-metric spaces with the aim to study existence of the unique fixed point for self mapping. Our established theorems extend and elaborate classical conclusions of standart metric supported by many examples and corollaries as a further completion of the results in the current literature.

1. Introduction

The most celebrated result of the theory of metric fixed points is the Banach contraction principle [1]. Due to its importance, several authors have obtained many interesting extensions and generalizations [2,5,8].

In 1931, for the first time quasi-metric spaces were introduced by Wilson [14], in such a way that without the requirement that the (asymmetric) metric d has to satisfy $d(x,y)=d(y,x)$. As such, any metric space is a quasi-metric space but the converse is not true. Various fixed point results were established on such spaces; see [7,9–12] and references therein. In quasi-metric spaces some notions, as convergence, compactness and completeness are different from those in metric case. Collins and zimer [3] have discussed these notions in the quasi-metric space.

Recently, Samet et al. [4] introduced a new concept of $\theta-$contraction and established some fixed point results for such mappings in complete generalized metric spaces and generalized the results of Banach contraction on such space.

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Very recently, Zheng et al. [13] introduced a new concept of $\theta-\phi-$contraction and established some fixed point results for such mappings in complete metric spaces and generalized the results of Brower and Kannan.

In this paper, aspired by the notion of Samet et al [4] and the notion introduced by Zheng et al. [13], we present a new notion of generalized $\theta-$contraction and $\theta-\phi-$contraction and establish various fixed point theorems for such mappings in complete quasi-metric spaces. The results presented in the paper improve and extend the corresponding results of Kannan. [5] and Reich [8].

2. Preliminaries

Definition 2.1. *Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y, z \in X$ satisfies*

(i) $d(x, y) = d(y, x) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) \leq d(x, z) + d(z, y)$. *(Triangular Inequality)*

Then (X, d) is called an quasi-metric space.

Definition 2.2. *[3]. Let (X, d) is a quasi-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$.*

(i) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ right (left) converges to x if and only if*

$$\lim_{n\to+\infty} d(x, x_n) = \lim_{n\to+\infty} d(x_n, x) = 0.$$

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ right Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$, for all $m > n \geq N$ such that $d(x_n, x_m) < \varepsilon$.*

(iii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ left Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$, for all $m > n \geq N$, such that $d(x_m, x_n) < \varepsilon$.*

Lemma 2.1. *[3]. Let (X, d) be a quasi-metric space and $\{x_n\}_n$ be a sequence in X. If $\{x_n\}_n$ right converges to $x \in X$ and left converges to $y \in X$, then $x = y$.*

Definition 2.3. *[3]. Let (X, d) be a quasi-metric space. X is said to be right (left) complete if every right (left) Cauchy sequence $\{x_n\}_n$ in X right (left) converges to $x \in X$.*

Definition 2.4. *[3]. Let (X, d) be a quasi-metric space. X is said to be complete if X is right and left complete.*

The following definition was given by Samet et al in [4].

Definition 2.5. *[4] Let Θ_C be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ such that*

(θ_1) *θ is increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x < y$, $\theta(x) < \theta(y)$ $\forall x, y \in X$;*

(θ_2) *For each sequence $x_n \in]0, +\infty[$,*

$$\lim_{n\to\infty} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) *θ is continuous.*

Definition 2.6. *[4] Let Θ_G be the family of all functions $\theta :]0,+\infty[\to]1,+\infty[$ such that*

(θ_1) *θ is increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x < y$, $\theta(x) < \theta(y)$ $\forall x, y \in X$;*

(θ_2) *For each sequence $x_n \in]0,+\infty[$,*

$$\lim_{n\to\infty} x_n = 0, \quad \textit{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) *there exist $r \in]0,1[$ and $l > 0$ such that $\lim_{n\to\infty} \frac{\theta(t)-1}{t^r} = l$;*

(θ_4) *θ is continuous.*

In [13]. Zheng Presented the concept of $\theta - \phi-$contraction on metric spaces and proved the following nice result.

Definition 2.7. *[13] Let Φ be the family of all functions ϕ: $[1,+\infty[\to [1,+\infty[$, such that*

(ϕ_1) *ϕ is increasing;*

(ϕ_2) *For each $t \in]1,+\infty[$, $lim_{n\to\infty}\phi^n(t) = 1$;*

(ϕ_3) *ϕ is continuous.*

Lemma 2.2. *[13] If $\phi \in \Phi$. Then $\phi(1)$=1, and $\phi(t) < t$ for all $t \in]1+\infty[$.*

Definition 2.8. *[13]. Let (X, d) be a metric space and $T : X \to X$ be a mapping.*
T is said to be a $\theta - \phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \varphi[\theta(N(x, y))],$$

where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Theorem 2.1. *[13]. Let (X, d) be an complete metric space and let $T : X \to X$ be an $\theta - \phi$-contraction. Then T has a unique fixed point.*

3. Main result

In this paper, we presented the concept θ-contraction and $\theta - \phi$-contraction of quasi-metric space and we prove some fixed point results for such spaces. Also, we derive some useful corollaries of these results.

Theorem 3.1. *Let (X, d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exists $\theta \in \Theta_G$ and $r \in]0,1[$ such that for all $x, y \in X$*

$$\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Rightarrow \theta[d(Tx, Ty)] \leq [\theta(M(x, y))]^r, \tag{3.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

and

$$d(y, x) \leq d\left(T^2y, x\right)$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X, we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.
If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ and $d(x_{n_0+1}, x_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Then we assume that $d(x_n, x_{n+1}) > 0$ or $d(x_{n+1}, x_n) > 0$.
Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.2}$$

Applying (3.1) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &= \theta(d(Tx_{n-1}, Tx_n)) \\ &\leq [\theta(M(x_n, x_{n-1}))]^r,\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max(d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned}$$

Suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ for some positive integer n, we have

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_n, x_{n+1}))]^r < \theta(d(x_n, x_{n+1})),$$

which is a contradiction. Hence

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^r \leq \ldots \leq [\theta(d(x_0, x_1))]^{r^n} \tag{3.3}$$

Since $r \in]0, 1[$, we obtain

$$\theta(d(x_n, x_{n+1})) < \theta(d(x_{n-1}, x_n)).$$

By (θ_1), we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.4}$$

Applying (3.1) with $x = x_n$ and $y = x_{n-1}$, we obtain

$$\begin{aligned}\theta(d(x_{n+1}, x_n)) &= \theta(d(Tx_n, Tx_{n-1})) \\ &\leq [\theta(M(x_n, x_{n-1}))]^r\end{aligned}$$

where

$$\begin{aligned}M(x_n, x_{n-1}) &= \max(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max(d(x_{n-1}, x_n), d(x_n, x_{n-1})\}.\end{aligned}$$

Suppose that $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$ for some $n \in \mathbb{N}$.
Case 1 : $d(x_n, x_{n-1}) \geq d(x_{n-1}, x_n)$, we get

$$\begin{aligned}\theta(d(x_n, x_{n-1})) &\leq \theta(d(x_{n+1}, x_n)) \\ &\leq [\theta(d(x_n, x_{n-1}))]^r \\ &< \theta(d(x_n, x_{n-1})).\end{aligned}$$

Which is a contradiction.

Case 2 : $d(x_n, x_{n-1}) < d(x_{n-1}, x_n)$, we get

$$\theta(d(x_{n+1}, x_n)) \leq [\theta(d(x_{n-1}, x_n))]^r.$$

Since $d(y, x) \leq d\left(T^2y, x\right)$, so $d(x_{n-1}, x_n) \leq d(x_{n+1}, x_n)$. Which implies that

$$\begin{aligned}\theta(d(x_{n+1}, x_n)) &\leq [\theta(d(x_{n-1}, x_n))]^r \\ &\leq [\theta(d(x_{n+1}, x_n))]^r \\ &< \theta(d(x_{n+1}, x_n)),\end{aligned}$$

which is a contradiction. Hence

$$\theta(d(x_{n+1}, x_n)) \leq [\theta(d(x_n, x_{n-1}))]^r \leq \ldots \leq [\theta(d(x_1, x_0))]^{r^n} \tag{3.5}$$

Since $r \in]0, 1[$ and (θ_1), we conclude that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \tag{3.6}$$

From (3.4), the sequence $d(x_n, x_{n+1})_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \alpha. \tag{3.7}$$

Assume that $\alpha > 0$. By property of θ and using (3.3), we obtain

$$1 < \theta(\alpha) \leq \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{r^n} \tag{3.8}$$

Letting $\lim_{n\to\infty}$ in (3.8) and using (θ_2), we get

$$1 < \theta(\alpha) \leq \lim_{n\to+\infty} [\theta(d(x_0, x_1))]^{r^n}.$$

Therefore,

$$1 < \theta(\alpha) \leq 1$$

Which is a contradiction. Thus, $\alpha = 0$, then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.9}$$

From (3.6), the sequence $d(x_{n+1}, x_n)_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lambda. \tag{3.10}$$

Assume that $\lambda > 0$. By property of θ and using (3.5), we obtain

$$1 < \theta(\lambda) \leq \theta(d(x_{n+1}, x_n)) \leq [\theta(d(x_1, x_0))]^{r^n} \tag{3.11}$$

Letting $\lim_{n\to\infty}$ in (3.11) and using (θ_2), we get

$$1 < \theta(\lambda) \leq \lim_{n\to+\infty} [\theta(d(x_1, x_0))]^{r^n}.$$

Therefore,

$$1 < \theta(\alpha) \leq 1$$

Which is a contradiction. Thus, $\lambda = 0$, then

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{3.12}$$

Step 2 : We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
Firstly we show $\{x_n\}_{n\in\mathbb{N}}$ is right-Cauchy sequence i.e. $\lim_{n,m\to\infty} d\left(x_n, x_m\right) = 0$.
From condition (θ_3), there exist $k \in]0,1[$ and $l > 0$ such that

$$\lim_{n\to\infty} \frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} = l$$

Suppose that $l < \infty$. In this case, let $A = \frac{l}{2}$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} - l| \leq A \ for\ all\ n \geq n_0.$$

This implies that

$$\frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} \geq A \ for\ all\ n \geq n_0.$$

Then

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq Bn\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \ for\ all\ n \geq n_0.$$

Where $A = \frac{1}{B}$
Now, suppose that $l = \infty$. Let $B > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k}| \geq B \ for\ all\ n \geq n_0.$$

This implies that

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \ for\ all\ n \geq n_0.$$

Where $A = \frac{1}{B}$.
Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \ for\ all\ n \geq n_0.$$

By continuing this process we have,

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\left(\theta\left(d\left(x_0, x_1\right)\right)\right)^{r^n} - 1\right] \ for\ all\ n \geq n_0. \tag{3.13}$$

Letting $n \to \infty$ in (3.13), we obtain

$$\lim_{n\to\infty} n\left[d\left(x_n, x_{n+1}\right)^k\right] = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}},\ for\ all\ n \geq n_1. \tag{3.14}$$

Now, by triangular inequality and using (3.14), we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m),\ for\ all\ m > n \geq n_1 \\ &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + ... + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

From the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is right-Cauchy sequence in (X, d).

Secondly we show $\{x_n\}_{n\in\mathbb{N}}$ is left-Cauchy sequence i.e. $\lim_{m,n\to\infty} d(x_m, x_n) = 0$

Applying (3.1) with $x = x_n$ and $y = x_{n-1}$, then. From condition (θ_3), there exist $k \in]0, 1[$ and $l > 0$ such that

$$\lim_{n\to\infty} \frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} = l.$$

Suppose that $l < \infty$. In this case, let $H = \frac{l}{2}$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} - l| \leq H\ for\ all\ n \geq n_0.$$

This implies that

$$\frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} \geq H\ for\ all\ n \geq n_0.$$

Then

$$n\left[d(x_{n+1}, x_n)^k\right] \leq Mn\left[\theta(d(x_{n+1}, x_n)) - 1\right]\ for\ all\ n \geq n_0.$$

Where $H = \frac{1}{M}$ Suppose that $l = \infty$. Let $M > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k}| \geq M\ for\ all\ n \geq n_0.$$

This implies that

$$n\left[d(x_{n+1}, x_n)^k\right] \leq Hn\left[\theta(d(x_{n+1}, x_n)) - 1\right]\ for\ all\ n \geq n_0.$$

Where $H = \frac{1}{M}$.

Thus, in all cases, there exist $H > 0$ and $n \in \mathbb{N}$ such that

$$n\left[d(x_{n+1}, x_n)^k\right] \leq An\left[\theta(d(x_{n+1}, x_n)) - 1\right]\ for\ all\ n \geq n_0.$$

By continuing this process we have,

$$n\left[d\left(x_{n+1},x_n\right)^k\right]\leq Hn\left[\left(\theta\left(d\left(x_1,x_0\right)\right)\right)^{r^n}-1\right]\ for\ all\ n\geq n_0. \tag{3.15}$$

Letting $n\to\infty$ in (3.15), we obtain

$$\lim_{n\to\infty} n\left[d\left(x_{n+1},x_n\right)^k\right]=0.$$

Thus, there exists $n_1\in\mathbb{N}$ such that

$$d\left(x_{n+1},x_n\right)\leq\frac{1}{n^{\frac{1}{k}}},\ for\ all\ n\geq n_1. \tag{3.16}$$

Now, by triangular inequality and using (3.16), we get

$$\begin{aligned}d\left(x_m,x_n\right)&\leq d\left(x_m,x_{m+1}\right)+d\left(x_{m+1},x_{m+2}\right)+...+d\left(x_{n-1},x_n\right),\ for\ all\ n>m\geq n_1\\&\leq\frac{1}{m^{\frac{1}{k}}}+\frac{1}{(m+1)^{\frac{1}{k}}}+...+\frac{1}{(n-1)^{\frac{1}{k}}}\\&\leq\sum_{i=m}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

From the convergence of the series $\sum_{i=m}^{\infty}\frac{1}{i^{\frac{1}{k}}}$, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is left-Cauchy sequence in (X,d).

Finally, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in complete quasi-metric space (X,d). By completeness of (X,d), there exists z, $w\in X$ such that

$$\lim_{n\to\infty} d\left(x_n,z\right)=0 \text{ and } \lim_{n\to\infty} d\left(w,x_n\right)=0.$$

By Lemma (2.3), we get $z=w$.

Step 3: we prove that $z=Tz$, i.e. $d\left(Tz,z\right)=0$ and $d\left(z,Tz\right)=0$.

Arguing by contradiction, we assume that $d\left(Tz,z\right)>0$ or $d\left(z,Tz\right)>0$.

First assume that $d\left(z,Tz\right)>0$. By triangular inequality we get

$$d\left(Tx_n,Tz\right)\leq d\left(Tx_n,z\right)+d\left(z,Tz\right) \tag{3.17}$$

and

$$d\left(z,Tz\right)\leq d\left(z,Tx_n\right)+d\left(Tx_n,Tz\right) \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\lim_{n\to+\infty} d\left(Tx_n,Tz\right)=d\left(z,Tz\right). \tag{3.19}$$

So, there exists $n_0\in\mathbb{N}$ such that

$$d\left(Tx_n,Tz\right)\geq d\left(z,Tz\right)>0\ for\ all\ n\geq n_0.$$

and we have

$$\max\{d\left(Tx_n,Tz\right),d\left(Tz,Tx_n\right)\}>0.$$

Applying (3.1) with $x = x_n$ and $y = z$, we obtain

$$\theta\left(d\left(Tx_n, Tz\right)\right) \leq \left[\theta\left(M\left(x_n, z\right)\right)\right]^r, \tag{3.20}$$

where

$$M\left(x_n, z\right) = \max\left\{d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, z\right)\right\}.$$

and

$$\lim_{n\to+\infty} M\left(x_n, z\right) = d\left(z, Tz\right). \tag{3.21}$$

Taking the limit as $n \to \infty$ in (3.20) and using the properties of θ, we obtain we obtain

$$\begin{aligned}
\lim_{n\to+\infty} \theta\left(d\left(Tx_n, Tz\right)\right) &= \theta\left(\lim_{n\to+\infty} d\left(Tx_n, Tz\right)\right) \\
&= \theta\left(d\left(z, Tz\right)\right) \\
&\leq \left[\theta\left(\lim_{n\to+\infty} M\left(x_n, z\right)\right)\right]^r \\
&= \left[\theta\left(d\left(z, Tz\right)\right)\right]^r \\
&< \theta\left(d\left(z, Tz\right)\right).
\end{aligned}$$

which is contradiction.
If $d\left(Tz, z\right) > 0$, by similar method, we get contradiction. Therefore $d\left(z, Tz\right) = 0$ and $d\left(Tz, z\right) = 0$, Hence $z = Tz$.
Step 4. Uniqueness.
Suppose that there are two distinct point $z, u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z, u) = d(Tz, Tu) > 0$ or $d(u, z) = d(Tu, Tz) > 0$.
Applying (3.1) with $x = z$ and $y = u$, we obtain

$$\theta\left(d(z, u)\right) \leq \left[\theta\left(M(z, u)\right)\right]^r,$$

where

$$M(z, u) = \max\left\{d\left(z, u\right), d\left(z, Tz\right), d\left(u, Tu\right)\right\} = d(z, u)$$

which implies that $\theta\left(d(z, u)\right) < \theta\left(d(z, u)\right)$. Is a contradiction, thus, $z = u$. □

Example 3.1. *Let $X = [1, +\infty[$. Define $d : X \times X \to [0, +\infty[$ by*

$$d(x, y) = \max\{y - x, 0\} \textit{ for all } x, y \in X.$$

Then (X, d) is a complete quasi-metric space.
Define mapping $T : X \to X$ by

$$T(x) = \sqrt{x}.$$

Then, $T(x) \in [1, +\infty[$. *Let* $\theta(t) = e^{\sqrt{t}}$, $r = \frac{1}{2}$. *It obvious that* $\theta \in \Theta$ *and* $r \in]0, 1[$.
Let $x, y \in [1, +\infty[$, *then we have*

$$d(y,x) = \max\{x - y, 0\} \text{ and } d(T^2y, x) = \max\{x - y^{\frac{1}{4}}, 0\}.$$

So,

$$\max\{x - y, 0\} \leq \max\{y - y^{\frac{1}{4}}, 0\},$$

which implies that

$$d(y,x) \leq d(T^2y, x) \text{ for all } x, y \in X.$$

On the other hand

$$d(Tx, Ty) = d\left(\sqrt{x}, \sqrt{y}\right) = \max\{\sqrt{y} - \sqrt{x}, 0\},$$

and

$$\begin{aligned} M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\} \\ &= \max\left\{\max\{y - x, 0\}, \max\{\sqrt{x} - x, 0\}, \max\{\sqrt{y} - y, 0\}\right\}. \end{aligned}$$

First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Leftrightarrow y > x$. *Hence*

$$d(Tx, Ty) = \sqrt{y} - \sqrt{x}, \ \theta(d(Tx,Ty) = e^{\sqrt{\sqrt{y} - \sqrt{x}}}$$

and

$$\begin{aligned} M(x,y) &= \max\left\{y - x, \ \sqrt{x} - x, \ \sqrt{y} - y\right\} \\ &= y - x. \end{aligned}$$

Then, we have

$$[\theta(d(x,y))]^{\frac{1}{2}} = \left[e^{\sqrt{y-x}}\right]^{\frac{1}{2}} = e^{\sqrt{\sqrt{y-x}}}.$$

On the other hand

$$\theta(d(Tx,Ty) - [\theta(d(x,y))]^{\frac{1}{2}} = e^{\sqrt{\sqrt{y} - \sqrt{x}}} - e^{\sqrt{\sqrt{y-x}}}.$$

Since $x, y \in [1, +\infty[$, *then*

$$\sqrt{y} - \sqrt{x} \leq \sqrt{y - x}.$$

Since $e^{\sqrt{x}}$ *is increasing for all* $x \geq 0$. *Hence*

$$e^{\sqrt{\sqrt{y} - \sqrt{x}}} - e^{\sqrt{\sqrt{y-x}}}$$

which implies that

$$\begin{aligned} \theta(d(Tx,Ty) &\leq [\theta(d(x,y))]^{\frac{1}{2}} \\ &\leq [\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))]^{\frac{1}{2}} \end{aligned}$$

Hence, the condition (3.1) *is satisfied. Therefore,* T *has a unique fixed point* $z = 1$.

If we remove our condition $d(y,x) \leq d(Ty^2, x)$ *for all* $x, y \in X$., it may be that T does not admit a fixed point.

Example 3.2. *Let* $X = \left[\frac{1}{4}, \frac{3}{5}\right]$. *Define* $d : X \times X \to [0, +\infty[$ *by*

$$d(x, y) = \max\{y - x, 0\} \text{ for all } x, y \in X.$$

Then (X, d) *is a complete quasi-metric space.*
Define mapping $T : X \to X$ *by*

$$T(x) = \frac{\sqrt{x} + 1}{4}$$

Then, $T(x) \in \left[\frac{1}{4}, \frac{3}{5}\right]$. *Let* $\theta(t) = e^{\sqrt{t}}$, $r = \frac{1}{2}$. *It obvious that* $\theta \in \Theta$ *and* $r \in]0, 1[$.
Let $x, y \in \left[\frac{1}{4}, \frac{3}{5}\right]$, *then we have*

$$d(y, x) = \max\{x - y, 0\} \text{ and } d(T^2 y, x) = \max\{x - \frac{1}{4}\left[\sqrt{\frac{\sqrt{y} + 1}{4}} + 1\right], 0\}.$$

If $x > y$ *and* $y = \frac{1}{4}$. *So,*

$$\max\{x - y, 0\} = x - \frac{1}{4} > \max\{x - \frac{1}{4}\left[\sqrt{\frac{\sqrt{y} + 1}{4}} + 1\right], 0\}.$$

which implies that

$$d(y, x) > d(T^2 y, x).$$

On the other hand

$$d(Tx, Ty) = d\left(\frac{\sqrt{x} + 1}{4}, \frac{\sqrt{y} + 1}{4}\right) = \max\{\frac{\sqrt{y} - \sqrt{x}}{4}, 0\},$$

and

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty)\} \\ &= \max\left\{\max\{y - x, 0\}, \max\{\frac{\sqrt{x} + 1}{4} - x, 0\}, \max\{\frac{\sqrt{y} + 1}{4} - y, 0\}\right\}. \end{aligned}$$

First observe that $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Leftrightarrow y > x$. *Hence*

$$d(Tx, Ty) = \frac{\sqrt{y} - \sqrt{x}}{4}, \; \theta(d(Tx, Ty) = e^{\frac{\sqrt{\sqrt{y} - \sqrt{x}}}{2}}$$

and

$$\begin{aligned} M(x, y) &= \max\left\{y - x, \frac{\sqrt{x} + 1}{4} - x, \frac{\sqrt{y} + 1}{4} - y\right\} \\ &\geq y - x. \end{aligned}$$

Then, we have

$$[\theta(d(x, y))] = e^{\sqrt{\sqrt{y - x}}}.$$

On the other hand

$$\theta(d(Tx, Ty) - \sqrt{[\theta(d(x, y))]} = e^{\frac{\sqrt{\sqrt{y} - \sqrt{x}}}{2}} - e^{\sqrt{\sqrt{y - x}}}.$$

Since $x, y \in \left[\frac{1}{4}, \frac{3}{5}\right]$ and the function e^t is increasing for all $t \in \left[\frac{1}{4}, \frac{3}{5}\right]$, then

$$e^{\frac{\sqrt{\sqrt{y}-\sqrt{x}}}{2}} \leq e^{\sqrt{\sqrt{y-x}}}.$$

Which implies that

$$\begin{aligned}\theta(d(Tx, Ty) &\leq [\theta(d(x,y))]^{\frac{1}{2}} \\ &\leq [\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))]^{\frac{1}{2}}\end{aligned}$$

Hence, T has no fixed point.

Theorem 3.2. *Let (X, d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exists $\phi \in \phi$ and $\theta \in \Theta$ such that for all $x, y \in X$*

$$\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(M(x,y))] \tag{3.22}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

and

$$d(y,x) \leq d\left(T^2 y, x\right)$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X, we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ and $d(x_{n_0+1}, x_{n_0}) = 0$, then x_{n_0} is a fixed point of T.
Then, we assume that $d(x_n, x_{n+1}) > 0$ or $d(x_{n+1}, x_n) > 0$. Then $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} > 0$
Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.23}$$

Applying (3.22) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &= \theta(d(Tx_{n-1}, Tx_n)) \\ &\leq \phi[\theta(M(x_{n-1}, x_n))]\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max(d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned}$$

Suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ for some positive integer n, we have

$$\theta(d(x_n, x_{n+1})) \leq \phi[\theta(d(x_n, x_{n+1}))]$$

By Lemma (2.9), we obtain

$$\theta(d(x_n, x_{n+1})) < \theta(d(x_n, x_{n+1})).$$

Which is a contradiction, then

$$\theta\left(d\left(x_n,x_{n+1}\right)\right)\leq\phi\left[\theta\left(d\left(x_{n-1},x_n\right)\right)\right]\leq\ldots\leq\phi^n\left[\theta\left(d\left(x_0,x_1\right)\right)\right] \tag{3.24}$$

By Lemma (2.9), we obtain

$$\theta\left(d\left(x_n,x_{n+1}\right)\right)<\theta\left(d\left(x_{n-1},x_n\right)\right).$$

By (θ_1), we have

$$d\left(x_n,x_{n+1}\right)<d\left(x_{n-1},x_n\right). \tag{3.25}$$

Applying (3.22) with $x=x_n$ and $y=x_{n-1}$, we obtain

$$\begin{aligned}\theta\left(d\left(x_{n+1},x_n\right)\right)&=\theta\left(d\left(Tx_n,Tx_{n-1}\right)\right)\\&\leq\phi\left[\theta\left(M\left(x_n,x_{n-1}\right)\right)\right]\end{aligned}$$

where

$$\begin{aligned}M\left(x_n,x_{n-1}\right)&=\max\{d\left(x_n,x_{n-1}\right),d\left(x_n,x_{n+1}\right),d\left(x_{n-1},x_n\right)\}\\&=\max\{d\left(x_{n-1},x_n\right),d\left(x_n,x_{n-1}\right)\}.\end{aligned}$$

Suppose that $d\left(x_n,x_{n-1}\right)\leq d\left(x_{n+1},x_n\right)$ for some $n\in\mathbb{N}$.
Case 1 : $d\left(x_n,x_{n-1}\right)\geq d\left(x_{n-1},x_n\right)$, we get

$$\begin{aligned}\theta\left(d\left(x_n,x_{n-1}\right)\right)&\leq\theta\left(d\left(x_{n+1},x_n\right)\right)\\&\leq\phi\left[\theta\left(d\left(x_n,x_{n-1}\right)\right)\right]\\&<\theta\left(d\left(x_n,x_{n-1}\right)\right).\end{aligned}$$

Which is a contradiction.
Case 2 : $d\left(x_n,x_{n-1}\right)<d\left(x_{n-1},x_n\right)$, we get

$$\theta\left(d\left(x_{n+1},x_n\right)\right)\leq\phi\left[\theta\left(d\left(x_{n-1},x_n\right)\right)\right]$$

Since $d\left(y,x\right)\leq d\left(T^2y,x\right)$, so $d\left(x_{n-1},x_n\right)\leq d\left(x_{n+1},x_n\right)$ Which implies that

$$\begin{aligned}\theta\left(d\left(x_{n+1},x_n\right)\right)&\leq\phi\left[\theta\left(d\left(x_{n-1},x_n\right)\right)\right]\\&\leq\phi\left[\theta\left(d\left(x_{n+1},x_n\right)\right)\right]\\&<\theta\left(d\left(x_{n+1},x_n\right)\right),\end{aligned}$$

which is a contradiction. Hence

$$\theta\left(d\left(x_{n+1},x_n\right)\right)\leq\phi\left[\theta\left(d\left(x_n,x_{n-1}\right)\right)\right]\leq\ldots\leq\phi^n\left[\theta\left(d\left(x_1,x_0\right)\right)\right] \tag{3.26}$$

By Lemma (2.9) and (θ_1), we conclude that

$$d\left(x_{n+1},x_n\right)<d\left(x_n,x_{n-1}\right). \tag{3.27}$$

From (3.25), the sequence $d\left(x_n, x_{n+1}\right)_{n\in\mathbb{N}}$ is monotone nonincreasing . So there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = \alpha. \tag{3.28}$$

Letting $\lim_{n\to\infty}$ in (3.24) and using (ϕ_2) and (θ_3), we get

$$1 \leq \lim_{n\to+\infty} \theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \lim_{n\to+\infty} \phi^n\left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right]$$

Thus, $\lim_{n\to+\infty} \theta\left(d\left(x_n, x_{n+1}\right)\right) = 1$, then by (θ_2) implies that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{3.29}$$

From (3.27), the sequence $d\left(x_{n+1}, x_n\right)_{n\in\mathbb{N}}$ is monotone nonincreasing . So there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = \lambda. \tag{3.30}$$

Letting $\lim_{n\to\infty}$ in (3.26) and using (ϕ_2) and (θ_3), we get

$$1 \leq \lim_{n\to+\infty} \theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \lim_{n\to+\infty} \phi^n\left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right]$$

Thus, $\lim_{n\to+\infty} \theta\left(d\left(x_{n+1}, x_n\right)\right) = 1$, then by (θ_2) implies that

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{3.31}$$

Step 2 : We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Firstly we show $\{x_n\}_{n\in\mathbb{N}}$ is right-Cauchy sequence. If otherwise there exists an $\varepsilon > 0$ and sequences $\left(n_{(k)}\right)_k$ and $\left(m_{(k)}\right)_k$ such that, for all positive integers k, $\left(n_{(k)}\right) > \left(m_{(k)}\right) > k$,

$$d\left(m_{(k)}, n_{(k)}\right) \leq \varepsilon \tag{3.32}$$

and

$$d\left(m_{(k)}, n_{(k)-1}\right) < \varepsilon \tag{3.33}$$

By triangular inequality, we obtain

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq d\left(x_{m_{(k)}}, x_{n(k)-1}\right) + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right) \\ &< \varepsilon + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right) \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{3.34}$$

Now, by triangular inequality, we have

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{m(k)}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \tag{3.35}$$

$$\leq d\left(x_{m_{(k)+1}}, x_{m(k)}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right). \tag{3.36}$$

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right) \tag{3.37}$$

$$\leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right) \tag{3.38}$$

Letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.39}$$

By (3.39), let $B = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) - \varepsilon| \leq B \ \forall n \geq n_0.$$

This implies that

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \geq B > 0 \ \forall n \geq n_0.$$

Applying (3.22) with $x = x_{m_{(k)}}$ and $y = x_{m_{(k)}}$, we have

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{m_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right], \tag{3.40}$$

where

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)}+1}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)\right\}.$$

Therefore by (3.34) and (3.29), we get that

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{3.41}$$

Letting $k \to \infty$ in (3.40) and using (3.41), (ϕ_3), (θ_3) and Lemma (2.9), we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta(\varepsilon)\right] < \theta(\varepsilon)$$

which is a contradiction .

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a right-Cauchy sequence in (X, d).

Secondly we prove that $\{x_n\}_n \in \mathbb{N}$ is a left-Cauchy sequence, if otherwise there exists an $\varepsilon > 0$ and sequences $\left(n_{(k)}\right)_k$ and $\left(m_{(k)}\right)_k$ such that, for all positive integers k, $\left(n_{(k)}\right) > \left(m_{(k)}\right) > k$,

$$d\left(n_{(k)}, m_{(k)}\right) \leq \varepsilon \tag{3.42}$$

and

$$d\left(n_{(k)-1}, m_{(k)}\right) < \varepsilon \tag{3.43}$$

By triangular inequality, we obtain

$$\begin{aligned}\varepsilon \leq d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) &\leq d\left(x_{n_{(k)}}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m_{(k)}}\right)\\ &< d\left(x_{n(k)}, x_{n_{(k)-1}}\right) + \varepsilon\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \varepsilon. \tag{3.44}$$

Now, by triangular inequality, we have

$$d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) \leq d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \tag{3.45}$$

$$\leq d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) \tag{3.46}$$

and

$$d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) \leq d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right) + d\left(x_{n(k)+1}, x_{m_{(k)}}\right) \tag{3.47}$$

$$\leq d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{m(k)+1}\right) + d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) \tag{3.48}$$

Letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.49}$$

By (3.49), let $A = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) - \varepsilon| \leq A \ \forall n \geq n_1.$$

This implies that

$$d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) \geq A > 0 \ \forall n \geq n_1.$$

Applying (3.22) with $x = x_{n_{(k)}}$ and $y = x_{m_{(k)}}$, we have

$$\theta\left(d\left(x_{n_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(M\left(x_{n_{(k)}}, x_{m_{(k)}}\right)\right)\right] \tag{3.50}$$

where

$$M\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \max\left\{d\left(x_{n_{(k)}}, x_{m_{(k)}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right\}.$$

Therefore by (3.29) and (3.44), we get that

$$\lim_{k\to+\infty} M\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \varepsilon. \tag{3.51}$$

Letting $k \to \infty$ in (3.48) using (3.49) and Lemma (2.9), we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta(\varepsilon)\right] < \theta(\varepsilon),$$

which is a contradiction. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a left-Cauchy sequence in (X, d).
Hence, by completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{n\to+\infty} d\left(x_n, z\right) = \lim_{n\to+\infty} d\left(u, x_n\right) = 0. \tag{3.52}$$

So, from Lemma (2.3), we get $z = u$ and hence

$$\lim_{n\to+\infty} d\left(x_n, z\right) = \lim_{n\to+\infty} d\left(z, x_n\right) = 0.$$

Step 3: we prove that $z = Tz$, i.e. $d\left(Tz, z\right) = 0$ and $d\left(z, Tz\right) = 0$.
Arguing by contradiction, we assume that $d\left(Tz, z\right) > 0$ or $d\left(z, Tz\right) > 0$.
First assume that $d\left(z, Tz\right) > 0$. As in the proof of Theorem (3.1), we get

$$\lim_{n\to+\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right). \tag{3.53}$$

So there exists $n_0 \in \mathbb{N}$ such that

$$d\left(Tx_n, Tz\right) \geq d\left(z, Tz\right) > 0 \ for\ all\ n \geq n_0.$$

Applying (3.22) with $x = x_n$ and $y = z$, we obtain

$$\theta\left(d\left(Tx_n, Tz\right)\right) \leq \phi\left[\theta\left(M\left(x_n, z\right)\right)\right], \tag{3.54}$$

where

$$M\left(x_n, z\right) = \max\left\{d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, z\right)\right\}.$$

Since $\lim_{n\to+\infty} d\left(x_n, x_{n+1}\right) = d\left(x_n, z\right) = 0$, we obtain that

$$\lim_{n\to+\infty} M\left(x_n, z\right) = d\left(z, Tz\right). \tag{3.55}$$

Taking the limit as $n \to \infty$ in (3.54) and using the properties of ϕ and θ, we obtain we obtain

$$\begin{aligned}
\lim_{n\to+\infty} \theta\left(d\left(Tx_n, Tz\right)\right) &= \theta\left(\lim_{n\to+\infty} d\left(Tx_n, Tz\right)\right) \\
&= \theta\left(d\left(z, Tz\right)\right) \\
&\leq \phi\left[\theta\left(\lim_{n\to+\infty} M\left(x_n, z\right)\right)\right] \\
&= \phi\left[\theta\left(d\left(z, Tz\right)\right)\right] \\
&< \theta\left(d\left(z, Tz\right)\right).
\end{aligned}$$

which is contradiction.
If $d\left(Tz, z\right) > 0$, by similar method, we get contradiction. Therefore $d\left(z, Tz\right) = 0$ and $d\left(Tz, z\right) = 0$, Hence $z = Tz$.
Step 4. Uniqueness.
Suppose that there are two distinct point $z, u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z, u) = d(Tz, Tu) > 0$ or $d(u, z) = d(Tu, Tz) > 0$.
Applying (3.22) with $x = z$ and $y = u$, we obtain

$$\theta\left(d(z, u)\right) \leq \phi\left[\theta\left(M(z, u)\right)\right],$$

where

$$M(z, u) = \max\left\{d\left(z, u\right), d\left(z, Tz\right), d\left(u, Tu\right)\right\} = d(z, u)$$

which implies that $\theta\left(d(z, u)\right) < \theta\left(d(z, u)\right)$. Is a contradiction, thus, $z = u$. □

Corollary 3.1. *Let (X, d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exists $\theta \in \Theta_C$ and $r \in \left]0, 1\right[$ such that for all $x, y \in X$*

$$\max\{d\left(Tx, Ty\right), d\left(Ty, Tx\right)\} > 0 \Rightarrow \theta\left[d\left(Tx, Ty\right)\right] \leq \left[\theta\left(M\left(x, y\right)\right)\right]^r, \tag{3.56}$$

where

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), d\left(x, Tx\right), d\left(y, Ty\right)\right\}.$$

and

$$d\left(y, x\right) \leq d\left(T^2 y, x\right)$$

Then T has a unique fixed point.

Proof. Let $\phi(t) = t^k$, for all $t \in [1, +\infty[$. It is obvious that $\phi \in \Phi$ and, we have

$$\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(M(x, y))]. \tag{3.57}$$

Hence T satisfies in assumption of Theorem (3.4) and is the unique fixed point of T. □

Corollary 3.2. *Let (X, d) be a complete quasi-metric space, there exists $\alpha \in \left]0, \frac{1}{2}\right[$ for any $x, y \in X$, $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$, we have*

$$d(Tx, Ty) \leq \alpha[d(Tx, x) + d(y, Ty)].$$

Then T has a fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = t^{2\alpha}$ for all $t \in [1, +\infty[$.
It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. Therefore,

$$\begin{aligned}
\theta(d(Tx, Ty)) &= e^{d(Tx, Ty)} \\
&\leq e^{\alpha(d(Tx, x) + d(y, Ty))} \\
&= e^{2\alpha\left(\frac{d(Tx, x) + d(y, Ty)}{2}\right)} \\
&= \left[e^{\left(\frac{d(Tx, x) + d(y, Ty)}{2}\right)}\right]^{2\alpha} \\
&= \phi\left[\theta\left(\frac{d(Tx, x) + d(y, Ty)}{2}\right)\right] \\
&\leq \phi[\theta(\max\{d(x, y), d(Tx, x), d(y, Ty)\})]
\end{aligned}$$

Therefore, from Theorem 3.4, T has a unique fixed point $x \in X$. □

Corollary 3.3. *Let (X, d) be a complete quasi-metric space, there exists $\lambda \in \left]0, \frac{1}{3}\right[$ for any $x, y \in X$, $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$, we have*

$$d(Tx, Ty) \leq \alpha[d(x, y) + d(Tx, x) + d(y, Ty)].$$

Then T has a fixed point.

Proof. Let $\theta(t) = e^t$ for all t $\in]0, +\infty[$, and $\phi(t) = t^{3\lambda}$ for all t $\in [1, +\infty[$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. Therefore,

$$\begin{aligned}
\theta(d(Tx, Ty)) &= e^{d(Tx, Ty)} \\
&\leq e^{3\lambda\frac{(d(x, y) + d(Tx, x) + d(y, Ty))}{3}} \\
&= \left[e^{\frac{(d(x, y) + d(Tx, x) + d(y, Ty))}{3}}\right]^{3\lambda}
\end{aligned}$$

$$= \phi\left[\theta\left(\left(\frac{(d(x,y)+d(Tx,x)+d(y,Ty))}{3}\right)\right)\right]$$
$$\leq \phi\left[\theta\left(\max\{d(x,y), d(Tx,x), d(y,Ty)\}\right)\right].$$

Therefore, from Theorem 3.4, T has a unique fixed point $x \in X$. □

Example 3.3. *Let* $X = [1,+\infty[$. *Define* $d : X \times X \to [0,+\infty[$ *by*

$$d(x,y) = \max\{y-x, 0\} \text{ for all } x, y \in X.$$

Then (X,d) *is a complete quasi-metric space.*
Define mapping $T : X \to X$ *by*

$$T(x) = \frac{\sqrt{x}+1}{2}$$

Then, $T(x) \in [1,+\infty[$. *Let* $\theta(t) = \sqrt{t}+1, \phi(t) = \frac{t+1}{2}$. *It obvious that* $\theta \in \Theta$ *and* $\phi \in \Phi$.
Let $x, y \in [1,+\infty[$, *then we have*

$$d(y,x) = \max\{x-y, 0\} \text{ and } d(T^2y,x) = \max\{x - \sqrt{\frac{\sqrt{y}+1}{8}} - \frac{1}{2}, 0\}.$$

So,

$$\max\{x-y,0\} \leq \max\{x - \sqrt{\frac{\sqrt{y}+1}{8}} - \frac{1}{2}, 0\},$$

which implies that

$$d(y,x) \leq d(T^2y,x) \text{ for all } x, y \in X.$$

On the other hand

$$d(Tx,Ty) = d\left(\frac{\sqrt{x}+1}{2}, \frac{\sqrt{y}+1}{2}\right) = \max\{\frac{\sqrt{y}-\sqrt{x}}{2}, 0\},$$

and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$
$$= \max\left\{\max\{y-x,0\}, \max\{\frac{\sqrt{x}+1}{2} - x, 0\}, \max\{\frac{\sqrt{y}+1}{2} - y, 0\}\right\}.$$

First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Leftrightarrow y > x$. *Hence*

$$d(Tx,Ty) = \frac{\sqrt{y}-\sqrt{x}}{2}, \ \theta(d(Tx,Ty) = \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} + 1$$

and

$$M(x,y) = y - x.$$

Then, we have

$$\phi\left[\theta(d(x,y))\right] = \frac{\sqrt{y-x}}{2} + 1.$$

On the other hand

$$\theta(d(Tx,Ty) - \phi\left[\theta(d(x,y))\right] = \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} + 1 - \frac{\sqrt{y-x}}{2} + 1$$
$$= \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} - \frac{\sqrt{y-x}}{2}.$$

Since $x, y \in [1, +\infty[$, *then*

$$\sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} - \frac{\sqrt{y-x}}{2} \leq 0.$$

Which implies that

$$\theta(d(Tx,Ty) \leq \phi\left[\theta(d(x,y))\right]$$
$$\leq \phi\left[\theta(\max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\right\}, d\left(y,Tx\right))\right]$$

Hence, the condition (3.22) *is satisfied. Therefore, T has a unique fixed point $z = 1$.*

If we remove our condition $d(y,x) \leq d(Ty^2, x)$ *for all* $x, y \in X$, it may be that T does not admit a fixed point.

Example 3.4. *Let* $X = \left[\frac{1}{4}, \frac{1}{2}\right]$. *Define* $d : X \times X \to [0, +\infty[$ *by*

$$d(x,y) = \max\{y - x, 0\} \text{ for all } x, y \in X.$$

Then (X, d) is a complete quasi-metric space.
Define mapping $T : X \to X$ *by*

$$T(x) = \frac{\sqrt{x}+4}{16}$$

Then, $T(x) \in \left[\frac{1}{4}, \frac{1}{2}\right]$. *Let* $\theta(t) = \sqrt{t} + 1$, $\phi(t) = \frac{t+1}{2}$. *It obvious that* $\theta \in \Theta$ *and* $\phi \in \Phi$.
Let $x, y \in \left[\frac{1}{4}, \frac{1}{2}\right]$, *then we have*

$$d(y,x) = \max\{x - y, 0\} \text{ and } d(T^2y, x) = \max\{x - \frac{1}{16}\left[\sqrt{\frac{\sqrt{y}+4}{16}} + 4\right], 0\}.$$

If $x > y$ *and* $y = \frac{1}{4}$. *So,*

$$\max\{x - y, 0\} = x - \frac{1}{4} > \max\{x - \frac{1}{16}\left[\sqrt{\frac{\sqrt{y}+4}{16}} + 4\right], 0\}.$$

which implies that

$$d(y,x) > d(T^2y, x).$$

On the other hand

$$d(Tx,Ty) = d\left(\frac{\sqrt{x}+4}{16}, \frac{\sqrt{y}+4}{16}\right) = \max\{\frac{\sqrt{y}-\sqrt{x}}{16}, 0\},$$

and

$$\begin{aligned} M(x,y) &= \max\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\} \\ &= \max\left\{\max\{y-x,0\}, \max\{\frac{\sqrt{x}+4}{16}-x,0\}, \max\{\frac{\sqrt{y}+4}{16}-y,0\}\right\}. \end{aligned}$$

First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \Leftrightarrow y > x$. *Hence*

$$d(Tx,Ty) = \frac{\sqrt{y}-\sqrt{x}}{16},\ \theta(d(Tx,Ty) = \sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}+1$$

and

$$\begin{aligned} M(x,y) &= \max\left\{y-x, \frac{\sqrt{x}+4}{16}-x, \frac{\sqrt{y}+4}{16}-y\right\} \\ &\geq y-x. \end{aligned}$$

Then, we have

$$\phi\left[\theta(d(x,y))\right] = \frac{\sqrt{y-x}}{2}+1.$$

On the other hand

$$\begin{aligned} \theta(d(Tx,Ty) - \phi\left[\theta(d(x,y))\right] &= \sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}+1-\frac{\sqrt{y-x}}{16}-1 \\ &= \sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}-\frac{\sqrt{y-x}}{2}. \end{aligned}$$

Since $x,y \in \left[\frac{1}{4},\frac{1}{2}\right]$, *then*

$$\sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}-\frac{\sqrt{y-x}}{2} \leq 0.$$

Which implies that

$$\begin{aligned} \theta(d(Tx,Ty) &\leq \phi\left[\theta(d(x,y))\right] \\ &\leq \phi\left[\theta(\max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\right\}, d\left(y,Tx\right))\right] \end{aligned}$$

Hence, T has no fixed point.

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