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Evaluating Employee Performance: An Approach on Łukasiewicz Intuitionistic Fuzzy Sets in *BM*-Algebras

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Abstract. In contemporary human resource management, performance evaluations are often influenced by subjectivity and uncertainty, posing challenges to fairness and accuracy. This study introduces a mathematically grounded approach to employee performance assessment by integrating Łukasiewicz logic with intuitionistic fuzzy set theory, framed within the structure of *BM*-algebras. We construct and examine Łukasiewicz intuitionistic fuzzy subalgebras ($\mathfrak{LIFA}$) and ideals ($\mathfrak{LIFI}$), developing a set of theoretical results to define their properties and interactions. Through illustrative examples, we demonstrate the logical consistency and applicability of these constructs. The proposed model employs min-max normalization and fuzzy reasoning to facilitate equitable, transparent, and adaptable evaluations. Beyond workplace settings, this framework holds particular promise for research-oriented educational institutions by fostering inclusive assessment strategies and supporting a more dynamic and responsive learning environment. Moreover, the model's potential to be scaled and shared across collaborative networks underscores its relevance to collective capacity-building and institutional development.

1. Introduction

An essential part of effective human resource management is employee performance evaluation. Vital decisions regarding the organization, including succession planning, training, pay, and promotions, are based on it. Employees are evaluated according to several criteria in traditional performance reviews, including technical proficiency, leadership, communication skills, timeliness, and client interaction. However, because human judgment is qualitative, these assessments

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frequently contain a high degree of subjectivity, ambiguity, and inconsistency. Furthermore, the involvement of several evaluators or criteria with differing degrees of importance adds to the complexity. To manage such ambiguity, we utilize mathematical models, such as set theory, to provide a framework for analysis. A fuzzy set is a type of set theory that deals with inclusive degrees and was introduced by Professor Zadeh [9]. In 1983, Atanassov [1] broadened the thought of Zadeh's fuzzy sets, which are today referred to as intuitionistic fuzzy sets. This collection consists of elements with an exclusive degree and an inclusive degree, as in a fuzzy set. Chaira [2] extended the definition and defined the operations on fuzzy sets and intuitionistic fuzzy sets for applications in decision-making problems. In the core domain of mathematics, algebra employs the formal manipulation of abstract symbols and arithmetic operations, rather than specific numerical values. Several algebraic structures have been developed in the context of general or universal algebra. It explains the basis of propositional calculus. As an aspect of that, mathematicians delivered the theory of BCK/BCI-algebras. Apart from these two algebraic structures, there were various algebraic structures, namely $BCC/BCH/B/BE$-algebras, etc. In 2007, Kim introduced the theory of BM-algebras [8], which is a specialization of B-algebras. Jan Łukasiewicz was a Polish logical thinker and theologian who performed many improvements in propositional logic called Łukasiewicz logic or Łukasz logic. It is a non-traditional and vastly valued logic of Łukasiewicz t-norm. Making use of the theory of Łukasz t-norm in BCK/BCI-algebras, Jun and Ahn introduced Łukasiewicz fuzzy sets along with the concept of Łukasiewicz fuzzy subalgebras [6]. They subsequently formulated the notion of Łukasiewicz fuzzy ideals within the same algebraic framework [5], and further advanced this line of inquiry through a foundational study on Łukasiewicz fuzzy BE-algebras and BE-filters, offering valuable insights into their structural properties and applications within fuzzy algebraic systems [7]. Additionally, they addressed the interrelations among these fuzzy constructs. In a related direction, Jana and Pal [4] proposed a practical algorithm for solving decision-making problems based on bipolar intuitionistic fuzzy soft sets. Complementing these applied perspectives, Gokila and Jansirani [3] examined the structure of Łukasiewicz fuzzy BM-algebras and BM-ideals, contributing to the theoretical advancement of fuzzy algebraic frameworks through rigorous definitions and illustrative examples.

This study introduces Łukasiewicz intuitionistic fuzzy ideals in BM-algebras, with extensions to subalgebras, and explores several distinctive attributes, operations, and relations among them, along with specific instances. Finally, we have evaluated an employee performance evaluation model by integrating Łukasiewicz intuitionistic fuzzy set, using min-max normalization to deliver an extensive and equitable appraisal system.

Symbols	Representations
$\mathfrak{FS}$	*F*uzzy *S*et
$\mathfrak{IFS}$	*I*ntuitionistic *F*uzzy *S*et
$\mathfrak{LIFS}$	Łukasiewicz *I*ntuitionistic *F*uzzy *S*et
$\mathfrak{FA}$	*F*uzzy *S*ubalgebra
$\mathfrak{IFA}$	*I*ntuitionistic *F*uzzy *S*ubalgebra
$\mathfrak{LIFA}$	Łukasiewicz *I*ntuitionistic *F*uzzy *S*ubalgebra
$\mathfrak{FI}$	*F*uzzy *I*deal
$\mathfrak{IFI}$	*I*ntuitionistic *F*uzzy *I*deal
$\mathfrak{LIFI}$	Łukasiewicz *I*ntuitionistic *F*uzzy *I*deal
$\mathfrak{IFP}$	*I*ntuitionistic *F*uzzy point
$\mathfrak{UB}$	*U*pper *B*ound
$\mathfrak{LB}$	*L*ower *B*ound
$\mathfrak{CT}$	*C*omparison *T*able

2. Preliminaries

To establish the foundation for our proposed framework, this section introduces the fundamental concepts and notations necessary for understanding $\mathfrak{LIFS}$s within the context of *BM*-algebras. We begin by recalling essential definitions related to *BM*-algebras and $\mathfrak{IFS}$s, which serve as the structural and logical basis for our study. The properties and operations of these algebraic systems are critical for formulating Łukasiewicz intuitionistic fuzzy subalgebras ($\mathfrak{LIFA}$s) and ideals ($\mathfrak{LIFI}$s), which are investigated in the subsequent sections. Throughout, we adopt standard notation and terminology to maintain consistency with prior literature and facilitate the development of our theoretical results.

Definition 2.1. *BM-algebra is the set of elements of $\check{\mathfrak{G}}$, that meets the given axioms under a binary operator "$\rightsquigarrow$" and a fixed element "0":*

$$(BM_1) \quad \acute{\alpha} \rightsquigarrow 0 = \acute{\alpha}$$
$$(BM_2) \quad (\acute{\gamma} \rightsquigarrow \acute{\alpha}) \rightsquigarrow (\acute{\gamma} \rightsquigarrow \acute{\beta}) = \acute{\beta} \rightsquigarrow \acute{\alpha}, \quad \forall\, \acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}.$$

Proposition 2.1. *Every BM-algebra satisfies*

(i) $\acute{\alpha} \rightsquigarrow \acute{\alpha} = 0$
(ii) $0 \rightsquigarrow (0 \rightsquigarrow \acute{\alpha}) = \acute{\alpha}$
(iii) $0 \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = \acute{\beta} \rightsquigarrow \acute{\alpha}$
(iv) $(\acute{\alpha} \rightsquigarrow \acute{\gamma}) \rightsquigarrow (\acute{\beta} \rightsquigarrow \acute{\gamma}) = \acute{\alpha} \rightsquigarrow \acute{\beta}$
(v) $\acute{\alpha} \rightsquigarrow \acute{\beta} = 0 \Leftrightarrow \acute{\beta} \rightsquigarrow \acute{\alpha} = 0, \quad \forall\, \acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}.$

Definition 2.2. *A $\mathfrak{FS}$ $\hat{\hat{\mathcal{U}}}$ is called $\mathfrak{FA}$ of $\breve{\mathfrak{G}}$ if it meets the criteria*

$$(\mathfrak{FU}_1) \quad \hat{\hat{\mathcal{U}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \min\left\{\hat{\hat{\mathcal{U}}}(\acute{\alpha}), \hat{\hat{\mathcal{U}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

Definition 2.3. *A $\mathfrak{FS}$ $\hat{\hat{\mathcal{U}}}$ is called $\mathfrak{FI}$ of $\breve{\mathfrak{G}}$ if it meets the criteria*

$$(\mathfrak{FI}_1) \quad \hat{\hat{\mathcal{U}}}(0) \geq \hat{\hat{\mathcal{U}}}(\acute{\alpha})$$

$$(\mathfrak{FI}_2) \quad \hat{\hat{\mathcal{U}}}(\acute{\alpha}) \geq \min\left\{\hat{\hat{\mathcal{U}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \hat{\hat{\mathcal{U}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

Definition 2.4. *An $\mathfrak{IFS}$ $\breve{\mathbb{H}}$ in BM-algebra $\breve{\mathfrak{G}}$ is of the type,*

$$\breve{\mathbb{H}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}), \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \breve{\mathfrak{G}}\right\}$$

in which $\ddot{\varphi}_{\breve{\mathbb{H}}} : \breve{\mathfrak{G}} \to [0,1]$ and $\ddot{\psi}_{\breve{\mathbb{H}}} : \breve{\mathfrak{G}} \to [0,1]$ refer to inclusive and exclusive degree under the condition that

$$0 \leq \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}) + \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \leq 1, \quad \forall\, \acute{\alpha} \in \breve{\mathfrak{G}}.$$

The set can also be denoted as $\breve{\mathbb{H}} = \left(\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}}\right)$.

Definition 2.5. *An $\mathfrak{IFS}$ $\breve{\mathbb{H}} = (\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}})$ in $\breve{\mathfrak{G}}$ be of the kind*

$$\ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\beta}) = \begin{cases} \dot{\delta} \in (0,1] & \text{if } \acute{\beta} = \acute{\alpha} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\beta}) = \begin{cases} \dot{\sigma} \in [0,1) & \text{if } \acute{\beta} = \acute{\alpha} \\ 1 & \text{otherwise} \end{cases}$$

is said to be $\mathfrak{IFP}$ with support $\acute{\alpha}$ and membership value $\dot{\delta}$ and non-membership value $\dot{\sigma}$. It is denoted by $[\acute{\alpha}/\dot{\delta}]$ and $[\acute{\alpha}/\dot{\sigma}]$, respectively.

Definition 2.6. *For an $\mathfrak{IFS}$ $\breve{\mathbb{H}} = (\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}})$ in $\breve{\mathfrak{G}}$, we state that an $\mathfrak{IFP}$ of inclusive $[\acute{\alpha}/\delta]$ and exclusive $[\acute{\alpha}/\dot{\sigma}]$ degree is*

- *(i) contained in $\breve{\mathbb{H}}$, noted by $[\acute{\alpha}/\dot{\delta}] \in \ddot{\varphi}_{\breve{\mathbb{H}}}$ and $[\acute{\alpha}/\dot{\sigma}] \in \ddot{\psi}_{\breve{\mathbb{H}}}$, if $\ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \geq \dot{\delta}$ and $\ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \leq \dot{\sigma}$.*
- *(ii) quasi-coincident with $\breve{\mathbb{H}}$, if $\ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}) + \delta > 1$, then $[\acute{\alpha}/\delta]_q \ddot{\varphi}_{\breve{\mathbb{H}}}$, and if $\ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha}) + \dot{\sigma} < 1$, then $[\acute{\alpha}/\dot{\sigma}]_q \ddot{\psi}_{\breve{\mathbb{H}}}$.*

Definition 2.7. *An $\mathfrak{IFS}$ $\breve{\mathbb{H}} = (\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}})$ in a BM-algebra $\breve{\mathfrak{G}}$ is known as $\mathfrak{IFA}$ of $\breve{\mathfrak{G}}$ if it meets the criteria*

$$(\mathfrak{IFA}_1) \quad \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \min\left\{\ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}), \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\beta})\right\}$$

$$(\mathfrak{IFA}_2) \quad \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \max\left\{\ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha}), \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

Definition 2.8. *An $\mathfrak{IFS}$ $\breve{\mathbb{H}} = (\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}})$ in a BM-algebra $\breve{\mathfrak{G}}$ is known as $\mathfrak{IFI}$ of $\breve{\mathfrak{G}}$ if it meets the criteria*

$$(\mathfrak{IFI}_1) \quad \ddot{\varphi}_{\breve{\mathbb{H}}}(0) \geq \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \quad \text{and} \quad \ddot{\psi}_{\breve{\mathbb{H}}}(0) \leq \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha})$$

$$(\mathfrak{IFI}_2) \quad \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{\breve{\mathbb{H}}}(\acute{\beta})\right\}$$

$$(\mathfrak{IFI}_3) \quad \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha}) \leq \max\left\{\ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\breve{\mathbb{H}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

3. Łukasiewicz Intuitionistic Fuzzy Subalgebras in *BM*-algebras

This section presents a formal development of Łukasiewicz intuitionistic fuzzy subalgebras ($\mathfrak{LIFA}$s) within the algebraic framework of *BM*-algebras. Building upon the foundational concepts introduced in the previous section, we aim to extend the theory of $\mathfrak{IFS}$s by incorporating Łukasiewicz logic to capture graded membership and non-membership more expressively. We provide a precise definition of $\mathfrak{LIFA}$ and investigate its structural properties through a series of lemmas and theorems. These results offer insight into the behavior and algebraic coherence of $\mathfrak{LIFA}$s, laying the groundwork for further analysis and practical application in performance evaluation systems.

Definition 3.1. *An $\mathfrak{LIFS}$ of $\check{\mathbb{H}}$ in $\check{\mathfrak{G}}$ with $\varepsilon \in [0,1]$ is defined by,*

$$\mathbb{L}^{\varepsilon}_{\check{\mathbb{H}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}^{\varepsilon}_{\check{\mathbb{H}}}, \ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$$

where $\ddot{\varphi}^{\varepsilon}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1]$, $\acute{\alpha} \mapsto \max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1\}$ refers to the inclusive degree and $\ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1]$, $\acute{\alpha} \mapsto \min\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon, 1\}$ refers to the exclusive degree respectively, and

$$0 \le \ddot{\varphi}^{\varepsilon}_{\check{\mathbb{H}}} + \ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}} \le 1.$$

Lemma 3.1. *If $\check{\mathbb{H}}$ is an $\mathfrak{IFS}$ in $\check{\mathfrak{G}}$ and $\varepsilon \in (0,1)$, then its $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ satisfies*

(i) $\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) \ge \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) \Rightarrow \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \ge \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})$ *(3.1)*

(ii) $\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) \le \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) \Rightarrow \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \le \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}), \quad \forall\, \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}.$ *(3.2)*

Proof. Suppose $\check{\mathbb{H}}$ is an $\mathfrak{IFS}$ in $\check{\mathfrak{G}}$ and $\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) \ge \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta})$. Then

$$\begin{aligned}\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &= \max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1\} \\ &\ge \max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\} = \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}).\end{aligned}$$

Thus, $\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \ge \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})$. Similarly, suppose $\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) \le \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta})$. Then,

$$\begin{aligned}\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &= \min\{1, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon\} \\ &\le \min\{1, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon\} = \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}).\end{aligned}$$

Thus, $\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \le \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})$. □

Definition 3.2. *An $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ in $\check{\mathfrak{G}}$ is said to be an $\mathfrak{LIFA}$ of a BM-algebra $\check{\mathfrak{G}}$ if it meets the criteria*

$$(\mathfrak{LIFA}_1) \quad [\acute{\alpha}/\dot{\delta}_a], [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow \left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) / \min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \tag{3.3}$$

$$(\mathfrak{LIFA}_2) \quad [\acute{\alpha}/\dot{\sigma}_a], [\acute{\beta}/\dot{\sigma}_b] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow \left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) / \max\{\dot{\sigma}_a, \dot{\sigma}_b\}\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \tag{3.4}$$

$\forall \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$, $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ *and* $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$.

Example 3.1. *A set in BM-algebra $\check{\mathfrak{G}} = \{0, \ddot{\xi}_1, \ddot{\xi}_2, \ddot{\xi}_3\}$ owns the "$\rightsquigarrow$" operation in the following table:*

$\rightsquigarrow$	0	$\ddot{\varsigma}_1$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_3$
0	0	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$
$\ddot{\varsigma}_1$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$
$\ddot{\varsigma}_2$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_3$
$\ddot{\varsigma}_3$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0

Thus, $\breve{\mathfrak{G}}$ serves as a BM-algebra. Defining an $\mathfrak{IFS}$ $\breve{\mathbb{H}} = (\ddot{\varphi}_{\breve{\mathbb{H}}}, \ddot{\psi}_{\breve{\mathbb{H}}})$ in $\breve{\mathfrak{G}}$ as follows:

$$\ddot{\varphi}_{\breve{\mathbb{H}}} : \breve{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.86 & \text{if } \acute{\alpha} \in \{0, \ddot{\varsigma}_1\} \\ 0.65 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \\ 0.31 & \text{if } \acute{\alpha} = \ddot{\varsigma}_3 \end{cases} \quad \text{and} \quad \ddot{\psi}_{\breve{\mathbb{H}}} : \breve{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.09 & \text{if } \acute{\alpha} \in \{0, \ddot{\varsigma}_1\} \\ 0.25 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \\ 0.65 & \text{if } \acute{\alpha} = \ddot{\varsigma}_3 \end{cases}$$

If it is taken that $\varepsilon = 0.62$, then the $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\breve{\mathbb{H}}}$ of $\breve{\mathbb{H}}$ in $\breve{\mathfrak{G}}$ is provided as follows:

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}} : \breve{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.48 & \text{if } \acute{\alpha} \in \{0, \ddot{\varsigma}_1\} \\ 0.27 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \\ 0.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_3 \end{cases} \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}} : \breve{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.71 & \text{if } \acute{\alpha} \in \{0, \ddot{\varsigma}_1\} \\ 0.87 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \\ 1.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_3 \end{cases}$$

Typically, it is verified that $Ł^{\varepsilon}_{\breve{\mathbb{H}}}$ is an $\mathfrak{LIFA}$ of a BM-algebra $\breve{\mathfrak{G}}$.

Theorem 3.1. *Every $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\breve{\mathbb{H}}}$ is an $\mathfrak{LIFA}$ of a BM-algebra $\breve{\mathfrak{G}}$ if and only if it fulfills:*

(i) $\forall \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$, $\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}$ *(3.5)*

(ii) $\forall \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$, $\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}$ *(3.6)*

Proof. Let $\breve{\mathbb{H}}$ be an $\mathfrak{IFS}$ in *BM*-algebra $\breve{\mathfrak{G}}$. Assume that $Ł^{\varepsilon}_{\breve{\mathbb{H}}}$ is a $\mathfrak{LIFA}$ of a *BM*-algebra $\breve{\mathfrak{G}}$. Let $\acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$ and it is noted that

$$\left[\acute{\alpha}/\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}} \quad \text{and} \quad \left[\acute{\beta}/\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}.$$

From (3.3), it is evident that

$$\left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}},$$

and hence

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

Similarly note that

$$\left[\acute{\alpha}/\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}} \quad \text{and} \quad \left[\acute{\beta}/\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

From (3.4), it is evident that

$$\left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}},$$

and hence

$$\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\breve{\mathbb{H}}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}.$$

In contrast, let's state, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ fulfills (3.5) and (3.6). Also, let $\acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ also that

$$[\acute{\alpha}/\dot{\delta}_a] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \dot{\delta}_a \quad \text{and} \quad \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_b,$$

which implies from (3.5) that

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \geq \min\{\dot{\delta}_a, \dot{\delta}_b\}.$$

Hence, $\left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) / \min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$ ensures that

$$[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\beta/\dot{\sigma}_b] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \dot{\sigma}_a \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_b,$$

which implies from (3.6) that

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \leq \max\{\dot{\sigma}_a, \dot{\sigma}_b\}.$$

Hence, $\left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) / \max\{\dot{\sigma}_a, \dot{\sigma}_b\}\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Therefore, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFA}$ of a *BM*-algebra $\breve{\mathfrak{G}}$. □

Theorem 3.2. *If* $\check{\mathbb{H}}$ *is an* $\mathfrak{IFA}$ *of* $\breve{\mathfrak{G}}$*, then* $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *in* $\breve{\mathfrak{G}}$ *is a* $\mathfrak{LIFA}$ *of BM-algebra* $\breve{\mathfrak{G}}$.

Proof. Let $\check{\mathbb{H}}$ be an $\mathfrak{IFA}$ of $\breve{\mathfrak{G}}$. Let $\acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ be ensures that

$$[\acute{\alpha}/\dot{\delta}_a] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \dot{\delta}_a \quad \text{and} \quad \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_b.$$

Similarly, let $\acute{\alpha}, \acute{\beta} \in \breve{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$ ensures that

$$[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\beta}/\dot{\sigma}_b] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \dot{\sigma}_a \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_b.$$

Thus

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) &= \max\left\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon - 1\right\} \\
&\geq \max\left\{0, \min\left\{\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}), \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta})\right\} + \varepsilon - 1\right\} \qquad [(3.3)] \\
&= \max\left\{0, \min\left\{\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\right\}\right\} \\
&= \min\left\{\max\left\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1\right\}, \max\left\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\right\}\right\} \\
&= \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\
&\geq \min\{\dot{\delta}_a, \dot{\delta}_b\}.
\end{aligned}$$

So, $\left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) \,/\, \min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) &= \min\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon, 1\right\} \\
&\leq \min\left\{\max\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}), \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta})\right\} + \varepsilon, 1\right\} \qquad [(3.4)] \\
&= \min\left\{\max\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon\right\}, 1\right\} \\
&= \max\left\{\min\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon, 1\right\}, \min\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon, 1\right\}\right\} \\
&= \max\left\{\ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}}(\acute{\alpha}), \ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}}(\acute{\beta})\right\} \\
&\leq \max\{\dot{\sigma}_a, \dot{\sigma}_b\}.
\end{aligned}$$

So, $\left[(\acute{\alpha} \rightsquigarrow \acute{\beta}) \,/\, \max\{\dot{\sigma}_a, \dot{\sigma}_b\}\right] \in \ddot{\psi}^{\varepsilon}_{\check{\mathbb{H}}}$. Hence, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFA}$ of a *BM*-algebra $\check{\mathfrak{G}}$. □

The reverse portion of the theorem is falsified with the proof of an example below.

Example 3.2. *A set in a BM-algebra* $\check{\mathfrak{G}} = \{0, \ddot{\varsigma}_1, \ddot{\varsigma}_2\}$ *owns the* $\rightsquigarrow$ *operation in the following table:*

$\rightsquigarrow$	0	$\ddot{\varsigma}_1$	$\ddot{\varsigma}_2$
0	0	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$
$\ddot{\varsigma}_1$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_2$
$\ddot{\varsigma}_2$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0

Defining an $\mathfrak{IFS}$ $\check{\mathbb{H}} = (\ddot{\varphi}_{\check{\mathbb{H}}}, \ddot{\psi}_{\check{\mathbb{H}}})$ *in* $\check{\mathfrak{G}}$ *as established*

$$\ddot{\varphi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.76 & \text{if } \acute{\alpha} = 0 \\ 0.32 & \text{if } \acute{\alpha} = \ddot{\varsigma}_1 \\ 0.40 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \end{cases} \quad \text{and} \quad \ddot{\psi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.16 & \text{if } \acute{\alpha} = 0 \\ 0.65 & \text{if } \acute{\alpha} = \ddot{\varsigma}_1 \\ 0.59 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \end{cases}$$

Provided that $\varepsilon = 0.41$, *then the* $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}} = \left(\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}\right)$ *of* $\check{\mathbb{H}}$ *in* $\check{\mathfrak{G}}$ *is formed in the way*

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \widetilde{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.17 & \text{if } \acute{\alpha} = 0 \\ 0.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_1 \\ 0.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \end{cases} \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1], \quad \alpha \mapsto \begin{cases} 0.57 & \text{if } \acute{\alpha} = 0 \\ 1.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_1 \\ 1.00 & \text{if } \acute{\alpha} = \ddot{\varsigma}_2 \end{cases}$$

Typically, it is verified that $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *is an* $\mathfrak{LIFA}$ *of a BM-algebra* $\check{\mathfrak{G}}$. *But* $\check{\mathbb{H}}$ *is not an* $\mathfrak{IFA}$ *of* $\check{\mathfrak{G}}$ *because of*

$$\ddot{\varphi}_{\check{\mathbb{H}}}(0 \rightsquigarrow \ddot{\varsigma}_2) = \ddot{\varphi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_1) = 0.32 \ngeq 0.40 = \min\left\{\ddot{\varphi}_{\check{\mathbb{H}}}(0), \ddot{\varphi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_2)\right\},$$

$$\ddot{\psi}_{\check{\mathbb{H}}}(0 \rightsquigarrow \ddot{\varsigma}_2) = \ddot{\psi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_1) = 0.65 \nleq 0.59 = \max\left\{\ddot{\psi}_{\check{\mathbb{H}}}(0), \ddot{\psi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_2)\right\}.$$

Lemma 3.2. *If* $\check{\mathbb{H}}$ *is an* $\mathfrak{IFA}$ *of* $\check{\mathfrak{G}}$, *then its* $\mathfrak{LIFA}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *satisfies*

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \geq \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}}. \tag{3.7}$$

Proof. Let $\check{\mathbb{H}}$ is an $\mathfrak{IFA}$ of $\check{\mathfrak{G}}$. Consider,

$$\ddot{\varphi}_{\check{\mathbb{H}}}(0) = \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}), \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha})\right\} = \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}}.$$

It is inferred from (3.1) that

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \geq \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}}.$$

Similarly, consider the degree of non-membership:

$$\ddot{\psi}_{\check{\mathbb{H}}}(0) = \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\alpha}) \leq \max\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}), \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha})\right\} = \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}}.$$

It is inferred from (3.2) that

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \leq \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}}.$$

Thus, the $\mathfrak{LIFA}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ satisfies the condition (3.7). □

4. Łukasiewicz Intuitionistic Fuzzy Ideals in *BM*-Algebras

In this section, we extend the theoretical framework by introducing the concept of Łukasiewicz intuitionistic fuzzy ideals ($\mathfrak{LIFI}$s) in *BM*-algebras. While subalgebras capture internal consistency under specific operations, ideals serve as critical structures for analyzing the behavior of algebraic systems under approximation and filtering processes. By integrating the principles of Łukasiewicz logic and $\mathfrak{IFS}$ theory, we define $\mathfrak{LIFI}$s and examine their algebraic properties. The proposed framework allows for a more nuanced treatment of uncertainty and partial belonging in ideal-related contexts. Several theorems and illustrative examples are provided to clarify the conditions under which a fuzzy set qualifies as a $\mathfrak{LIFI}$ and to demonstrate the interplay between $\mathfrak{LIFI}$s and $\mathfrak{LIFA}$s in the broader algebraic structure.

Definition 4.1. *Let $\check{\mathbb{H}}$ be an $\mathfrak{IFS}$ in $\check{\mathfrak{G}}$. Then its $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ in $\check{\mathfrak{G}}$ is called an $\mathfrak{LIFI}$ of a BM-algebra $\check{\mathfrak{G}}$ if it satisfies*

$$(LFI_1) \quad \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \text{ is an } \mathfrak{UB} \text{ of } \left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \mid \acute{\alpha} \in \check{\mathfrak{G}}\right\} \tag{4.1}$$

$$(LFI_2) \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \text{ is an } \mathfrak{LB} \text{ of } \left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \mid \acute{\alpha} \in \check{\mathfrak{G}}\right\} \tag{4.2}$$

$$(LFI_3) \quad \left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\delta}_a, [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [\acute{\alpha}/\min\{\dot{\delta}_a, \dot{\delta}_b\}] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}\right] \tag{4.3}$$

$$(LFI_4) \quad \left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\sigma}_a, [\acute{\beta}/\dot{\sigma}_b] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [\acute{\alpha}/\max\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}\right] \tag{4.4}$$

$\forall \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ with $\dot{\delta}_a, \dot{\delta}_b \in (0, 1]$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0, 1)$.

Example 4.1. *A set in a BM-algebra $\check{\mathfrak{G}} = \{0, \ddot{\varsigma}_1, \ddot{\varsigma}_2, \ddot{\varsigma}_3, \ddot{\varsigma}_4\}$ owns the "$\rightsquigarrow$" operation in the following table:*

$\rightsquigarrow$	0	$\ddot{\varsigma}_1$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_4$
0	0	$\ddot{\varsigma}_4$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$
$\ddot{\varsigma}_1$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_4$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$
$\ddot{\varsigma}_2$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_4$	$\ddot{\varsigma}_3$
$\ddot{\varsigma}_3$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0	$\ddot{\varsigma}_4$
$\ddot{\varsigma}_4$	$\ddot{\varsigma}_4$	$\ddot{\varsigma}_3$	$\ddot{\varsigma}_2$	$\ddot{\varsigma}_1$	0

Defining an $\mathfrak{IFS}$ $\check{\mathbb{H}} = \left(\ddot{\varphi}_{\check{\mathbb{H}}}, \ddot{\psi}_{\check{\mathbb{H}}}\right)$ *in* $\check{\mathfrak{G}}$ *as provided*

$$\ddot{\varphi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.91 & \text{if } \acute{\alpha} = 0 \\ 0.63 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_1, \dddot{\varsigma}_2\} \\ 0.79 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_3, \dddot{\varsigma}_4\} \end{cases} \qquad \ddot{\psi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.02 & \text{if } \acute{\alpha} = 0 \\ 0.36 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_1, \dddot{\varsigma}_2\} \\ 0.21 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_3, \dddot{\varsigma}_4\} \end{cases}$$

If we take $\varepsilon = 0.58$, *then the* $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}} = \left(\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}\right)$ *of* $\check{\mathbb{H}}$ *in* $\check{\mathfrak{G}}$ *is given by the way*

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.49 & \text{if } \acute{\alpha} = 0 \\ 0.21 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_1, \dddot{\varsigma}_2\} \\ 0.37 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_3, \dddot{\varsigma}_4\} \end{cases} \qquad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1], \quad \acute{\alpha} \mapsto \begin{cases} 0.60 & \text{if } \acute{\alpha} = 0 \\ 0.94 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_1, \dddot{\varsigma}_2\} \\ 0.79 & \text{if } \acute{\alpha} \in \{\dddot{\varsigma}_3, \dddot{\varsigma}_4\} \end{cases}$$

and it is essential to make sure that $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *is an* $\mathfrak{LIFI}$ *of a BM-algebra* $\check{\mathfrak{G}}$.

Theorem 4.1. *Every* $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *is an* $\mathfrak{LIFI}$ *of* $\check{\mathfrak{G}}$ *if and only if it fulfills the conditions:*

(i) $\left[\acute{\alpha}/\dot{\delta}_a\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow \left[0/\dot{\delta}_a\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ *(4.5)*

(ii) $[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [0/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ *(4.6)*

(iii) $\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}),\ \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}$ *(4.7)*

(iv) $\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}),\ \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}$ *(4.8)*

$\forall \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$, $\dot{\delta}_a \in (0,1]$ *and* $\dot{\sigma}_a \in [0,1)$.

Proof. For instance, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is a $\mathfrak{LIFI}$ of a *BM*-algebra $\check{\mathfrak{G}}$. Let $\acute{\alpha} \in \check{\mathfrak{G}}$ and $\dot{\delta}_a \in (0,1]$ also that $\left[\acute{\alpha}/\dot{\delta}_a\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Utilizing (4.1), leads to

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \geq \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \dot{\delta}_a,$$

and so $[0/\dot{\delta}_a] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a \in [0,1)$ also that $[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Utilizing (4.2), leads to

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(0) \leq \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \dot{\sigma}_a,$$

and so $[0/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Note that

$$\left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta})\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad \left[\acute{\beta}/\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}.$$

From (4.3), it clear,

$$\left[\acute{\alpha}/\min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}},$$

and hence

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}.$$

Similarly, let

$$\left[(\acute{\alpha} \rightsquigarrow \beta)/\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta})\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad \left[\acute{\beta}/\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}.$$

From (4.4), it clear

$$\left[\acute{\alpha}/\max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}},$$

and hence

$$\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\}, \quad \forall\, \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}.$$

Conversely, let us consider $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ satisfies (4.5), (4.6), (4.7) and (4.8). Since

$$\left[\acute{\alpha}/\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}},$$

we have

$$\left[0/\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and so} \quad \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0) \geq \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}} \text{ by (4.5).}$$

Hence, $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0)$ is an $\mathfrak{UB}$ of $\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \mid \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Similarly, since

$$\left[\acute{\alpha}/\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}},$$

we have

$$\left[0/\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha})\right] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and so} \quad \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0) \leq \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}), \quad \forall\, \acute{\alpha} \in \check{\mathfrak{G}} \text{ by (4.6).}$$

Hence, $\ddot{\psi}_{\check{\mathbb{H}}}(0)$ is an $\mathfrak{LB}$ of $\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \mid \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Also, let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ ensures that

$$[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \quad [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \dot{\delta}_a$ and $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_b$, which imply from (4.7) that

$$\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \geq \min\{\dot{\delta}_a, \dot{\delta}_b\}.$$

So, $\left[\acute{\alpha}/\min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$ ensures that

$$[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\beta}/\dot{\sigma}_b] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then

$$\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \dot{\sigma}_a \quad \text{and} \quad \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_b,$$

which implies from (4.8) that

$$\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \leq \max\{\dot{\sigma}_a, \dot{\sigma}_b\}.$$

So, $[\acute{\alpha}/\max\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Therefore, $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFI}$ of a *BM*-algebra $\check{\mathfrak{G}}$. □

Lemma 4.1. *Every $\mathfrak{LIFI}$ $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ of $\check{\mathfrak{G}}$ satisfies the condition if $\acute{\alpha} \leq \acute{\beta}$ and $\acute{\beta} \rightsquigarrow \acute{\alpha} = 0$, then*

(i) $[\acute{\beta}/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [\acute{\alpha}/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}, \quad \forall \dot{\delta}_a \in (0,1]$ *(4.9)*

(ii) $[\acute{\beta}/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \quad \forall \acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}, \quad \forall \dot{\sigma}_a \in [0,1).$ *(4.10)*

Proof. Let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\delta}_a \in (0,1]$ also that $\acute{\alpha} \leq \acute{\beta}, \acute{\beta} \rightsquigarrow \acute{\alpha} = 0$ and $[\acute{\beta}/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Then $(\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$ and $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_a$ so,

$$\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \tilde{\beta}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} = \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} = \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_a.$$

Hence, $[\acute{\alpha}/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a \in [0,1)$ be so that $\acute{\alpha} \leq \acute{\beta}, \acute{\beta} \rightsquigarrow \acute{\alpha} = 0$ and $[\acute{\beta}/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Then $(\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$ and $\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_a$ so,

$$\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} = \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} = \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_a.$$

Hence, $[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Thus, (4.9) and (4.10) are verified. □

Lemma 4.2. *Every $\mathfrak{LIFI}$ $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ of a BM-algebra $\check{\mathfrak{G}}$ fulfills the condition if $\acute{\alpha} \rightsquigarrow \acute{\beta} \leq \acute{\gamma}$ and $\acute{\gamma} \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$, then*

$$\text{(i)} \quad \left[\acute{\beta}/\dot{\delta}_a\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, \left[\acute{\gamma}/\dot{\delta}_b\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow \left[\acute{\alpha}/\min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \tag{4.11}$$

$$\text{(ii)} \quad \left[\acute{\beta}/\dot{\sigma}_a\right] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}, [\acute{\gamma}/\dot{\sigma}_b] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \Rightarrow [\acute{\alpha}/\max\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \tag{4.12}$$

$\forall \acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}$, $\dot{\delta}_a, \dot{\delta}_b \in (0, 1]$ *and* $\forall$ $\dot{\sigma}_a, \dot{\sigma}_b \in [0, 1)$.

Proof. Let $\acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0, 1]$ also that $\acute{\alpha} \rightsquigarrow \acute{\beta} \leq \acute{\gamma}$, $\acute{\gamma} \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$,

$$[\acute{\beta}/\dot{\delta}_a] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\gamma}/\dot{\delta}_b] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Then, $(\acute{\alpha} \rightsquigarrow \acute{\beta}) \rightsquigarrow \acute{\gamma} = 0$, $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_a$ and $\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma}) \geq \dot{\delta}_b$. Hence,

$$\begin{aligned} \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &\geq \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &\geq \min\left\{\min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}((\acute{\alpha} \rightsquigarrow \acute{\beta}) \rightsquigarrow \acute{\gamma}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\}, \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &= \min\left\{\min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\}, \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &= \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &\geq \min\left\{\dot{\delta}_b, \dot{\delta}_a\right\} \end{aligned}$$

and so $\left[\acute{\alpha}/\min\left\{\dot{\delta}_a, \dot{\delta}_b\right\}\right] \in \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0, 1)$ also that $\acute{\gamma} \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$, $[\acute{\beta}/\dot{\sigma}_a] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$ and $[\acute{\gamma}/\dot{\sigma}_b] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Then, $(\acute{\alpha} \rightsquigarrow \acute{\beta}) \rightsquigarrow \acute{\gamma} = 0$, $\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_a$ and $\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma}) \leq \dot{\sigma}_b$. Hence,

$$\begin{aligned} \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &\leq \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &\leq \max\left\{\max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}((\acute{\alpha} \rightsquigarrow \acute{\beta}) \rightsquigarrow \acute{\gamma}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\}, \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &= \max\left\{\max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(0), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\}, \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &= \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\ &\leq \max\{\dot{\sigma}_b, \dot{\sigma}_a\} \end{aligned}$$

and so $[\acute{\alpha}/\max\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}$. Thus, (4.11) and (4.12) are verified. □

Remark 4.1. *If $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ is a $\mathfrak{LIIF}$ of a BM-algebra $\check{\mathfrak{G}}$, then it meets the following inequalities:*

$$\text{(i)} \quad \acute{\alpha} \leq \acute{\beta} \text{ and } \acute{\beta} \rightsquigarrow \acute{\alpha} = 0 \Rightarrow \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \geq \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \text{ and } \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) \leq \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \tag{4.13}$$

$$\begin{aligned} \text{(ii)} \quad \acute{\alpha} \rightsquigarrow \acute{\beta} \leq \acute{\gamma} \text{ and } \acute{\gamma} \rightsquigarrow \acute{\beta} = 0 \Rightarrow \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &\geq \min\left\{\ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}), \ddot{\varphi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\} \text{ and} \\ \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &\leq \max\left\{\ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}), \ddot{\psi}_{\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\gamma})\right\} \end{aligned} \tag{4.14}$$

$\forall \acute{\alpha}, \acute{\beta}, \acute{\gamma} \in \check{\mathfrak{G}}$.

Theorem 4.2. *Every $\mathfrak{LIFS}$ $\text{Ł}^{\varepsilon}_{\check{\mathbb{H}}}$ in $\check{\mathfrak{G}}$ is an $\mathfrak{LIFI}$ of $\check{\mathfrak{G}}$ if $\check{\mathbb{H}}$ is an $\mathfrak{IFI}$ of $\check{\mathfrak{G}}$.*

Proof. For instance, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFS}$ of an $\mathfrak{IFI}$ $\check{\mathbb{H}}$ in $\check{\mathfrak{G}}$. Let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ also that $\left[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\delta}_a\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \left[\acute{\beta}/\dot{\delta}_b\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Then $\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \geq \dot{\delta}_a$ and $\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \geq \dot{\delta}_b$. Thus,

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &= \max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1\} \\
&\geq \max\{0, \min\{\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{\check{\mathbb{H}}}(\check{\beta})\} + \varepsilon - 1\} \qquad [(4.7)] \\
&= \max\left\{0, \min\left\{\ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon - 1, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\right\}\right\} \\
&= \min\left\{\max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon - 1\}, \max\{0, \ddot{\varphi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\}\right\} \\
&= \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\
&\geq \min\{\dot{\delta}_a, \dot{\delta}_b\}.
\end{aligned}$$

So, $\left[\acute{\alpha}/\min\{\dot{\delta}_a, \dot{\delta}_b\}\right] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Similarly, let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$ also that $\left[\left(\acute{\alpha} \rightsquigarrow \acute{\beta}\right)/\dot{\sigma}_a\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, \left[\acute{\beta}/\dot{\sigma}_b\right] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Then $\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) \leq \dot{\sigma}_a$ and $\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta}) \leq \dot{\sigma}_b$. Thus,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha}) &= \min\{0, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha}) + \varepsilon - 1\} \\
&\leq \min\{0, \max\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta})\} + \varepsilon - 1\} \qquad [(4.8)] \\
&= \min\left\{0, \max\left\{\ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon - 1, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\right\}\right\} \\
&= \max\left\{\min\left\{0, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}) + \varepsilon - 1\right\}, \min\left\{0, \ddot{\psi}_{\check{\mathbb{H}}}(\acute{\beta}) + \varepsilon - 1\right\}\right\} \\
&= \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\alpha} \rightsquigarrow \acute{\beta}), \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\acute{\beta})\right\} \\
&\leq \max\{\dot{\sigma}_a, \dot{\sigma}_b\}.
\end{aligned}$$

So, $[\acute{\alpha}/\max\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Hence, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFI}$ of a *BM*-algebra $\check{\mathfrak{G}}$. □

The reverse portion of the theorem is falsified with the proof of an example below.

Example 4.2. *A set in a BM-algebra* $\check{\mathfrak{G}} = \{0, \ddot{\zeta}_1, \ddot{\zeta}_2, \ddot{\zeta}_3\}$ *owns the "*$\rightsquigarrow$*" operation in the following table:*

$\rightsquigarrow$	0	$\ddot{\zeta}_1$	$\ddot{\zeta}_2$	$\ddot{\zeta}_3$
0	0	$\ddot{\zeta}_3$	$\ddot{\zeta}_2$	$\ddot{\zeta}_1$
$\ddot{\zeta}_1$	$\ddot{\zeta}_1$	0	$\ddot{\zeta}_3$	$\ddot{\zeta}_2$
$\ddot{\zeta}_2$	$\ddot{\zeta}_2$	$\ddot{\zeta}_1$	0	$\ddot{\zeta}_3$
$\ddot{\zeta}_3$	$\ddot{\zeta}_3$	$\ddot{\zeta}_2$	$\ddot{\zeta}_1$	0

Provided $\check{\mathfrak{G}}$ *is a BM-algebra. Defining an* $\mathfrak{IFS}$ $\check{\mathbb{H}}$ *in* $\check{\mathfrak{G}}$ *as follows*

$$\ddot{\varphi}_{\check{\mathbb{H}}}: \check{\mathfrak{G}} \to [0,1],\ \acute{\alpha} \mapsto \begin{cases} 0.80 & \text{if } \acute{\alpha} = 0 \\ 0.39 & \text{if } \acute{\alpha} \in \{\ddot{\zeta}_1, \ddot{\zeta}_2\} \\ 0.25 & \text{if } \acute{\alpha} = \ddot{\zeta}_3 \end{cases} \quad \text{and} \quad \ddot{\psi}_{\check{\mathbb{H}}}: \check{\mathfrak{G}} \to [0,1],\ \acute{\alpha} \mapsto \begin{cases} 0.11 & \text{if } \acute{\alpha} = 0 \\ 0.52 & \text{if } \acute{\alpha} = \ddot{\zeta}_1 \\ 0.39 & \text{if } \acute{\alpha} = \ddot{\zeta}_2 \\ 0.45 & \text{if } \acute{\alpha} = \ddot{\zeta}_3 \end{cases}$$

Then $\check{\mathbb{H}}$ *is not an* $\mathfrak{IFI}$ *of* $\check{\mathfrak{G}}$ *because of*

$$\ddot{\varphi}_{\check{\mathbb{H}}}(\ddot{\zeta}_1) = 0.25 \ngeq 0.39 = \min\{\ddot{\varphi}_{\check{\mathbb{H}}}(\ddot{\zeta}_3 \rightsquigarrow \ddot{\zeta}_1), \ddot{\varphi}_{\check{\mathbb{H}}}(\ddot{\zeta}_1)\},$$

$$\ddot{\psi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_1) = 0.52 \nleq 0.39 = \max\{\ddot{\psi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_1 \rightsquigarrow \ddot{\varsigma}_2), \ddot{\psi}_{\check{\mathbb{H}}}(\ddot{\varsigma}_2)\}.$$

Given that $\varepsilon = 0.61$, then the $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ of $\check{\mathbb{H}}$ in $\check{\mathfrak{G}}$ is provided as below:

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1],\ \acute{\alpha} \mapsto \begin{cases} 0.41 & \text{if } \acute{\alpha} = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1],\ \acute{\alpha} \mapsto \begin{cases} 0.72 & \text{if } \acute{\alpha} = 0 \\ 1 & \text{otherwise} \end{cases}$$

Thus, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFI}$ of a BM-algebra $\check{\mathfrak{G}}$.

Theorem 4.3. *Every $\mathfrak{LIFI}$ of a BM-algebra $\check{\mathfrak{G}}$ is an $\mathfrak{LIFA}$ of a BM-algebra $\check{\mathfrak{G}}$.*

Proof. For instance, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFI}$ of a *BM*-algebra $\check{\mathfrak{G}}$. Let $\acute{\alpha}, \acute{\beta} \in \mathscr{G}$ and $\dot{\delta}_a, \dot{\delta}_b \in (0,1]$ be so that

$$[\acute{\alpha}/\dot{\delta}_a] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} \quad \text{and} \quad [\acute{\beta}/\dot{\delta}_b] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}.$$

Since $\acute{\alpha} \rightsquigarrow \acute{\beta} \leq \acute{\alpha}$ and $\acute{\alpha} \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$, we have $[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\delta}_a] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.9). Hence, $[\acute{\alpha}/\min\{\dot{\delta}_a, \dot{\delta}_b\}] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.3), and so $[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\min\{\dot{\delta}_a, \dot{\delta}_b\}] \in \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.9). Similarly, let $\acute{\alpha}, \acute{\beta} \in \check{\mathfrak{G}}$ and $\dot{\sigma}_a, \dot{\sigma}_b \in [0,1)$ also that $[\acute{\alpha}/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}, [\acute{\beta}/\dot{\sigma}_b] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$. Since $\acute{\alpha} \rightsquigarrow \acute{\beta} \leq \acute{\alpha}$ and $\acute{\alpha} \rightsquigarrow (\acute{\alpha} \rightsquigarrow \acute{\beta}) = 0$, we have $[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\dot{\sigma}_a] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.10). Hence, $[\acute{\alpha}/\min\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.4), and so $[(\acute{\alpha} \rightsquigarrow \acute{\beta})/\min\{\dot{\sigma}_a, \dot{\sigma}_b\}] \in \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}$ by (4.10). Therefore, $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ is an $\mathfrak{LIFA}$ of a *BM*-algebra $\check{\mathfrak{G}}$. □

The reverse portion of the theorem is falsified with the proof of an example below.

Example 4.3. *A set in a BM-algebra $\check{\mathfrak{G}} = \{0, \dot{\varsigma}_1, \dot{\varsigma}_2\}$ owns the "$\rightsquigarrow$" operation in the following table:*

$\rightsquigarrow$	0	$\dot{\varsigma}_1$	$\dot{\varsigma}_2$
0	0	$\dot{\varsigma}_2$	$\dot{\varsigma}_1$
$\dot{\varsigma}_1$	$\dot{\varsigma}_1$	0	$\dot{\varsigma}_2$
$\dot{\varsigma}_2$	$\dot{\varsigma}_2$	$\dot{\varsigma}_1$	0

Defining an $\mathfrak{IFS}$ $\check{\mathbb{H}}$ in $\check{\mathfrak{G}}$ as after

$$\ddot{\varphi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \acute{\alpha} \mapsto \begin{cases} 0.80 & \text{if } \acute{\alpha} = 0 \\ 0.39 & \text{if } \acute{\alpha} = \dot{\varsigma}_1 \\ 0.25 & \text{if } \acute{\alpha} = \dot{\varsigma}_2 \end{cases} \quad \text{and} \quad \ddot{\psi}_{\check{\mathbb{H}}} : \check{\mathfrak{G}} \to [0,1], \acute{\alpha} \mapsto \begin{cases} 0.80 & \text{if } \acute{\alpha} = 0 \\ 0.39 & \text{if } \acute{\alpha} = \dot{\varsigma}_1 \\ 0.25 & \text{if } \acute{\alpha} = \dot{\varsigma}_2 \end{cases}$$

Given that $\varepsilon = 0.58$, the $\mathfrak{LIFS}$ $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ of $\check{\mathbb{H}}$ in $\check{\mathfrak{G}}$ is provided as below

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1], \acute{\alpha} \mapsto \begin{cases} 0.49 & \text{if } \acute{\alpha} = 0 \\ 0.21 & \text{if } \acute{\alpha} = \dot{\varsigma}_1 \\ 0.37 & \text{if } \acute{\alpha} = \dot{\varsigma}_2 \end{cases} \quad \text{and} \quad \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}} : \check{\mathfrak{G}} \to [0,1], \acute{\alpha} \mapsto \begin{cases} 0.60 & \text{if } \acute{\alpha} = 0 \\ 0.94 & \text{if } \acute{\alpha} = \dot{\varsigma}_1 \\ 0.79 & \text{if } \acute{\alpha} = \dot{\varsigma}_2 \end{cases}$$

Typically, it is verified that $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *is an* $\mathfrak{LIFA}$ *of a BM-algebra* $\check{\mathfrak{G}}$*. But* $Ł^{\varepsilon}_{\check{\mathbb{H}}}$ *is not an* $\mathfrak{LIFI}$ *of a BM-algebra* $\check{\mathfrak{G}}$ *because of*

$$\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_2) = 0.09 \ngeq 0.3 = \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_2 \rightsquigarrow \grave{\zeta}_1), \ddot{\varphi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_1)\right\},$$

$$\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_2) = 0.09 \nleq 0.3 = \min\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_2 \rightsquigarrow \grave{\zeta}_1), \ddot{\psi}_{Ł^{\varepsilon}_{\check{\mathbb{H}}}}(\grave{\zeta}_1)\right\}.$$

5. Operations on Łukasiewicz Intuitionistic Fuzzy Sets

Definition 5.1. *If* $\grave{\mathfrak{P}}$ *and* $\grave{\mathfrak{Q}}$ *are two* $\mathfrak{IFS}$*s of* $\check{\mathfrak{G}}$*, then*

(i) $\grave{\mathfrak{P}} \subset \grave{\mathfrak{Q}} \Leftrightarrow \forall\, \acute{\alpha} \in \check{\mathfrak{G}},\ \ddot{\varphi}_{\grave{\mathbb{P}}}(\acute{\alpha}) \leq \ddot{\varphi}_{\grave{\mathbb{Q}}}(\acute{\alpha})$ *and* $\ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \geq \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})$

(ii) $\grave{\mathfrak{P}} = \grave{\mathfrak{Q}} \Leftrightarrow \forall \acute{\alpha} \in \check{\mathfrak{G}},\ \ddot{\varphi}_{\grave{\mathbb{P}}}(\acute{\alpha}) = \ddot{\varphi}_{\grave{\mathbb{Q}}}(\acute{\alpha})$ *and* $\ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) = \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})$

(iii) $\overline{\grave{\mathfrak{P}}} = \left\{\left(\acute{\alpha},\ \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}),\ \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(iv) $\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \min\{\ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\}, \max\{\ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(v) $\grave{\mathfrak{P}} \cup \grave{\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \max\{\ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\}, \min\{\ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(vi) $\grave{\mathfrak{P}} + \grave{\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}),\ \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(vii) $\grave{\mathfrak{P}} \cdot \grave{\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}),\ \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$.

Theorem 5.1. *If* $Ł^{\varepsilon}_{\grave{\mathfrak{P}}}$ *and* $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}$ *are two* $\mathfrak{LIFS}$*s of a BM-algebra* $\check{\mathfrak{G}}$*, then*

(i) $Ł^{\varepsilon}_{\grave{\mathfrak{P}}} \subset Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} \Leftrightarrow \forall \acute{\alpha} \in \check{\mathfrak{G}},\ \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \leq \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})$ *and* $\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \geq \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})$

(ii) $Ł^{\varepsilon}_{\grave{\mathfrak{P}}} = Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} \Leftrightarrow \forall \acute{\alpha} \in \check{\mathfrak{G}},\ \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) = \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})$ *and* $\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) = \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})$

(iii) $\overline{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}} = \left\{\left(\acute{\alpha},\ \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}),\ \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$.

Proof. It's obvious that (i), (ii), and (iii) are true. □

Theorem 5.2. *If* $Ł^{\varepsilon}_{\grave{\mathfrak{P}}}$ *and* $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}$ *are two* $\mathfrak{LIFS}$*s of a BM-algebra* $\check{\mathfrak{G}}$*, then*

(i) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right\}, \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(ii) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cup \grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \max\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right\}, \min\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$.

Proof. (i) Let $Ł^{\varepsilon}_{\grave{\mathfrak{P}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ and $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ be two $\mathfrak{LIFS}$s of $\check{\mathfrak{G}}$. Consider, $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Thus,

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\} \\
&= \max\left\{0, \min\left\{\ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\right\} + \varepsilon - 1\right\} \\
&= \max\left\{0, \min\left\{\ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \varepsilon - 1, \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\}\right\} \\
&= \min\left\{\max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \varepsilon - 1\right\}, \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\}\right\}, \\
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}\cap\mathfrak{Q}}}(\acute{\alpha}) &= \min\left\{1, \ddot{\psi}_{\mathfrak{P}\cap\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\} \\
&= \min\left\{1, \max\left\{\ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}), \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha})\right\} + \varepsilon\right\} \\
&= \min\left\{1, \max\left\{\ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon, \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\}\right\} \\
&= \max\left\{\min\left\{1, \ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon\right\}, \min\left\{1, \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\}\right\}, \\
\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}\cap\mathfrak{Q}}}(\acute{\alpha}) &= \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}.
\end{aligned}$$

Therefore,

$$Ł^{\varepsilon}_{\mathfrak{P}\cap\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \min\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}, \max\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}.$$

(ii) Let $Ł^{\varepsilon}_{\mathfrak{P}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ and $Ł^{\varepsilon}_{\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ be two $\mathfrak{LIFS}$s of $\check{\mathfrak{G}}$. Consider, $Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Thus,

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha}) &= \max\left\{0, \ddot{\varphi}_{\mathfrak{P}\cup\mathfrak{Q}}(\acute{\alpha}) + \varepsilon - 1\right\} \\
&= \max\left\{0, \max\left\{\ddot{\varphi}_{\mathfrak{P}}(\acute{\alpha}), \ddot{\varphi}_{\mathfrak{Q}}(\acute{\alpha})\right\} + \varepsilon - 1\right\} \\
&= \max\left\{0, \max\left\{\ddot{\varphi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon - 1, \ddot{\varphi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon - 1\right\}\right\} \\
&= \max\left\{\max\left\{0, \ddot{\varphi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon - 1\right\}, \max\left\{0, \ddot{\varphi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon - 1\right\}\right\}, \\
\ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha}) &= \max\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha}) &= \min\left\{1, \ddot{\psi}_{\mathfrak{P}\cup\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\} \\
&= \min\left\{1, \min\left\{\ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}), \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha})\right\} + \varepsilon\right\} \\
&= \min\left\{1, \min\left\{\ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon, \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\}\right\} \\
&= \min\left\{\min\left\{1, \ddot{\psi}_{\mathfrak{P}}(\acute{\alpha}) + \varepsilon\right\}, \min\left\{1, \ddot{\psi}_{\mathfrak{Q}}(\acute{\alpha}) + \varepsilon\right\}\right\}, \\
\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}}}(\acute{\alpha}) &= \min\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}.
\end{aligned}$$

Therefore,

$$Ł^{\varepsilon}_{\mathfrak{P}\cup\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \max\left\{\ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}, \min\left\{\ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right\}\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}.$$

□

Theorem 5.3. *If $Ł^{\varepsilon}_{\mathfrak{P}}$ and $Ł^{\varepsilon}_{\mathfrak{Q}}$ are two $\mathfrak{LIFS}$s of a BM-algebra $\check{\mathfrak{G}}$, then*

(i) $Ł^{\varepsilon}_{\mathfrak{P}+\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}).\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$

(ii) $Ł^{\varepsilon}_{\mathfrak{P}\cdot\mathfrak{Q}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{Ł^{\varepsilon}_{\mathfrak{Q}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}.$

Proof. (i) Let $Ł^{\varepsilon}_{\grave{\mathfrak{P}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ and $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ be two $\mathfrak{LIFS}$s of $\check{\mathfrak{G}}$. Consider, $Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Thus,

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\} \\
&= \max\left\{0, \left[\ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\right] + \varepsilon - 1\right\} \\
&= \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \varepsilon - 1\right\} + \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\} \\
&\quad - \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\}, \\
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) + \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}) - \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \min\left\{1, \ddot{\psi}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon\right\} \\
&= \min\left\{1, \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon\right\}, \\
\ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}}(\acute{\alpha}) &= \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}).
\end{aligned}$$

Therefore, $Ł^{\varepsilon}_{\grave{\mathfrak{P}}+\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) + \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}) - \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha})\right) \middle| \mathfrak{a} \in \mathfrak{G}\right\}$.

(ii) Let $Ł^{\varepsilon}_{\grave{\mathfrak{P}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ and $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$ be two $\mathfrak{LIFS}$s of $\check{\mathfrak{G}}$. Consider, $Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. Thus,

$$\begin{aligned}
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\} \\
&= \max\left\{0, \ddot{\varphi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon - 1\right\}, \\
\ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \min\{1, \ddot{\psi}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon\} \\
&= \min\{1, \{\ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\} + \varepsilon\} \\
&= \min\{1, \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \varepsilon\} + \min\{1, \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon\} - \min\{1, \ddot{\psi}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) + \varepsilon\}, \\
\ddot{\psi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}}}(\acute{\alpha}) &= \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{Q}}}(\acute{\alpha}).
\end{aligned}$$

Therefore, $Ł^{\varepsilon}_{\grave{\mathfrak{P}}\cdot\grave{\mathfrak{Q}}} = \left\{\left(\acute{\alpha}, \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}}(\acute{\alpha}) \cdot \ddot{\varphi}_{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}}(\acute{\alpha}), \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{P}}}(\acute{\alpha}) + \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{Q}}}(\acute{\alpha}) - \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{P}}}(\acute{\alpha}) \cdot \ddot{\psi}^{\varepsilon}_{\grave{\mathfrak{Q}}}(\acute{\alpha})\right) \middle| \acute{\alpha} \in \check{\mathfrak{G}}\right\}$. □

Example 5.1. *Let* $\check{\mathfrak{G}} = \{0, \ddot{\varsigma}_1, \ddot{\varsigma}_2, \ddot{\varsigma}_3, \ddot{\varsigma}_4, \ddot{\varsigma}_5\}$ *be a BM-algebra and let*

$$\grave{\mathfrak{P}} = \left\{\begin{array}{ll} (0, 0.7, 0.03), & (\ddot{\varsigma}_1, 0.5, 0.2), \\ (\ddot{\varsigma}_2, 0.55, 0.23), & (\ddot{\varsigma}_3, 0.43, 0.2), \\ (\ddot{\varsigma}_4, 0.48, 0.1), & (\ddot{\varsigma}_5, 0.61, 0.11) \end{array}\right\}, \quad \grave{\mathfrak{Q}} = \left\{\begin{array}{ll} (0, 0.76, 0.01), & (\ddot{\varsigma}_1, 0.47, 0.32), \\ (\ddot{\varsigma}_2, 0.61, 0.1), & (\ddot{\varsigma}_3, 0.55, 0.18), \\ (\ddot{\varsigma}_4, 0.41, 0.31), & (\ddot{\varsigma}_5, 0.4, 0.4) \end{array}\right\}$$

be two $\mathfrak{I}\mathfrak{F}\mathfrak{S}$s of $\breve{\mathfrak{G}}$. Taking $\varepsilon = 0.6$, we get the $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$ of $\grave{\mathfrak{P}}$ and $\grave{\mathfrak{Q}}$ respectively as follows:

$$Ł^{\varepsilon}_{\grave{\mathfrak{P}}} = \left\{ \begin{array}{ll} (0, 0.3, 0.63), & (\ddot{\zeta}_1, 0.1, 0.8), \\ (\ddot{\zeta}_2, 0.15, 0.83), & (\ddot{\zeta}_3, 0.03, 0.8), \\ (\ddot{\zeta}_4, 0.08, 0.7), & (\ddot{\zeta}_5, 0.21, 0.71) \end{array} \right\}, \quad Ł^{\varepsilon}_{\grave{\mathfrak{Q}}} = \left\{ \begin{array}{ll} (0, 0.36, 0.61), & (\ddot{\zeta}_1, 0.07, 0.92), \\ (\ddot{\zeta}_2, 0.21, 0.7), & (\ddot{\zeta}_3, 0.15, 0.78), \\ (\ddot{\zeta}_4, 0.01, 0.91), & (\ddot{\zeta}_5, 0, 1) \end{array} \right\}$$

Now, let us find some operations on the set $Ł^{\varepsilon}_{\grave{\mathfrak{P}}}$ and $Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}$.

(i) $\overline{Ł^{\varepsilon}_{\grave{\mathfrak{P}}}} = \{(0, 0.63, 0.3), (\ddot{\zeta}_1, 0.8, 0.1), (\ddot{\zeta}_2, 0.83, 0.15), (\ddot{\zeta}_3, 0.8, 0.03), (\ddot{\zeta}_4, 0.7, 0.08), (\ddot{\zeta}_5, 0.71, 0.21)\}$

$\overline{Ł^{\varepsilon}_{\grave{\mathfrak{Q}}}} = \{(0, 0.61, 0.36), (\ddot{\zeta}_1, 0.92, 0.07), (\ddot{\zeta}_2, 0.7, 0.21), (\ddot{\zeta}_3, 0.78, 0.15), (\ddot{\zeta}_4, 0.91, 0.01), (\ddot{\zeta}_5, 1, 0)\}$

(ii) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cap \grave{\mathfrak{Q}}} = \{(0, 0.3, 0.63), (\ddot{\zeta}_1, 0.07, 0.92), (\ddot{\zeta}_2, 0.15, 0.83), (\ddot{\zeta}_3, 0.03, 0.8), (\ddot{\zeta}_4, 0.01, 0.91), (\ddot{\zeta}_5, 0, 1)\}$

(iii) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cup \grave{\mathfrak{Q}}} = \{(0, 0.36, 0.61), (\ddot{\zeta}_1, 0.1, 0.8), (\ddot{\zeta}_2, 0.21, 0.7), (\ddot{\zeta}_3, 0.15, 0.78), (\ddot{\zeta}_4, 0.08, 0.7), (\ddot{\zeta}_5, 0.21, 0.71)\}$

(iv) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} + \grave{\mathfrak{Q}}} = \{(0, 0.55, 0.38), (\ddot{\zeta}_1, 0.16, 0.74), (\ddot{\zeta}_2, 0.33, 0.58), (\ddot{\zeta}_3, 0.18, 0.62), (\ddot{\zeta}_4, 0.09, 0.64), (\ddot{\zeta}_5, 0.21, 0.71)\}$

(v) $Ł^{\varepsilon}_{\grave{\mathfrak{P}} \cdot \grave{\mathfrak{Q}}} = \{(0, 0.11, 0.86), (\ddot{\zeta}_1, 0.01, 0.98), (\ddot{\zeta}_2, 0.03, 0.95), (\ddot{\zeta}_3, 0.005, 0.96), (\ddot{\zeta}_4, 0.001, 0.97), (\ddot{\zeta}_5, 0, 1)\}$.

6. A Utilization of Łukasiewicz Intuitionistic Fuzzy Sets in Evaluating Employee Performance

The $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$ offers a wide variety of demands for managing uncertainty in our daily lives. This illustration demonstrates how to utilize such an application to address a typical decision-making dilemma. We employ the idea of $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$ to explain a dominant decision-making dilemma and then present a list of guidelines for finding the most suitable item from the provided study (see Algorithm).

Definition 6.1. *$\mathfrak{C}\mathfrak{T}$ has $n \times n$ ubiquitous object names, such as $\ddot{\mu}_1, \ddot{\mu}_2, \ddot{\mu}_3, \ldots, \ddot{\mu}_n$ and the opportunities $\mathfrak{p}_{ij}$ is the cardinality of components for which the quantity being specified by $\mathfrak{p}_i \geq \mathfrak{p}_j$ in membership and $\mathfrak{p}_i \leq \mathfrak{p}_j$ in non-membership.*

Algorithm:

(1) The set $\acute{\mathfrak{Y}}$ be the attributes for performing the evaluation.
(2) Get a dataset to accomplish the evaluation.
(3) Using the standard normalization method, convert the given dataset to $\mathfrak{I}\mathfrak{F}\mathfrak{S}$.
(4) Convert the $\mathfrak{I}\mathfrak{F}\mathfrak{S}$ to $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$.
(5) Create a $\mathfrak{C}\mathfrak{T}$ Table 2–8 for the inclusive function $\acute{\alpha}$ and exclusive function $\acute{\beta}$ of $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$s.
(6) Calculate the score for both the inclusive and exclusive functions.
(7) To determine the final result, take the difference of inclusive and exclusive score.
(8) Find the maximum result.

Employee performance evaluation is an organized approach done by higher officials in various organizations to measure how efficiently an employee is executing their job responsibilities. It entails evaluating numerous performance standards. This evaluation cannot be done easily because of its Subjective judgment, Inconsistencies, Uncertainty, and hesitation. To overcome these challenges, we applied the theory of $\mathfrak{L}\mathfrak{I}\mathfrak{F}\mathfrak{S}$ to perform the evaluation.

Let $\acute{\mathfrak{X}} = \{\grave{e}_1 = Joseph, \grave{e}_2 = Ria, \grave{e}_3 = Kavin, \grave{e}_4 = Arya, \grave{e}_5 = Jehira\}$ be the set of all Employees in a company. The manager of the company needs to evaluate these employees for the the promotion based on their performance, which are defined by certain attributes, $\mathfrak{Y} = \{\ddot{\mu}_1 = \textit{Team work}, \ddot{\mu}_2 = \textit{Leadership}, \ddot{\mu}_3 = \textit{technical skill}, \ddot{\mu}_4 = \textit{Punctuality}, \ddot{\mu}_5 = \textit{Client Feedback}\}$. Table 1 presents the scores achieved by employees based on their performance in each criterion, as evaluated by their managers.

TABLE 1. Scores of each employee based on their performance

	$\ddot{\mu}_1$	$\ddot{\mu}_2$	$\ddot{\mu}_3$	$\ddot{\mu}_4$	$\ddot{\mu}_5$
$\grave{e}_1$	9.5	10	7.9	8	8.5
$\grave{e}_2$	8.8	9.1	10	7.3	7.1
$\grave{e}_3$	8.2	7.9	9.3	7.9	9.7
$\grave{e}_4$	7.3	8.8	8.4	8.5	10
$\grave{e}_5$	10	7.9	8.3	8.5	7.6

To construct the $\mathfrak{IFS}$s for the above data, let us use the standard min-max normalization

$$\alpha_{\grave{e}_i} = \frac{\ddot{\mu}_i - \ddot{\mu}_{\min}}{\ddot{\mu}_{\max} - \ddot{\mu}_{\min}}, \text{ where } i = 1, 2, 3, 4, 5.$$

Thus, the $\mathfrak{IFS}$s of each Employee based on their Performance are defined as

$$\grave{e}_i = \{(\ddot{\mu}_j, \alpha_{\grave{e}_i}, \beta_{\grave{e}_i}) \mid i, j = 1, 2, 3, 4, 5\}$$

$$\grave{e}_1 = \{(\ddot{\mu}_1, 0.83, 0.15), (\ddot{\mu}_2, 1, 0), (\ddot{\mu}_3, 0.28, 0.68), (\ddot{\mu}_4, 0.31, 0.62), (\ddot{\mu}_5, 0.48, 0.47)\}$$
$$\grave{e}_2 = \{(\ddot{\mu}_1, 0.59, 0.39), (\ddot{\mu}_2, 0.69, 0.29), (\ddot{\mu}_3, 1, 0), (\ddot{\mu}_4, 0.07, 0.87), (\ddot{\mu}_5, 0, 1)\}$$
$$\grave{e}_3 = \{(\ddot{\mu}_1, 0.38, 0.54), (\ddot{\mu}_2, 0.28, 0.67), (\ddot{\mu}_3, 0.76, 0.21), (\ddot{\mu}_4, 0.28, 0.65), (\ddot{\mu}_5, 0.9, 0.07)\}$$
$$\grave{e}_4 = \{(\ddot{\mu}_1, 0.07, 0.87), (\ddot{\mu}_2, 0.59, 0.36), (\ddot{\mu}_3, 0.45, 0.51), (\ddot{\mu}_4, 0.48, 0.45), (\ddot{\mu}_5, 1, 0)\}$$
$$\grave{e}_5 = \{(\ddot{\mu}_1, 1, 0), (\ddot{\mu}_2, 0.28, 0.65), (\ddot{\mu}_3, 0.41, 0.55), (\ddot{\mu}_4, 0.48, 0.42), (\ddot{\mu}_5, 0.17, 0.76)\}$$

Let us convert the $\mathfrak{IFS}$s to $\mathfrak{LIFS}$s by taking $\varepsilon = 0.5$ and the set is defined by

$$Ł^{\varepsilon}_{\grave{e}_i} = \{(\ddot{\mu}_j, \alpha_{Ł^{\varepsilon}_{\grave{e}_i}}, \beta_{Ł^{\varepsilon}_{\grave{e}_i}}) \mid i = 1, 2, 3, 4, 5\}$$

$$Ł^{\varepsilon}_{è_1} = \{(\ddot{\mu}_1, 0.33, 0.65), (\ddot{\mu}_2, 0.5, 0.5), (\ddot{\mu}_3, 0, 1), (\ddot{\mu}_4, 0, 1), (\ddot{\mu}_5, 0, 0.97)\}$$
$$Ł^{\varepsilon}_{è_2} = \{(\ddot{\mu}_1, 0.09, 0.89), (\ddot{\mu}_2, 0.19, 0.79), (\ddot{\mu}_3, 0.5, 0.5), (\ddot{\mu}_4, 0, 1), (\ddot{\mu}_5, 0, 1)\}$$
$$Ł^{\varepsilon}_{è_3} = \{(\ddot{\mu}_1, 0, 1), (\ddot{\mu}_2, 0, 1), (\ddot{\mu}_3, 0.26, 0.71), (\ddot{\mu}_4, 0, 1), (\ddot{\mu}_5, 0.4, 0.57)\}$$
$$Ł^{\varepsilon}_{è_4} = \{(\ddot{\mu}_1, 0, 1), (\ddot{\mu}_2, 0.09, 0.86), (\ddot{\mu}_3, 0, 1), (\ddot{\mu}_4, 0, 0.95), (\ddot{\mu}_5, 0.5, 0.5)\}$$
$$Ł^{\varepsilon}_{è_5} = \{(\ddot{\mu}_1, 0.5, 0.5), (\ddot{\mu}_2, 0, 1), (\ddot{\mu}_3, 0, 1), (\ddot{\mu}_4, 0, 0.92), (\ddot{\mu}_5, 0, 1)\}$$

Table 2. Tabular representation of $\alpha_{Ł^{\varepsilon}_{è_1}}$, $\alpha_{Ł^{\varepsilon}_{è_2}}$, $\alpha_{Ł^{\varepsilon}_{è_3}}$, $\alpha_{Ł^{\varepsilon}_{è_4}}$ and $\alpha_{Ł^{\varepsilon}_{è_5}}$

	$\ddot{\mu}_1$	$\ddot{\mu}_2$	$\ddot{\mu}_3$	$\ddot{\mu}_4$	$\ddot{\mu}_5$
$è_1$	0.33	0.5	0	0	0
$è_2$	0.09	0.19	0.5	0	0
$è_3$	0	0	0.26	0	0.4
$è_4$	0	0.09	0	0	0.5
$è_5$	0.5	0	0	0	0

Table 3. Comparison table for the above table

	$è_1$	$è_2$	$è_3$	$è_4$	$è_5$
$è_1$	5	4	3	4	4
$è_2$	3	5	4	4	4
$è_3$	3	2	5	3	4
$è_4$	3	2	4	5	4
$è_5$	4	3	3	3	5

Table 4. α score

	Row Total (r^{α^+})	**Column Total** (c^{α^+})	α **Score** ($r^{\alpha^+} - c^{\alpha^+}$)
$è_1$	20	18	2
$è_2$	20	16	4
$è_3$	17	19	−2
$è_4$	18	19	−1
$è_5$	18	21	−3

TABLE 5. Tabular representation of $\beta_{Ł^{\varepsilon}_{è_1}}$, $\beta_{Ł^{\varepsilon}_{è_2}}$, $\beta_{Ł^{\varepsilon}_{è_3}}$, $\beta_{Ł^{\varepsilon}_{è_4}}$ and $\beta_{Ł^{\varepsilon}_{è_5}}$

	$\ddot{\mu}_1$	$\ddot{\mu}_2$	$\ddot{\mu}_3$	$\ddot{\mu}_4$	$\ddot{\mu}_5$
$è_1$	0.65	0.5	1	1	0.97
$è_2$	0.89	0.79	0.5	1	1
$è_3$	1	1	0.71	1	0.57
$è_4$	1	0.86	1	0.95	0.5
$è_5$	0.5	1	1	0.92	1

TABLE 6. Comparison table for the above table

	$è_1$	$è_2$	$è_3$	$è_4$	$è_5$
$è_1$	5	4	3	3	3
$è_2$	2	5	4	3	3
$è_3$	3	2	5	2	2
$è_4$	3	2	4	5	3
$è_5$	3	3	3	3	5

TABLE 7. β score

	Row Total (r^{β^+})	**Column Total (c^{β^+})**	**β Score ($r^{\beta^+} - c^{\beta^+}$)**
$è_1$	18	16	2
$è_2$	17	16	1
$è_3$	14	19	−5
$è_4$	17	16	1
$è_5$	17	16	1

TABLE 8. Table for Final result.

	α score ($\ddot{\alpha}$)	**β score ($\dddot{\beta}$)**	**Final Result ($\ddot{\alpha} - \dddot{\beta}$)**
$è_1$	2	2	0
$è_2$	4	1	3
$è_3$	−2	−5	3
$è_4$	−1	1	−2
$è_5$	−3	1	−4

Decision: Employee Ria ($\mathfrak{e}_2$) and Kavin ($\mathfrak{e}_3$) are the best choices for the promotion.

7. Conclusion

This paper introduced a comprehensive framework for evaluating employee performance through the integration of $\mathfrak{LIFS}$s within *BM*-algebras. By formalizing the concepts of $\mathfrak{LIFA}$s and $\mathfrak{LIFI}$s and establishing their foundational properties through rigorous theorems and examples, we have shown that this algebraic structure effectively models uncertainty in human resource evaluation processes. The proposed model enhances traditional assessment techniques by incorporating degrees of membership and non-membership, enabling nuanced and flexible decision-making. The min-max normalization applied within this context ensures a balanced comparison across multiple performance indicators. Overall, this approach provides a mathematically robust, logically consistent, and practically applicable tool for organizations seeking to enhance fairness and precision in employee evaluations. Future work may focus on real-world deployment and comparative analysis with existing fuzzy-based methods to further validate its utility.

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