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A Novel ξ-Order Hölder Function Classes and Their Integral Transform

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Abstract. We introduce a function class $\mathcal{H}(\xi)$ ($\xi > 0$) capturing tail decay and Hölder regularity. For $h \in \mathcal{H}(\xi)$, its Fourier transform $\mathcal{F}[h]$ inherits Hölder continuity of order ξ and essential boundedness. For $\xi > 1$, derivatives of $\mathcal{F}[h]$ up to order $\lfloor \xi \rfloor$ are L^{∞}-bounded, with fractional Hölder continuity arising from h's decay. Our approach integrates multiscale analysis and Fourier multiplier theory, extending prior results on Hölder-Fourier correspondences. Novel integral estimates and phase cancellation methods resolve critical gaps in non-integer smoothness characterization. These results deepen the Fourier regularity analysis for non-integer ξ, offering tools for harmonic analysis and pseudo-differential operators.

1. Introduction

The study of Fourier transforms and their regularity properties has long been a central theme in harmonic analysis. Among these properties, the Hölder continuity of the Fourier transform provides a natural framework for understanding the interplay between the decay of a function and the smoothness of its transform. Classical results, such as the Riemann-Lebesgue lemma, establish fundamental connections between integrability conditions and continuity, but finer relationships involving Hölder or Lipschitz classes require more nuanced characterizations. In foundational works, [1] and [2] systematically established the theoretical links between Fourier transforms and function spaces, such as Hölder and Lipschitz spaces. Subsequent advancements by [3] extended the analysis of convolution operator mapping properties through wavelet and operator theory, while [4] deepened the study of multiplier behaviors for non-smooth kernels within the framework of pseudo-differential operators. Though these studies did not fully resolve the correspondence between function regularity and the Lipschitz order of Fourier transforms,

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they laid critical groundwork for modern analytical tools. This paper introduces a class of functions, denoted $\mathcal{H}(\xi)$ ($\xi > 0$), which unifies and generalizes prior approaches. The $\mathcal{H}(\xi)$-class is defined via integral conditions that quantify the tail behavior of functions in a manner compatible with Hölder regularity. Our primary objective is to demonstrate that the Fourier transform of any $\mathcal{H}(\xi)$-function inherits Hölder continuity of order ξ. Methodologically, this work is inspired by [5]'s insights into the convergence of multidimensional Fourier series, which emphasize decay-rate conditions as pivotal determinants of regularity, and aligns with [6]'s modern framework for Fourier analysis. Key results include sufficient conditions under which membership in $\mathcal{H}(\xi)$ guarantees Hölder regularity of the Fourier transform, even for non-integer orders $\xi > 1$. Specifically, if $h \in \mathcal{H}(\xi)$, its Fourier transform $\mathcal{F}[h]$ has Hölder continuity of order ξ and is essentially bounded, with derivatives up to order $\lfloor \xi \rfloor$ existing and satisfying boundedness conditions. Notably, for non-integer ξ, the fractional Holder component of $\mathcal{F}[h]$ arises naturally from the decay properties of h. These findings resonate with [7]'s systematic classification of function spaces and [8]'s theory of oscillatory integrals. The technical foundation of this work relies on novel integral estimates and duality arguments, extending [3]'s multiscale analysis methods. By integrating [6]'s modern Fourier multiplier theory, we bridge gaps between existing results and provide a unified framework for analyzing Hölder-Fourier correspondences. Furthermore, [9]'s profound insights into phase cancellation for high-frequency oscillatory integrals underpin the estimation techniques employed here.

2. Preliminaries

Definition 2.1. *For $\Omega \subset R^n$ open, set*

$$L_{\text{loc}}(\Omega) = \left\{ g : \Omega \to R \mid g \in L(\tilde{\Omega}), \forall \tilde{\Omega} \subset\subset \Omega \right\}$$

where $\tilde{\Omega} \subset\subset \Omega$ means that there exists K compact such that $\tilde{\Omega} \subset K \subset \Omega$. We say that $\tilde{\Omega}$ is compactly contained in Ω.

Definition 2.2. *Let $\xi > 0, g \in L_{\text{loc}}(\boldsymbol{R})$.Define the function:*

$$\phi(y,s) := \int_R s^{-1} g\left(ts^{-1}\right) e^{-its^{-1}y} \chi_{[1,\infty)} d\mu(t) \tag{2.1}$$

Let the family of functions $\mathcal{H}_1(\xi)$ and $\mathcal{H}_2(\xi)$ be defined as

$$\mathcal{H}_1(\xi) := \left\{ g \in L_{loc}(R) \mid \exists L_1 > 0,\ \forall y \in R,\ |\phi(y,s)| \leq L_1 |s|^{\xi} \right\} \tag{2.2}$$

$$\mathcal{H}_2(\xi) := \left\{ g \in L_{loc}(R) \mid \exists L_2 > 0,\ \forall y \in R,\ |\phi(y,-s)| \leq L_2 |s|^{\xi} \right\} \tag{2.3}$$

Now we define the collection of functions $\mathcal{H}(\xi)$ as

$$\mathcal{H}(\xi) := \mathcal{H}_1(\xi) \cap \mathcal{H}_2(\xi) \tag{2.4}$$

Definition 2.3. *Let $\gamma \in (0,1]$. A function $g : \Omega \subseteq R^n \to R^m$ satisfies γ-Hölder continuous on Ω (or is the Lipschitz condition of order γ) if there exists a constant $C > 0$ such that for all $x, y \in \Omega$,*

$$\|g(\boldsymbol{x}) - g(\boldsymbol{y})\| \leq C\|\boldsymbol{x} - \boldsymbol{y}\|^{\gamma}, \tag{2.5}$$

where $\|\cdot\|$ denotes the Euclidean norm. The constant C is called the Hölder constant, and γ is the Hölder exponent.

Remark 2.1. *When $\gamma = 1$, this reduces to standard Lipschitz continuity. For $\gamma \in (0,1)$, the term Hölder continuity is standard, though "Lipschitz of order γ" appears in some literature. Hölder continuity measures a function's modulus of smoothness. Smaller γ permits sharper oscillations, while $\gamma \to 1$ implies Lipschitz-like behavior.*

Definition 2.4. *Let $\Omega \subseteq R^n$ be an open domain, $k \in N$, and $\beta \in (0,1]$. The Hölder space $C^{k,\beta}(\Omega)$ consists of functions $g : \Omega \to R$ satisfying:*

- *(i) All θ-th order partial derivatives $D^{\theta}g$ (for multi-indices θ with $|\theta| \leq k$) exist and are continuous on Ω.*
- *(ii) For every multi-index θ with $|\theta| = k$, there exists $C_\theta > 0$ such that*

$$\|D^{\theta}g(\boldsymbol{x}) - D^{\theta}g(\boldsymbol{y})\| \leq C_{\theta}\|\boldsymbol{x} - \boldsymbol{y}\|^{\beta}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega,$$

where $\|\cdot\|$ denotes the Euclidean norm.

A function $g \in C^{k,\beta}(\Omega)$ is said to have $C^{k,\beta}$-regularity. For non-integer $\beta > 1$, let $\beta = j + \alpha$ where: $j = \lfloor\beta\rfloor$ (integer part), $\alpha = \beta - j \in (0,1)$. The fractional Hölder space $C^{\beta}(\Omega)$ is defined as $C^{\beta}(\Omega) := C^{j,\alpha}(\Omega)$. Functions in $C^{\beta}(\Omega)$ satisfy: j-th order derivatives exist and are continuous, $D^j g$ is α-Hölder continuous.

Definition 2.5. *Let $(R^n, \mathcal{A}, \mu)$ be a measure space. The space $L^{\infty}(R^n)$ consists of pointwise a.e.-equivalence classes of essentially bounded measurable functions $g : R^n \to R$ with norm*

$$\|g\|_{L^{\infty}} = \operatorname*{ess\,sup}_{R^n}|g|$$

Definition 2.6. *For real bumber $\beta > 1$, let $\beta = j + \alpha$ where $j = \lfloor\beta\rfloor$ (the integer part) and $\alpha = \beta - j \in (0,1)$. A function $g : R^n \to R$ is said to satisfy the β-Hölder continuous property if:*

- *(i) $D^{\boldsymbol{\gamma}} f \in L^{\infty}(R^n), \quad \forall\, |\boldsymbol{\gamma}| \leq j$.*
- *(ii) $D^{\boldsymbol{\gamma}} f \in C^{\beta}(R^n), \quad \forall\, |\boldsymbol{\gamma}| = j$.*

Remark 2.2. *when $\beta \in (0,1]$, we said that f have the β-Hölder continuous property if f satisfies the definition in Definition 2.3.*

3. Proof of lemmas

Lemma 3.1. *Suppose that $\Phi(u,v) \in L_{loc}(R)$ for all $v \in R$, and that $0 < m < t$ are real numbers. We define $\omega(v,t,s)$ and $\rho(v,s)$ as follows:*

$$\begin{aligned} \omega(v,t,s) &= \int_R u^t \Phi(u,v)\chi_{[0,s]} d\mu(u) \\ \rho(v,s) &= \int_R \Phi(u,v)\chi_{[s,\infty]} d\mu(u) \end{aligned} \tag{3.1}$$

Then, there exists a constant $c_1 > 0$ such that $|\omega(v,t,s)| \leq c_1|s|^m$ if and only if there exists a constant $c_2 > 0$ such that $|\rho(v,s)| \leq c_2|s|^{m-t}$.

Proof. (Forward Direction⇒) By hypothesis, $\exists c_1 > 0$ such that $|\omega(v,t,s)| \leq c_1 s^m$. Consider the tail integral via integration by parts:

$$\begin{aligned} \rho(v,s) &= \int_R \Phi(u,v)\chi_{[s,\infty]} d\mu(u) \\ &= \int_R u^{-t} u^t \Phi(u,v)\chi_{[s,\infty]} d\mu(u) \\ &= \left[u^{-t} w(v,t,u)\right]_{u=s}^{u=\infty} + t\int_R u^{-t-1} w(v,t,u)\chi_{[s,\infty]} d\mu(u) \end{aligned} \tag{3.2}$$

Using $|\omega(v,t,s)| \leq c_1|s|^m$ and $m < t$,we obtain the following estimate:

$$\left|\lim_{u\to\infty} u^{-t} w(v,t,u)\right| \leqslant \lim_{u\to\infty} u^{-t}|w(v,t,u)| \leqslant \lim_{u\to\infty} c_1 u^{m-t} \leq c_1 s^{m-t} \tag{3.3}$$

$$\left|s^{-t} w(v,t,s)\right| \leqslant s^{-t} c_1 s^m = c_1 s^{m-t} \tag{3.4}$$

$$\left|\int_R t u^{-t-1} w(v,t,u)\chi_{[s,\infty)} d\mu(u)\right| \leqslant \int_R c_1 u^m t u^{-t-1}\chi_{[s,\infty)} d\mu(u) < \frac{tc_1}{t-m} s^{m-t} \tag{3.5}$$

Combining (3.3), (3.4), (3.5) gives $|\rho(v,s)| \leq c_2|s|^{m-t}$ where $c_2 := 2c_1 + \frac{tc_1}{t-m}$.
(Reverse Direction⇐) By hypothesis, $|\rho(v,s)| \leq c_2|s|^{m-t}$. Rewrite the weighted integral:

$$\begin{aligned} w(v,t,s) &= \int_R u^t \Phi(u,v)\chi_{[0,s]} d\mu(u) \\ &= \left[u^t \rho(v,u)\right]_{u=0}^{u=s} - t\int u^{t-1}\rho(v,u)\chi_{[0,s]} d\mu(u) \end{aligned} \tag{3.6}$$

Bound each term:

$$\left|s^t \rho(v,s)\right| \leqslant |s|^t c_2 |s|^{m-t} = c_2|s|^m \tag{3.7}$$

$$\left|\lim_{u\to 0} u^t \rho(v,u)\right| \leqslant \lim_{u\to 0}\left|u^t \rho(v,u)\right| \leqslant \lim_{u\to 0} |u|^t\, c_2\, |u|^{m-t} \leqslant c_2|s|^m \tag{3.8}$$

$$\left|t\int_R u^{t-1}\rho(v,u)\chi_{[0,s]} d\mu(u)\right| \leq \int_R t\,|u|^{t-1}\, c_2\, |u|^{m-t}\,\chi_{[0,s]} d\mu(u) = \int_R t\,|u|^{m-1}\, c_2\chi_{[0,s]} d\mu(u) = \frac{c_2 t}{m}|s|^m \tag{3.9}$$

Thus,we have $\left|\omega(v,t,s)\right| \leqslant c_1|s|^m$,where $c_1 := 2c_2 + \frac{c_2 t}{m}$. □

Remark 3.1. *Let $\hat{w}(v,t,s)$ and $\hat{\rho}(v,s)$ be defined as follows:*

$$\hat{w}(v,t,s) := \int_R |u|^t \Phi(u,v)\chi_{[-s,0]}\, d\mu(u),$$

and

$$\hat{\rho}(v,s) := \int_R \Phi(u,v)\chi_{[-\infty,-s]}\, d\mu(u).$$

We can obtain the same result as in Lemma 3.1 for the symmetric negative half-line .

Lemma 3.2. *Let $\xi \in (0,2)$ and $\eta > 0$. Suppose $\Phi : R\times R \to C$ satisfies the condition $\left|\int_R \Phi\left(z\eta^{-1},v\right)\eta^{-1}\chi_{[1,\infty)}d\mu(z)\right| \leqslant L\left|\eta\right|^{\xi}$ for some constant $L>0$. Then, for the oscillatory integrals, it follows that: if $0<\xi<1$, we have $\left|\int_R \Phi\left(z\eta^{-1},v\right)\sin(z)\,\chi_{[0,1]}\eta^{-1}d\mu(z)\right| \leqslant \lambda|\eta|^{\xi}$ for some constant $\lambda>0$; and if $0<\xi<2$, we obtain $\left|\int_R \Phi\left(z\eta^{-1},v\right)\sin^2\frac{z}{2}\chi_{[0,1]}\eta^{-1}d\mu(z)\right| \leqslant \kappa|\eta|^{\xi}$ for some constant $\kappa>0$.*

Proof. By performing a variable substitution $t = z\eta^{-1}$ in equation (3.10) and applying the Bonnet mean value theorem in equation (3.12) (which relies on the fact that $\frac{\sin x}{x}$ is monotonic on the interval $[0,2]$), we obtain the following integral form:

$$\left|\int_R \Phi\left(z\eta^{-1},v\right)\cdot\sin z\cdot\eta^{-1}\chi_{[0,1]}d\mu(z)\right| \tag{3.10}$$

$$=\left|\int_R \Phi(t,v)\sin t\eta\cdot\chi_{[0,\eta^{-1}]}d\mu(t)\right| \tag{3.11}$$

$$=\eta\left|\int_R t\Phi(t,v)\frac{\sin th}{t\eta}\chi_{[0,\eta^{-1}]}d\mu(t)\right| \tag{3.12}$$

$$=\eta\left|\int_R t\Phi(t,v)\chi_{[0,p]}d\mu(t)\right|, 0<p<\frac{1}{\eta} \tag{3.13}$$

By applying a change of variables $t = z\eta^{-1}$ to $\left|\int_R \Phi\left(z\eta^{-1},v\right)\eta^{-1}\chi_{[1,\infty)}d\mu(z)\right|$, we obtain the following inequality:

$$\left|\int_R \Phi(t,v)\chi_{[\eta^{-1},\infty]}d\mu(t)\right| \le L\left|\frac{1}{\eta}\right|^{-\xi} \tag{3.14}$$

By applying Lemma 3.1, we can obtain an equivalent expression for equation (3.14),that is:There exists a constant $L_1>0$ such that

$$\left|\int t\Phi(t,v)\chi_{[0,\eta^{-1}]}d\mu(t)\right| \leqslant L_1\left|\frac{1}{\eta}\right|^{-\xi+1} \tag{3.15}$$

Combining equations (3.13) and (3.15) gives:

$$(3.13) \le \eta L_1|p|^{-\xi+1} \le \eta L_1\left|\frac{1}{\eta}\right|^{-\xi+1} \le L_1\left|\eta\right|^{\xi}$$

Thus, we have shown that

$$\left|\int_R \Phi\left(z\eta^{-1},v\right)\sin(z)\chi_{[0,1]}\eta^{-1}d\mu(z)\right| \leqslant \lambda|\eta|^{\xi}.$$

The same method can be applied to prove the second inequality.

□

Remark 3.2. *For* $\left|\int_R \Phi\left(z\eta^{-1}, v\right)\eta^{-1}\chi_{(-\infty,-1]}d\mu(z)\right| \leqslant L\left|\eta\right|^{\xi}$, *The analogous bounds hold for the negative interval.*

Lemma 3.3. *Let* $\xi > 0$. *If* $h \in \mathcal{H}(\xi)$, *then its Fourier transform* $\mathcal{F}[h](x) =: \int_R h(y)e^{-ixy}d\mu(y)$ *exists for all* $x \in R$ *and satisfies:* $\mathcal{F}[h] \in L^\infty(R)$.

Proof. For fixed $x_0 \in R$, we have

$$\begin{aligned} \mathcal{F}[h]\left(x_0\right) &= \int_R h(y)e^{-ix_0y}d\mu(y) \\ &= \int_{(-\infty,-1]\cup[-1,1]\cup[1,\infty)} h(y)e^{-ix_0y}d\mu(y) \end{aligned} \tag{3.16}$$

We will divide it into three parts for estimation.

(i) in the interval $[1, \infty)$, By substituting g with h in formula (2.1), setting $y = x_0$ and $s = 1$, and combining with formula (2.2), we obtain the following estimate:

$$\left|\int_R h(y)e^{-iyx_0}\chi_{[1,\infty]}d\mu(y)\right| = \left|\phi\left(x_0, 1\right)\right| \leq L_1$$

(ii) in the interval $[-1, 1]$, we have

$$\int_{[-1,1]} \left|h(y)e^{-ix_0y}\right| d\mu(y) = \int_{[-1,1]} |h(y)|d\mu(y) < \infty \text{ ,since } h \in H(\xi) \cap L_{\text{loc}}\left(R\right)$$

(iii) in the interval $(-\infty, -1]$.Combining with formula (2.3) ,we apply the same argument as in (i).we get:

$$\left|\int_R h(y)e^{-iyx_0}\chi_{[-\infty,-1]}d\mu(y)\right| = \left|\phi\left(x_0, -1\right)\right| \leq L_2$$

Noticing that the estimates in (i), (ii), and (iii) are independent of the choice of x_0, we deduce the existence of a constant Q such that:

$$\left|\mathcal{F}[h](x_0)\right| = \left|\int_R h(y)e^{-ix_0y}\, d\mu(y)\right| \leq Q, \quad \forall\, x_0 \in R. \tag{3.17}$$

Thus, we have established the required estimate, as desired. □

Lemma 3.4. *For* $\xi \in (0, 1)$, *let* $h \in \mathcal{H}(\xi)$. *Then, its Fourier transform* $\mathcal{F}[h]$ *is both bounded and Hölder continuous of order* ξ.

Proof. In Lemma 3.3, we have already shown that the Fourier transform of h exists and is bounded for all $x \in R$. Next, we only need to prove that it possesses the Hölder continuity property of order ξ.

For any $x, y \in R$ with $x \neq y$, we have:

$$\begin{aligned} \left|\mathcal{F}[h](x) - \mathcal{F}[h](y)\right| &= \left|\int_R h(t)\left(e^{-itx} - e^{-ity}\right)d\mu(t)\right| \\ &= \int_{(-\infty,-|x-y|^{-1})\cup(-|x-y|^{-1},|x-y|^{-1})\cup(|x-y|^{-1},\infty)} h(t)\left(e^{-itx} - e^{-ity}\right)d\mu(t) \end{aligned} \tag{3.18}$$

We will break down the estimation of equation (3.18) into three parts. For $t \in (|y-x|^{-1}, \infty)$, we have the following estimate:

$$\begin{aligned}
&\left|\int_{(|y-x|^{-1},\infty)} h(t)\left(e^{-itx}-e^{-ity}\right)d\mu(t)\right| \\
\leq& 2\left|\int_{(|y-x|^{-1},\infty)} h(t)e^{-ity}d\mu(t)\right| \\
=& 2\left|\int_R h\left(t|y-x|^{-1}\right)e^{-ity|y-x|^{-1}}\chi_{(1,\infty)}\,d\mu(t)\right| \\
=& 2|\phi(y,|y-x|)| \leq 2L_1|y-x|^{\xi}
\end{aligned} \tag{3.19}$$

since $h \in H(\xi) \subseteq H_1(\xi)$.

For $t \in (-\infty, -|y-x|^{-1})$, We can apply the same estimation technique from (3.19) to obtain the integral estimate on the symmetric interval since $h \in H(\xi) \subseteq H_2(\xi)$.That is:

$$\left|\int_{(-\infty,-|y-x|^{-1})} h(t)\left(e^{-itx}-e^{-ity}\right)d\mu(t)\right| \leq 2|\phi(y,-|y-x|)| \leq 2L_2|y-x|^{\xi} \tag{3.20}$$

For $t \in \left(-|y-x|^{-1}, |y-x|^{-1}\right)$, define $\Phi(u,v) = h(u)e^{-iuv}$, first, observe that $h \in H(\xi) \subset H_1(\xi)$ implies the following constraints:

$$\left|\int_R \Phi\left(ts^{-1}, y\right)s^{-1}\chi_{[1,\infty)}(t)\,d\mu(t)\right| \leqslant L_1|s|^{\xi}$$

On the other hand, due to symmetry, and without loss of generality, we only need to consider the case $t \in \left[0, |y-x|^{-1}\right]$. The treatment for $t \in \left[-(y-x)^{-1}, 0\right]$ follows in a similar manner. Additionally, we assume that $y > x$.

$$\begin{aligned}
&\left|\int_{(0,(y-x)^{-1})} h(t)e^{-itx}\left(1-e^{-it(y-x)}\right)d\mu(t)\right| \\
=&\left|\underbrace{\int_{(0,(y-x)^{-1})} e^{-itx} ih(t)\sin(t(y-x))d\mu(t)}_{I} + \underbrace{\int_{(0,(y-x)^{-1})} 2h(t)e^{-itx}\sin^2(\frac{t(y-x)}{2})\,d\mu(t)}_{II}\right|
\end{aligned} \tag{3.21}$$

According to Lemma 3.2, we have the following estimates for I and II:

$$\begin{aligned}
|I| =& \left|i\int_R \Phi\left(s(y-x)^{-1}, x\right)\cdot \sin s\cdot (y-x)^{-1}\chi_{[0,1]}d\mu(s)\right| \\
\leqslant& \lambda|y-x|^{\xi}
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
|II| =& 2\left|\int_R \Phi\left(s(y-x)^{-1}, x\right)\sin^2\frac{s}{2}(y-x)^{-1}\chi_{[0,1]}d\mu(s)\right| \\
\leqslant& 2\kappa|y-x|^{\xi}
\end{aligned} \tag{3.23}$$

Combining formulas (3.19),(3.20),(3.22) and (3.23), we can conclude that there exists a constant $B > 0$ such that the following holds:

$$|\mathcal{F}[h](x) - \mathcal{F}[h](y)| \le B|y-x|^{\xi}$$

which thus proves that $\mathcal{F}[h]$ exhibits the Hölder continuity property of order ξ. □

Lemma 3.5. *Let h be a locally integrable function, and suppose there exists a constant $C(y) > 0$,where C depends on y,such that for all $y \in R$, we have: $\left|\int_R uh(u)e^{-iuy}\chi_{[0,\infty)}\,d\mu(u)\right| \le C(y)$. Then, the following limit holds pointwise for each y:*
$\lim_{v\to 0^+}\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2} + i\sin u\right)v^{-2}\chi_{[0,1]}\,d\mu(u) = \int_R iuh(u)e^{-iuy}\chi_{[0,\infty)}\,d\mu(u)$.

Proof. We divide the integral $\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2} + i\sin u\right)v^{-2}\chi_{[0,1]}\,d\mu(u)$ into two parts, (a) and (b), and estimate them separately. In the following proof, the limit of (a) tends to 0, and it is mainly part (b) that contributes.

$$\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2} + i\sin u\right)v^{-2}\chi_{[0,1]}\,d\mu(u)$$
$$= \underbrace{\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2}\right)v^{-2}\chi_{[0,1]}\,d\mu(u)}_{(a)} \tag{a}$$
$$+ \underbrace{\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(i\sin u\right)v^{-2}\chi_{[0,1]}\,d\mu(u)}_{(b)} \tag{b}$$

For part (a),by taking some constant $v, L > 0$, we have:

$$\text{(a)} = \underbrace{\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2}\right)v^{-2}\chi_{[0,vL]}\,d\mu(u)}_{(a.1)}$$
$$+ \underbrace{\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2}\right)v^{-2}\chi_{[vL,1]}\,d\mu(u)}_{(a.2)}$$

For part $(a.2)$, since $\frac{\sin^2(x)}{x}$ is monotonically increasing and non-negative near $x = 0$, we apply the Bonnet mean value theorem and take L sufficiently large and v sufficiently small, we have:

$$\begin{aligned}
\left|(a.2)\right| &= \left|\int_R \frac{1}{2}h\left(uv^{-1}\right)e^{-iyuv^{-1}}\cdot\frac{\sin^2\frac{u}{2}}{\frac{u}{2}}\cdot uv^{-1}\cdot v^{-1}\chi_{[vL,1]}d\mu(u)\right| \\
&= \left|\frac{\sin^2\frac{u}{2}}{\frac{u}{2}}\right|_{u=1}\int_R \frac{1}{2}h\left(uv^{-1}\right)e^{-iyuv^{-1}}\cdot uv^{-1}\cdot v^{-1}\chi_{[p,1]}d\mu(u)\Bigg| \\
&\le \left|\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}uv^{-1}\cdot v^{-1}\chi_{[p,1]}d\mu(u)\right| \\
&\le \left|\int_R sh(s)e^{-iys}\chi_{[pv^{-1},v^{-1}]}d\mu(s)\right| < \varepsilon
\end{aligned} \tag{3.24}$$

for some constat p satisfying $L < \frac{p}{v} < \frac{1}{v}$. We obtain (3.24) arbitrarily small as long as v is sufficiently small because $\left|\int_R uh(u)e^{-iuy}\, d\mu(u)\right|$ is pointwise bounded for fixed y.
For part $(a.1)$, we have:

$$\begin{aligned}
&\left|\int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin^2 \frac{u}{2}\right) v^{-2} \chi_{[0,vL]}\, d\mu(u)\right| \\
=&\left|\int_R h(s) e^{-iys} \left(\sin^2 \frac{sv}{2}\right) v^{-1} \chi_{[0,L]} d\mu(s)\right|. \\
\leq&\int_R \frac{L}{2} \left|h(s)\right| \frac{\sin^2 \frac{sv}{2}}{\frac{sv}{2}} \chi_{[0,L]} d\mu(s) \longrightarrow 0 \text{ as } v \to 0^+
\end{aligned} \tag{3.25}$$

As $v \to 0^+$, (3.25) tends to zero by the Dominated Convergence Theorem, given that $h \in L_{\text{loc}}(R)$. By combining (3.24) and (3.25), we obtain that (a) tends to 0 as $v \to 0^+$,That is:

$$\lim_{v\to 0^+} \int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin^2 \frac{u}{2}\right) v^{-2} \chi_{[0,1]}\, d\mu(u) = 0 \tag{3.26}$$

From (3.26), we know that to prove the conclusion of the lemma, it suffices to show that the estimate in part (b) converges to the desired value. Now we estimate part (b).

$$\begin{aligned}
&\left|\int_R iuh(u)e^{-iuy} \chi_{[0,\infty)}\, d\mu(u) - \int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(i \sin u\right) v^{-2} \chi_{[0,1]}\, d\mu(u)\right| \\
=&\left|\int_R uh(u)e^{-iuy} \chi_{[0,\infty)}\, d\mu(u) - \int_R uh(u)e^{-iuy} \chi_{[0,A)}\, d\mu(u) \quad + \int_R uh(u)e^{-iuy} \chi_{[0,A)}\, d\mu(u)\right. \\
&\left. - \int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin u\right) v^{-2} \chi_{[0,vA]}\, d\mu(u) - \int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin u\right) v^{-2} \chi_{[vA,1]}\, d\mu(u)\right| \\
\leq&\underbrace{\left|\int_R uh(u)e^{-iuy} \chi_{[0,\infty)}\, d\mu(u) - \int_R uh(u)e^{-iuy} \chi_{[0,A)}\, d\mu(u)\right|}_{(b.1)} \\
&+\underbrace{\left|\int_R uh(u)e^{-iuy} \chi_{[0,A)}\, d\mu(u) - \int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin u\right) v^{-2} \chi_{[0,vA]}\, d\mu(u)\right|}_{(b.2)} \\
&+\underbrace{\left|\int_R h\left(uv^{-1}\right) e^{-iyuv^{-1}} \left(\sin u\right) v^{-2} \chi_{[vA,1]}\, d\mu(u)\right|}_{b.3}
\end{aligned} \tag{3.27}$$

Subsequently, we estimate $(b.1)$, $(b.2)$, and $(b.3)$ respectively.

For $(b.1)$, since the pointwise bound of expression $\int_R uh(u)e^{-iuy} \chi_{[0,\infty)}\, d\mu(u)$ holds, there exists A_0 such that for any $A \geq A_0$, expression $(b.1)$ is less than ϵ,i.e.,

$$\left|\int_R uh(u)e^{-iuy} \chi_{[0,\infty)}\, d\mu(u) - \int_R uh(u)e^{-iuy} \chi_{[0,A)}\, d\mu(u)\right| < \epsilon \quad \forall A \geq A_0 \tag{3.28}$$

For $(b.2)$, since h is locally integrable, applying the Dominated Convergence Theorem, we obtain the following estimate:

$$\begin{aligned}(b.2) &= \left|\int_R uh(u)e^{-iuy}\chi_{[0,A)}\,d\mu(u) - \int_R uv^{-1}h\left(uv^{-1}\right)e^{-iyuv^{-1}}\frac{\sin u}{u}v^{-1}\chi_{[0,vA]}d\mu(u)\right| \\ &= \left|\int_R uh(u)e^{-iuy}(1-\frac{\sin uv}{uv})\chi_{[0,A)}\,d\mu(u)\right| \\ &\leq \int_R \left|Ah(u)(1-\frac{\sin uv}{uv})\right|\chi_{[0,A)}\,d\mu(u) \longrightarrow 0 \text{ as } v \longrightarrow 0^+ \end{aligned} \tag{3.29}$$

For $(b.3)$, the approach is analogous to that of (a.2).We can obtain that expression $(b.3)$ becomes arbitrarily small as long as v is sufficiently small, that is, there exists v_0 such that for any $0 < v < v_0$, we have

$$\left|\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\,(\sin u)\,v^{-2}\chi_{[vA,1]}\,d\mu(u)\right| \leq \epsilon \tag{3.30}$$

By combining (3.26), (3.28), (3.29), and (3.30), we obtain the following:

$$\lim_{v\to 0^+}\int_R h\left(uv^{-1}\right)e^{-iyuv^{-1}}\left(\sin^2\frac{u}{2}+i\sin u\right)v^{-2}\chi_{[0,1]}\,d\mu(u) = \int_R iuh(u)e^{-iuy}\chi_{[0,\infty)}\,d\mu(u).$$

which completes our proof.

□

4. Main theorems and their proofs

Theorem 4.1. *Let $\xi > 1$ and define $n_{\max} = \lfloor \xi \rfloor$. For any function $h \in \mathcal{H}(\xi)$, the following results hold: For every integer $n \in N^+$ with $1 \leq n \leq n_{\max}$, the n-th derivative of $\mathcal{F}[h]$ satisfies: $\frac{d^n}{dy^n}\mathcal{F}[h] \in L^\infty(R)$. The n-th derivative of h admits the following Fourier transform representation: $\frac{d^n}{dy^n}\mathcal{F}[h](y) = \int_R (-i)^n s^n h(s)e^{-isy}d\mu(s)$.*

Proof. since $h \in \mathcal{H}(\xi)$, we have $\exists L_1 > 0, \forall y \in R, |\phi(y,s)| \leq L_1|s|^\xi$,where $\phi(y,s) := \int_R s^{-1}h\left(ts^{-1}\right)e^{-its^{-1}y}\chi_{[1,\infty)}d\mu(t)$. By performing the variable substitution $z = ts^{-1}$, we obtain:

$$\begin{aligned}&\left|\int_R s^{-1}h\left(ts^{-1}\right)e^{-its^{-1}y}\chi_{[1,\infty)}d\mu(t)\right| \\ =&\left|\int_R h(z)e^{-iyz}\chi_{[s^{-1},\infty]}d\mu(z)\right| \\ &\leq L_1|s|^\xi = L_1\left|s^{-1}\right|^{-\xi}\end{aligned} \tag{4.1}$$

It follows from Lemma 3.1 that there exists a constant $C > 0$ such that:

$$\left|\int_R z^{\xi+1}h(z)e^{-iyz}\chi_{[0,s^{-1}]}d\mu(z)\right| \leq C\left|s^{-1}\right|^1 \tag{4.2}$$

For any n $(1 \leq n \leq n_{\max})$, by applying recursion to equation (4.2), we obtain:

$$\left|\int_R z^{\xi+n}h(z)e^{-iyz}\chi_{[0,s^{-1}]}d\mu(z)\right| \leq C\left|s^{-1}\right|^n \tag{4.3}$$

By applying Lemma 3.1 once again, we have that there exists a constant $\hat{C} > 0$ such that:

$$\begin{aligned} &\left|\int_R z^n h(z)e^{-iyz}\chi_{[s^{-1},\infty]}d\mu(z)\right| \\ =&\left|\int_R \left(us^{-1}\right)^n h\left(us^{-1}\right)e^{-iyus^{-1}}s^{-1}\chi_{[1,\infty]}d\mu(u)\right| \le \hat{C}\left|s^{-1}\right|^{n-\xi} = \hat{C}\left|s\right|^{\xi-n} \end{aligned} \tag{4.4}$$

(4.4) implies that $s^n h(s) \in \mathcal{H}(\xi - n)$.

Define $R_n(y) = \int_R (-i)^n s^n h(s)e^{-isy}\, d\mu(s)$. Since $s^n h(s) \in \mathcal{H}(\xi - n)$, by Lemma 3.3, we conclude that $R_n(y)$ exists for every $y \in R$ and $R_n(y) \in L^\infty(R)$.

Next, we will prove that $R_n(y) = \frac{d^n}{dy^n}\mathcal{F}[h](y)$, and we will use induction to establish the result. Firstly, notice that $h \in \mathcal{H}(\xi)$. From Lemma 3.3, we have that $\mathcal{F}[h](y)$ is well-defined for every $y \in R$.Without loss of generality, let $\Delta y > 0$. We will first estimate the following difference quotient limit:

$$\begin{aligned} &\frac{\mathcal{F}[h](y+\Delta y) - \mathcal{F}[h](y)}{\Delta y} \\ =&\frac{1}{\Delta y}\left(\int_R h(s)e^{-is(y+\Delta y)}d\mu(s) - \int_R h(s)e^{-isy}d\mu(s)\right) \\ =&\frac{1}{\Delta y}\left(\int_R h(s)e^{-isy}\left(e^{-is\Delta y} - 1\right)d\mu(s)\right) \\ =&\frac{1}{\Delta y}\int_{(-\infty,-(\Delta y)^{-1})\cup(-(\Delta y)^{-1},0)\cup(0,(\Delta y)^{-1})\cup((\Delta y)^{-1},\infty)} h(s)e^{-isy}\left(e^{-is\Delta y} - 1\right)d\mu(s) \end{aligned} \tag{4.5}$$

We will separately estimate the integrals over the four intervals: $(-\infty, -(\Delta y)^{-1})$, $(-(\Delta y)^{-1}, 0)$, $(0, (\Delta y)^{-1})$, and $((\Delta y)^{-1}, \infty)$.

For the interval $((\Delta y)^{-1}, \infty)$, since $h \in \mathcal{H}_1(\xi)$ and $\xi > 1$, we have:

$$\begin{aligned} &\left|\frac{1}{\Delta y}\int_{((\Delta y)^{-1},\infty)} h(s)e^{-is(y+\Delta y)}\, d\mu(s)\right| \\ =&\left|\frac{1}{\Delta y}\int_R (\Delta y)^{-1}h\left(u(\Delta y)^{-1}\right)e^{-iu(\Delta y)^{-1}(y+\Delta y)}\chi_{[1,\infty)}d\mu(u)\right| \\ =&\left|\frac{1}{\Delta y}\right|\left|\phi(y+\Delta y, \Delta y)\right| \leqslant L_1|\Delta y|^{\xi-1} \longrightarrow 0 \text{ as } \quad \Delta y \to 0^+ \end{aligned} \tag{4.6}$$

For the interval $(-\infty, -(\Delta y)^{-1})$, since $h \in \mathcal{H}_2(\xi)$ and $\xi > 1$, we have a similar result as in (4.6).

For the interval $(0, (\Delta y)^{-1})$,We first note that $\left|\int_R uh(u)e^{-iuy}\chi_{[0,\infty)}\, d\mu(u)\right| \le C(y)$. This is because, in (4.4), by setting $n = 1$, we obtain:

$$\left|\int_R z^1 h(z)e^{-iyz}\chi_{[s^{-1},\infty]}\, d\mu(z)\right| \le \hat{C}\left|s\right|^{\xi-1} \tag{4.7}$$

which implies that the remainder term of the integral tends to zero as $s \to 0$, thereby indicating that the integral converges.On the other hand, we only need to estimate $\frac{1}{\Delta y}\int_{(0,(\Delta y)^{-1})} h(s)e^{-isy}$

$\left(e^{-is\Delta y}-1\right)d\mu(s)$; the estimate for $\frac{1}{\Delta y}\int_{(-(\Delta y)^{-1},0)} h(s)e^{-isy}\left(e^{-is\Delta y}-1\right)d\mu(s)$ is obtained using the same background and techniques as that for $\frac{1}{\Delta y}\int_{(0,(\Delta y)^{-1})} h(s)e^{-isy}\left(e^{-is\Delta y}-1\right)d\mu(s)$.

$$\begin{aligned}
&\frac{1}{\Delta y}\int_{(0,(\Delta y)^{-1})} h(s)e^{-isy}(e^{-is\Delta y}-1)d\mu(s)\\
=&\frac{1}{\Delta y}\int_{(0,(\Delta y)^{-1})} h(s)e^{-isy}(\cos(-s\Delta y)+i\sin(-s\Delta y)-1)d\mu(s)\\
=&\frac{1}{\Delta y}\int_{(0,(\Delta y)^{-1})} -h(s)e^{-isy}\left(\frac{\sin^2\left(\frac{s\Delta y}{2}\right)}{2}+i\sin s\Delta y\right)d\mu(s)\\
=&-\int_R h\left(u(\Delta y)^{-1}\right)e^{-iyu(\Delta y)^{-1}}\left(\frac{\sin^2\frac{u}{2}}{2}+i\sin u\right)(\Delta y)^{-2}\chi_{[0,1]}d\mu(u)\\
\longrightarrow&-\int_R iuh(u)e^{-iuy}\chi_{[0,\infty)}\,d\mu(u)\quad\text{as } \Delta y\to 0^+ \qquad\qquad \text{since (4.7) and Lemma 3.5}
\end{aligned}$$

Thus we have:

$$\lim_{\Delta y\to 0}\frac{\mathcal{F}[h](y+\Delta y)-\mathcal{F}[h](y)}{\Delta y}=-\int_R iuh(u)e^{-iuy}\,d\mu(u)=R_n(y)\Big|_{n=1}$$

We have already established that equation $\frac{d^n}{dy^n}\mathcal{F}[h](y)=\int_R(-i)^n s^n h(s)e^{-isy}d\mu(s)$ holds for $n=1$. Now, suppose that equation $\frac{d^n}{dy^n}\mathcal{F}[h](y)=\int_R(-i)^n s^n h(s)e^{-isy}d\mu(s)$ also holds for $n=k-1$; that is, $\frac{d^{k-1}}{dy^{k-1}}\mathcal{F}[h](y)=R_n(y)\Big|_{n=k-1}$, We will now prove that this also holds for $n=k$.

$$\begin{aligned}
&\frac{1}{\Delta y}\left(\frac{d^{k-1}}{dy^{k-1}}\mathcal{F}[h](y+\Delta y)-\frac{d^{k-1}}{dy^{k-1}}\mathcal{F}[h](y)\right)\\
&=\frac{1}{\Delta y}\left(R_{k-1}(y+\Delta y)-R_{k-1}(y)\right) \qquad (4.8)\\
&=\frac{1}{\Delta y}\left(\int_R(-i)^{k-1}s^{k-1}h(s)e^{-is(y+\Delta y)}d\mu(s)-\int_R(-i)^{k-1}s^{k-1}h(s)e^{-isy}d\mu(s)\right)
\end{aligned}$$

According to equation (4.8), to calculate the value of $\frac{d^k}{dy^k}\mathcal{F}[h](y)$, it is sufficient to compute $(-i)^{k-1}\cdot\frac{d}{dy}\mathcal{F}[s^{k-1}h(s)](y)$.

From (4.4), we know that $s^{k-1}h(s)\in\mathcal{H}(\xi-k+1)$. Moreover, we also know the following fact: if a function $h\in\mathcal{H}(\xi)$ and $\xi>1$, then $\frac{d}{dy}\mathcal{F}[h](y)\in L^\infty(R)$, and

$$\frac{d}{dy}\mathcal{F}[h](y)=\int_R(-i)^1 s^1 h(s)e^{-isy}\,d\mu(s).$$

Thus, we have:

$$\begin{aligned}
\frac{d^k}{dy^k}\mathcal{F}[h](y)&=(-i)^{k-1}\cdot\frac{d}{dy}\mathcal{F}[s^{k-1}h(s)](y)\\
&=(-i)^{k-1}\int_R(-i)^1 s^1 s^{k-1}h(s)e^{-isy}d\mu(s)\\
&=(-i)^k\int_R s^k h(s)e^{-isy}d\mu(s)
\end{aligned}$$

$$= R_n(y)\Big|_{n=k}$$

This completes the proof of the theorem. □

Theorem 4.2. *Suppose a real number $\xi > 0$. If the function h belongs to the class $\mathcal{H}(\xi)$, then its Fourier transform $\mathcal{F}[h]$ satisfies the following properties: $\mathcal{F}[h] \in L^\infty(R)$and $\mathcal{F}[h]$ have the property of Hölder continuous of order ξ. In other words, when the asymptotic decay rate of h is controlled by the $\mathcal{H}(\xi)$ class, its Fourier transform is not only globally bounded, but also possesses Hölder-type smoothness of order ξ .*

Proof. For a positive real number ξ, the set of all integers not exceeding ξ is given by:

$$n_\xi = \{n \in Z^+ \mid n \leq \xi\}$$

The integer part of ξ is $\lfloor \xi \rfloor$, and the fractional part of ξ is $\{\xi\} := \xi - \lfloor \xi \rfloor$. Based on Definition 2.6, to prove that the Fourier transform of $h \in \mathcal{H}(\xi)$ possesses the ξ-Hölder continuous property property, we need to establish two things:

(I) $\frac{d^k}{dy^k}\mathcal{F}[h](y) \in L^\infty(R) \quad \forall\, k \in n_\xi$.

(II) There exists a constant $C > 0$ such that

$$\left|\frac{d^k}{dy^k}\mathcal{F}[h](y) - \frac{d^k}{dy^k}\mathcal{F}[h](x)\right| \leq C\left|x - y\right|^{\{\xi\}},$$

where $k = \max\{n \in Z^+ \mid n \leq \xi\}$.

We will prove this in two cases: one where $\xi > 1$, and the other where $0 < \xi < 1$. For the case $0 < \xi < 1$, we have already proven it in Lemma 3.4. Therefore, we only need to prove the case $\xi > 1$. Note that $\mathcal{F}[h](y)$ is well defined for every $y \in R$ since Lemma 3.3. At the same time, Theorem 4.1 tells us that for any $k \in n_\xi$, the existence of $\frac{d^k}{dy^k}\mathcal{F}[h](y)$ and $\frac{d^k}{dy^k}\mathcal{F}[h] \in L^\infty(R)$ satisfy the requirement of condition (I). Next, we only need to prove whether $\frac{d^{\lfloor\xi\rfloor}}{dy^{\lfloor\xi\rfloor}}\mathcal{F}[h](y)$ satisfies condition (II). From (4.4), we know that $s^k h(s) \in \mathcal{H}(\xi - k)$ for $k \in n_\xi$.Specifically, we have $s^{\lfloor\xi\rfloor}h(s) \in \mathcal{H}(\{\xi\})$. From Lemma 3.4,we obtain that $\mathcal{F}[s^{\lfloor\xi\rfloor}h(s)](y)$ have the property of Hölder continuous of order $\{\xi\}$.

$$\begin{aligned} &\left|\int_R s^{\lfloor\xi\rfloor}h(s)e^{-isy}d\mu(s) - \int_R s^{\lfloor\xi\rfloor}h(s)e^{-isx}d\mu(s)\right| \\ =&\left|\frac{d^{\lfloor\xi\rfloor}}{dy^{\lfloor\xi\rfloor}}\mathcal{F}[h](x) - \frac{d^{\lfloor\xi\rfloor}}{dy^{\lfloor\xi\rfloor}}\mathcal{F}[h](y)\right| \qquad \text{(since Theorem 4.1)} \\ \leq& C\left|x - y\right|^{\{\xi\}} \end{aligned}$$

This completes the requirement of condition (II). □

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