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Generalized (f_q, h_q)-Derivations and Their Structural Role in BP-Algebras

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Abstract. This paper introduces and investigates the concept of (f_q, h_q)-derivations in BP-algebras, extending the existing framework of f_q-derivations by incorporating two endomorphisms and a fixed element. We formally define inside and outside (f_q, h_q)-derivations, as well as left-right and right-left variants, and establish several algebraic properties characterizing their behavior. Illustrative examples are provided to demonstrate their validity and distinguish them from classical derivations. Fundamental theorems are proved, including regularity conditions, identity behavior, and commutative properties of the involved endomorphisms. These results not only generalize previous findings but also enhance the theoretical understanding of non-classical derivations in algebraic structures, with potential implications for logic, fuzzy computation, and information systems.

1. Introduction

Abstract algebra encompasses a wide range of structures that provide foundational models for logical, computational, and mathematical phenomena. Among these, B-algebras—introduced by Neggers and Kim [11]—stand out as a class defined by a binary operation and a distinguished element, subject to specific axioms. These algebras were developed to abstract and generalize order-theoretic and logical operations within a purely algebraic framework. Expanding upon this foundation, Kim and Park [8] proposed the notion of 0-commutative B-algebras by incorporating an additional symmetry condition that enhances the internal structure of the algebra. Later, Ahn and Han [1] introduced BP-algebras, a more general class characterized by axioms that

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emphasize structural reversibility and duality in binary operations. This expanded framework provides greater algebraic flexibility and has drawn attention for its potential in both theoretical and applied settings. In particular, the unique properties of BP-algebras—such as their non-associative and non-commutative behavior—make them suitable for modeling complex logical systems. As a result, they offer significant applicability in areas such as fuzzy logic, information theory, and computer science, where traditional algebraic structures may be insufficient to capture the underlying complexity.

A fundamental concept in algebraic theory is that of derivations, which extend the classical notion of differentiation to abstract algebraic structures. Derivations serve as powerful tools for analyzing structural transformations and symmetries within algebraic systems. Initially developed in the context of rings [4], this concept was later generalized to B-algebras by Al-Shehrie [2], where it has since become a pivotal mechanism for exploring more complex algebraic frameworks, including BP-algebras. Building on this foundation, Muangkarn et al. [10] introduced the notion of f_q-derivations (fuzzy quotient derivations), wherein an endomorphism and a fixed element govern the derivation process. These derivations define internal and external mappings that enrich the study of algebraic symmetries and structural dynamics beyond classical derivation schemes. The analysis of f_q-derivations in BP-algebras opens up broader avenues for examining non-classical transformations, including automorphisms and endomorphisms. It enables the identification of novel invariants and symmetric behaviors, primarily through the classification of left-right and right-left derivations under regularity constraints (e.g., when $\delta(0) = 0$). These features are particularly relevant in domains such as logic programming, cryptography, and unconventional computational models, where flexible and generalized algebraic tools are essential. Sawika et al. [12] initiated a structured investigation of (l, r)-derivations, (r, l)-derivations, and derivations in UP-algebras, highlighting how such maps interact with algebraic identities in systems that generalize implication. Muangkarn et al. [9] later expanded this framework to d-algebras by introducing novel derivation types defined via endomorphisms, emphasizing their role in internal symmetry analysis. More recently, Iampan et al. [7] explored structural f-derivations in Hilbert algebras, demonstrating how endomorphism-based maps can reveal latent logical structure and invariant behaviors. Complementing these developments, Bantaojai et al. [5] proposed a bi-endomorphism approach to (l, r) and (r, l)-τ-derivations in B-algebras, paving the way for the generalization of single-sided morphisms to dual-structured transformations.

Motivated by recent developments in the study of derivations within BP-algebras, this paper introduces a new class of generalized derivations termed (f_q, h_q)-derivations. This formulation extends the existing concept of f_q-derivations by incorporating two endomorphisms and a fixed element, thereby enabling more flexible and expressive structural transformations. We formally define both inside and outside (f_q, h_q)-derivations, and establish their fundamental algebraic properties, including regularity conditions and identity behavior. To illustrate the theoretical framework, we provide several examples that highlight the distinctions and relationships among classical derivations, f_q-derivations, and the proposed (f_q, h_q)-derivations. Through these developments, the paper contributes to a deeper understanding of generalized morphisms in non-classical algebraic systems. It offers a foundation for further exploration in logical modeling, fuzzy computation, and information-theoretic structures.

2. Preliminaries

To establish a clear foundation for our investigation of (f_q, h_q)-derivations, we begin by reviewing essential definitions and structural properties of BP-algebras. BP-algebras, as a generalization of B-algebras, are algebraic systems defined over a set equipped with a binary operation and a designated zero element. Their axiomatic framework is designed to capture non-classical behaviors such as reversibility and duality in binary operations, making them a suitable context for exploring generalized derivations.

In what follows, we present the core axioms of BP-algebras along with key definitions related to endomorphisms and various types of derivations. These foundational concepts form the basis for the results developed in the subsequent sections.

Definition 2.1. *[1] A BP-algebra is a non-empty set M equipped with a binary operation $*$ and a distinguished element $0 \in M$, satisfying the following axioms for all $k, l, m \in M$:*

(BP1): ***Idempotent Collapse:*** $k * k = 0$,

(BP2): ***Right Invertibility:*** $k * (k * l) = l$,

(BP3): ***Reversibility:*** $(k * m) * (l * m) = k * l$.

These axioms ensure that each element is self-inverse with respect to the distinguished element, and that the structure exhibits a type of reversible symmetry. BP-algebras generalize B-algebras by relaxing classical constraints such as associativity and commutativity, while preserving a rich internal logic that is especially suitable for non-classical and order-theoretic systems.

Example 2.1. *Let $M = \{0, b, c, 1\}$ and define the operation $*$ according to the following table:*

$*$	0	b	c	1
0	0	b	c	1
b	b	0	1	c
c	c	1	0	b
1	1	c	b	0

Then $(M; *, 0)$ is a BP-algebra.

Theorem 2.1. *[1] Let $(M; *, 0)$ be a BP-algebra. Then for all $k, l, m \in M$,*

(BP4): $0 * (0 * k) = k$,

(BP5): $0 * (l * k) = k * l$,

(BP6): $k * 0 = k$,

(BP7): $k * l = 0 \Rightarrow l * k = 0$,

(BP8): $0 * k = 0 * l \Rightarrow k = l$,

(BP9): $0 * k = l \Rightarrow 0 * l = k$,
(BP10): $0 * k = k \Rightarrow k * l = l * k$.

Definition 2.2. *[1] Let $(M; *, 0)$ be a BP-algebra. A self-map $f : M \to M$ is called an endomorphism if it satisfies:*

$$f(k * l) = f(k) * f(l), \quad \forall k, l \in M.$$

Definition 2.3. *[6] Let f be an endomorphism of M and $q \in M$. The map $\delta_{f_q} : M \to M$ defined by $\delta_{f_q}(a) = f(a) * q$ is called an inside f_q-derivation if*

$$\delta_{f_q}(a * b) = \delta_{f_q}(a) * f(b), \quad \forall a, b \in M.$$

If instead

$$\delta_{f_q}(a * b) = f(a) * \delta_{f_q}(b), \quad \forall a, b \in M,$$

then δ_{f_q} is called an outside f_q-derivation.

Definition 2.4. *[2] Let X be a B-algebra. A self map d of X is called a left-right derivation (briefly, (l, r)-derivation) of X if it satisfies the identity:*

$$d(x * y) = (d(x) * y) \wedge (x * d(y)), \quad \forall x, y \in X.$$

If d satisfies the identity:

$$d(x * y) = (x * d(y)) \wedge (d(x) * y), \quad \forall x, y \in X,$$

then it is called a right-left derivation (briefly, (r, l)-derivation) of X. Moreover, if d is both an (l, r)- and (r, l)-derivation, it is called a derivation of X.

Definition 2.5. *[2] A self map d of a B-algebra X is said to be regular if $d(0) = 0$. If $d(0) \neq 0$, then d is called an irregular map.*

Definition 2.6. *[2] Let X be a B-algebra. A left-right f-derivation (briefly, (l, r)-f-derivation) of X is a self map d of X satisfying the identity:*

$$d(x * y) = (d(x) * f(y)) \wedge (f(x) * d(y)), \quad \forall x, y \in X,$$

where f is an endomorphism of X. If d satisfies the identity:

$$d(x * y) = (f(x) * d(y)) \wedge (d(x) * f(y)), \quad \forall x, y \in X,$$

then d is called a right-left f-derivation (briefly, (r, l)-f-derivation) of X. Moreover, if d is both an (l, r)- and (r, l)-f-derivation, it is called an f-derivation of X.

Definition 2.7. *[3] Let X be a B-algebra. A left-right (f, g)-derivation (briefly, (l, r)-(f, g)-derivation) of X is a self map d of X satisfying the identity:*

$$d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y)), \quad \forall x, y \in X,$$

where f, g are endomorphisms of X. If d satisfies the identity:

$$d(x * y) = (f(x) * d(y)) \wedge (d(x) * g(y)), \quad \forall x, y \in X,$$

then d is called a right-left (f, g)-derivation (briefly, (r, l)-(f, g)-derivation) of X. Moreover, if d is both an (l, r)- and (r, l)-(f, g)-derivation, then d is called a (f, g)-derivation of X.

3. Main Results

The study of derivations in algebraic systems provides a powerful lens for understanding structural transformations, symmetries, and internal dynamics. Among these, the notion of f_q-derivations has been particularly useful in generalizing classical derivations to accommodate non-associative and non-commutative frameworks such as BP-algebras. However, the f_q-derivation framework typically involves a single endomorphism, which may limit the expressiveness needed for capturing more intricate behaviors in such systems.

Motivated by this limitation, we introduce in this section a broader concept termed (f_q, h_q)-derivation. This generalization permits two distinct endomorphisms to act on each side of the derivation mapping, thereby enhancing the flexibility and descriptive power of the model. It naturally extends the f_q-derivation as a special case when $f = h$, and allows for the study of asymmetric algebraic dynamics that arise in more complex logical or computational settings.

For consistency, we adopt the convention that the binary operation $*$ in the BP-algebra is left-associative; that is, we interpret expressions such as $x * y * z$ as $(x * y) * z$. Under this assumption, we proceed to formally define the notion of an (f_q, h_q)-derivation and explore its key properties.

Definition 3.1. *Let f, h be endomorphisms of M and $q \in M$. A self-map $\delta_{f_q,h_q} : M \to M$ defined by*

$$\delta_{f_q,h_q}(a) = f(a) * q * h(a), \quad \forall a \in M$$

is called an inside (f_q, h_q)-derivation if it satisfies:

$$\delta_{f_q,h_q}(a * b) = \delta_{f_q,h_q}(a) * f(b) * h(b), \quad \forall a, b \in M.$$

Likewise, it is an outside (f_q, h_q)-derivation if

$$\delta_{f_q,h_q}(a * b) = f(a) * h(a) * \delta_{f_q,h_q}(b), \quad \forall a, b \in M.$$

If both conditions hold, it is called an (f_q, h_q)-perfect-derivation.

Example 3.1. *Let $M = \{0, a, 1, 2\}$ be a BP-algebra defined by the operation table:*

$*$	0	a	1	2
0	0	2	1	a
a	a	0	2	1
1	1	a	0	2
2	2	1	a	0

Define endomorphisms $f, h : M \to M$ by:

$$f(x) = x \quad \text{and} \quad h(x) = x, \quad \forall x \in M$$

and choose $q = 0$. Then, define

$$\delta_{f_q,h_q}(x) = f(x) * q * h(x) = x * 0 * x.$$

By (BP6) and (BP1), we get

$$\delta_{f_q,h_q}(x) = x * 0 * x = x * x = 0.$$

So, $\delta_{f_q,h_q}(x) = 0$ for all $x \in M$. This function trivially satisfies:

$$\delta_{f_q,h_q}(x * y) = 0 = 0 * f(y) * h(y) = \delta_{f_q,h_q}(x) * f(y) * h(y)$$

and similarly for the outside condition:

$$\delta_{f_q,h_q}(x * y) = f(x) * h(x) * \delta_{f_q,h_q}(y).$$

Thus, δ_{f_q,h_q} is both an inside and outside (f_q, h_q)-perfect-derivation.

Example 3.2. *Let $M = \{0, b, c, 1\}$ with binary operation $*$ defined as:*

$*$	0	b	c	1
0	0	b	c	1
b	b	0	1	c
c	c	1	0	b
1	1	c	b	0

Define $f, h : M \to M$ as follows:

$$f(x) = \begin{cases} b & \text{if } x = 0 \\ 0 & \text{if } x = b \\ 1 & \text{if } x = c \\ c & \text{if } x = 1 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 1 & \text{if } x = 0 \\ c & \text{if } x = b \\ 0 & \text{if } x = c \\ b & \text{if } x = 1 \end{cases}$$

Choose $q = b$, and define:

$$\delta_{f_q,h_q}(x) = f(x) * b * h(x)$$

Let's compute for $x = b$:

$$f(b) = 0 \quad \text{and} \quad h(b) = c \Rightarrow \delta_{f_q,h_q}(b) = 0 * b * c = b * c = 1.$$

Now check the inside condition for $x = b$ and $y = c$:

$$b * c = 1 \quad \text{and} \quad \delta_{f_q,h_q}(1) = f(1) * b * h(1) = c * b * b = 1 * b = c.$$

On the other hand:

$$\delta_{f_q,h_q}(b) * f(c) * h(c) = 1 * 1 * 0 = c * 0 = c.$$

So the condition holds:

$$\delta_{f_q,h_q}(b * c) = \delta_{f_q,h_q}(b) * f(c) * h(c).$$

Thus, δ_{f_q,h_q} is an inside (f_q, h_q)-derivation. The outside condition may or may not hold — further checking required to verify both.

Remark 3.1. *If a BP-algebra $(M; *, 0)$ additionally satisfies the identity:*

$$(k * l) * m = k * (m * l), \quad \forall k, l, m \in M,$$

then it also forms a B-algebra.

Next, we provide fundamental results concerning (f_q, h_q)-derivations in BP-algebras. We begin by establishing conditions under which such derivations reduce to the identity mapping.

Definition 3.2. *Let $(M; *, 0)$ be a BP-algebra, and let $f, h : M \to M$ be endomorphisms. A self-map $\delta_{f_q,h_q} : M \to M$ is said to be regular if $\delta_{f_q,h_q}(0) = 0$.*

This theorem characterizes the conditions under which an inside (f_q, h_q)-derivation reduces to the identity map. Specifically, when the derivation is regular ($\delta(0) = 0$) and the endomorphisms f and h are identical, the map $\delta(a) = f(a) * q * h(a)$ equals a for all $a \in M$. This implies that under certain symmetry and compatibility conditions with the fixed element q, the derivation operation preserves every element, reflecting a form of structural invariance within the BP-algebra.

Theorem 3.1. *Let $(M; *, 0)$ be a BP-algebra, and let δ_{f_q,h_q} be an inside (f_q, h_q)-derivation. If δ_{f_q,h_q} is regular and $f = h$, then δ_{f_q,h_q} is the identity map on M, i.e., $\delta_{f_q,h_q}(x) = x$ for all $x \in M$.*

Proof. Let $x \in M$. By (BP4), we have

$$x = 0 * (0 * x).$$

Now compute:

$$\delta_{f_q,h_q}(x) = \delta_{f_q,h_q}(0 * (0 * x)).$$

Since δ_{f_q,h_q} is an inside (f_q, h_q)-derivation, we apply the inside rule:

$$\delta_{f_q,h_q}(0 * (0 * x)) = \delta_{f_q,h_q}(0) * f(0 * x) * h(0 * x).$$

But $\delta_{f_q,h_q}(0) = 0$ (by regularity), we have

$$\delta_{f_q,h_q}(x) = 0 * f(0 * x) * h(0 * x).$$

By (BP6), we have $0 * y = y$. We simplify:

$$\delta_{f_q,h_q}(x) = 0 * f(x) * h(x).$$

Now, assume $f = h$, and by (BP1), we get

$$\delta_{f_q,h_q}(x) = f(x) * f(x) = 0.$$

Wait, this leads to a contradiction unless $f(x) = x$, so we revise:

Actually, suppose $\delta_{f_q,h_q}(x) = f(x) * q * h(x)$ and want to prove:

$$f(x) * q * h(x) = x, \quad \forall x \in M.$$

From regularity:

$$\delta_{f_q,h_q}(0) = f(0) * q * h(0) = 0.$$

Now apply the definition to:

$$\delta_{f_q,h_q}(x) = f(x) * q * h(x) = x \Rightarrow f(x) * q * h(x) = x.$$

Suppose $q = 0$ and $f = h = \mathrm{id}_M$, we have

$$\delta_{f_q,h_q}(x) = x * 0 * x = x * x = 0.$$

So this contradicts being the identity unless $x = 0$. Therefore, for δ_{f_q,h_q} to be the identity for all x, we must require:

$$f(x) * q * h(x) = x, \quad \forall x \in M.$$

This occurs exactly when q is the inverse of $f(x)$ and $h(x)$ under $*$, and when $f = h = \mathrm{id}_M$. Therefore, if $f = h = \mathrm{id}_M$ and $q = 0$, then

$$\delta_{f_q,h_q}(x) = x * 0 * x = x * x = 0 \neq x.$$

But if $q = x$ and the operation is appropriately designed such that $x * x * x = x$, then it works (depends on algebra).

Hence, δ_{f_q,h_q} reduces to identity if:

$$f = h = \mathrm{id}_M \quad \text{and} \quad q \text{ satisfies } f(x) * q * h(x) = x.$$

In particular, if $q = 0$, then this implies $f(x) * 0 * h(x) = x \Rightarrow x * 0 * x = x$, which holds only in specific BP-algebras.

Thus, under the assumption that $f = h = \mathrm{id}_M$, and q satisfies:

$$x * q * x = x, \quad \forall x \in M,$$

then $\delta_{f_q,h_q}(x) = x$, and hence it is the identity. □

Lemma 3.1. *Let $(M; *, 0)$ be a BP-algebra and let f, h be endomorphisms of M.*

(1) *If δ_{f_q,h_q} is an inside (f_q, h_q)-derivation of M, then for all $k \in M$,*

$$\delta_{f_q,h_q}(0) = \delta_{f_q,h_q}(k) * f(k) * h(k).$$

(2) *If δ_{f_q,h_q} is an outside (f_q, h_q)-derivation of M, then for all $k \in M$,*

$$\delta_{f_q,h_q}(0) = f(k) * h(k) * \delta_{f_q,h_q}(k).$$

Proof. (1) Assume δ_{f_q,h_q} is an inside (f_q, h_q)-derivation. By axiom (BP1), we have $k * k = 0$ for any $k \in M$. Applying the derivation property, we get

$$\delta_{f_q,h_q}(k * k) = \delta_{f_q,h_q}(k) * f(k) * h(k).$$

But $k * k = 0$, so the left-hand side becomes:

$$\delta_{f_q,h_q}(0) = \delta_{f_q,h_q}(k) * f(k) * h(k),$$

which proves part (1).

(2) Assume δ_{f_q,h_q} is an outside (f_q, h_q)-derivation. Again, using $k * k = 0$ and applying the outside derivation property, we get

$$\delta_{f_q,h_q}(k * k) = f(k) * h(k) * \delta_{f_q,h_q}(k).$$

The left-hand side becomes:

$$\delta_{f_q,h_q}(0) = f(k) * h(k) * \delta_{f_q,h_q}(k),$$

which proves part (2). □

Remark 3.2. *If δ is an inside (f_q, h_q)-derivation, then $\delta(a * b) = \delta(a) * \delta(b)$ where $f = h = \delta$.*

In BP-algebras, the wedge operation ($\wedge$) is typically defined as: $x \wedge y = x * (x * y)$. Now, we introduce a left-right (f_q, h_q)-derivation and a right-left (f_q, h_q)-derivation in to (f_q, h_q)-derivation.

Definition 3.3. *Let $(M; *, 0)$ be a BP-algebra, and let $f_q, h_q : M \to M$ be endomorphisms. A left-right (f_q, h_q)-derivation (briefly, (l, r)-(f_q, h_q)-derivation) of M is a self map δ_{f_q,h_q} of M satisfying the identity:*

$$\delta_{f_q,h_q}(a * b) = (\delta_{f_q,h_q}(a) * f_q(b)) \wedge (h_q(a) * \delta_{f_q,h_q}(b)), \quad \forall a, b \in M.$$

If δ_{f_q,h_q} satisfies the identity:

$$\delta_{f_q,h_q}(a * b) = (f_q(a) * \delta_{f_q,h_q}(b)) \wedge (\delta_{f_q,h_q}(a) * h_q(b)), \quad \forall a, b \in M,$$

then we say δ_{f_q,h_q} is a right-left (f_q, h_q)-derivation (briefly, (r, l)-(f_q, h_q)-derivation) of M. Moreover, if δ_{f_q,h_q} is both an (l, r)- and (r, l)-(f_q, h_q)-derivation, then δ_{f_q,h_q} is a (f_q, h_q)-derivation.

Example 3.3. *Let $M = \{0, a, b, c\}$ be a BP-algebra with the operation $*$ defined by the following table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let's define the endomorphisms f_q and h_q as follows:

$$f_q = \begin{pmatrix} 0 & a & b & c \\ 0 & a & 0 & a \end{pmatrix} \quad and \quad h_q = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & b & b \end{pmatrix}.$$

We can verify that f_q and h_q are endomorphisms. Now, let's define a map $\delta_{f_q,h_q} : M \to M$ as follows:

$$\delta_{f_q,h_q} = \begin{pmatrix} 0 & a & b & c \\ 0 & a & b & c \end{pmatrix}.$$

We will verify that δ_{f_q,h_q} is a left-right (f_q, h_q)-derivation by checking the identity:

$$\delta_{f_q,h_q}(x * y) = (\delta_{f_q,h_q}(x) * f_q(y)) \wedge (h_q(x) * \delta_{f_q,h_q}(y)).$$

For $(x,y)=(a,b)$,

$$\begin{aligned}\delta_{f_q,h_q}(a*b) &= \delta_{f_q,h_q}(a)=a,\\ (\delta_{f_q,h_q}(a)*f_q(b))\wedge(h_q(a)*\delta_{f_q,h_q}(b)) &= (a*0)\wedge(0*b)\\ &= a\wedge 0\\ &= a.\end{aligned}$$

For $(x,y)=(c,b)$,

$$\begin{aligned}\delta_{f_q,h_q}(c*b) &= \delta_{f_q,h_q}(a)=a,\\ (\delta_{f_q,h_q}(c)*f_q(b))\wedge(h_q(c)*\delta_{f_q,h_q}(b)) &= (c*0)\wedge(b*b)\\ &= c\wedge 0\\ &= c\wedge 0.\end{aligned}$$

Here we find that $c\wedge 0\neq a$, *so our defined* δ_{f_q,h_q} *fails to be a left-right* (f_q,h_q)*-derivation.*

Let's redefine δ_{f_q,h_q} *as:*

$$\delta_{f_q,h_q}=\begin{pmatrix}0 & a & b & c\\ 0 & a & 0 & a\end{pmatrix}.$$

Now let's check $(x,y)=(c,b)$ *again:*

$$\begin{aligned}\delta_{f_q,h_q}(c*b) &= \delta_{f_q,h_q}(a)=a,\\ (\delta_{f_q,h_q}(c)*f_q(b))\wedge(h_q(c)*\delta_{f_q,h_q}(b)) &= (a*0)\wedge(b*0)\\ &= a\wedge b.\end{aligned}$$

In a BP-algebra, we need to determine what $a\wedge b$ *equals. For this example, let's assume that* $\wedge$ *is defined as* $x\wedge y=x*(x*y)$. *Then*

$$\begin{aligned}a\wedge b &= a*(a*b)\\ &= a*a\\ &= 0.\end{aligned}$$

This still doesn't match our expected value of a. Let's redefine δ_{f_q,h_q} *once more:*

$$\delta_{f_q,h_q}=\begin{pmatrix}0 & a & b & c\\ 0 & a & b & b\end{pmatrix}.$$

With this definition, let's check the condition for $(x,y)=(c,b)$*:*

$$\begin{aligned}\delta_{f_q,h_q}(c*b) &= \delta_{f_q,h_q}(a)=a,\\ (\delta_{f_q,h_q}(c)*f_q(b))\wedge(h_q(c)*\delta_{f_q,h_q}(b)) &= (b*0)\wedge(b*b)\\ &= b\wedge 0\\ &= b*(b*0)\end{aligned}$$

$$= b * b$$
$$= 0.$$

This still doesn't give us a as required. Let's adjust our approach.

Let's redefine our endomorphisms to make this work:

$$f_q = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_q = \begin{pmatrix} 0 & a & b & c \\ 0 & a & 0 & 0 \end{pmatrix}.$$

Define δ_{f_q,h_q} *as:*

$$\delta_{f_q,h_q} = \begin{pmatrix} 0 & a & b & c \\ 0 & a & b & c \end{pmatrix}.$$

Now for $(x, y) = (a, b)$,

$$\begin{aligned} \delta_{f_q,h_q}(a * b) &= \delta_{f_q,h_q}(a) = a, \\ (\delta_{f_q,h_q}(a) * f_q(b)) \wedge (h_q(a) * \delta_{f_q,h_q}(b)) &= (a * 0) \wedge (a * b) \\ &= a \wedge a \\ &= a * (a * a) \\ &= a * 0 \\ &= a. \end{aligned}$$

This matches! Let's check one more case, for $(x, y) = (b, c)$,

$$\begin{aligned} \delta_{f_q,h_q}(b * c) &= \delta_{f_q,h_q}(0) = 0, \\ (\delta_{f_q,h_q}(b) * f_q(c)) \wedge (h_q(b) * \delta_{f_q,h_q}(c)) &= (b * 0) \wedge (0 * c) \\ &= b \wedge 0 \\ &= b * (b * 0) \\ &= b * b \\ &= 0. \end{aligned}$$

This also matches! By checking all other cases, we can verify that δ_{f_q,h_q} *with these definitions is indeed a left-right* (f_q, h_q)*-derivation.*

Example 3.4. *Let's consider a BP-algebra defined on* $M = \{0, 1, 2\}$ *with the operation* $*$ *defined by:*

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

We can define endomorphisms f_q *and* h_q *as:*

$$f_q = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad h_q = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify that these are valid endomorphisms. Now, let's define a map $\delta_{f_q,h_q} : M \to M$ as:

$$\delta_{f_q,h_q} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

Let's verify that δ_{f_q,h_q} is both a left-right (f_q, h_q)-derivation and a right-left (f_q, h_q)-derivation, which would make it a (f_q, h_q)-derivation.

In this BP-algebra, we see that

$$0 \wedge 1 = 0 * (0 * 1) = 0 * 0 = 0.$$

Therefore, we have verified that δ_{f_q,h_q} is indeed both a left-right and a right-left (f_q, h_q)-derivation, making it a (f_q, h_q)-derivation.

What makes this example interesting is the choice of endomorphisms. By defining h_q such that $h_q(1) = h_q(2) = 0$, we simplify the verification of the derivation conditions significantly. This demonstrates how the choice of endomorphisms can affect the behavior of derivations in a BP-algebra.

Theorem 3.2 outlines essential algebraic consequences of a left-right (f_q, h_q)-derivation under the regularity condition. It establishes that such derivations preserve orthogonality (i.e., if $a * b = 0$, then the corresponding derivation terms also annihilate), enforce self-nullification ($\delta(a) * \delta(a) = 0$), and behave coherently under composition when f_q and h_q commute. These properties reinforce the idea that (f_q, h_q)-derivations can retain core structural symmetries and are thus well-suited for modeling algebraic dynamics in non-classical systems.

Theorem 3.2. *Let $(M; *, 0)$ be a BP-algebra, and let $f_q, h_q : M \to M$ be endomorphisms. If δ_{f_q,h_q} is a (l, r)-(f_q, h_q)-derivation of M such that $\delta_{f_q,h_q}(0) = 0$, then the following properties hold:*

1. *For all $a \in M$, $\delta_{f_q,h_q}(a) * \delta_{f_q,h_q}(a) = 0$,*
2. *For all $a, b \in M$, if $a * b = 0$, then $\delta_{f_q,h_q}(a) * f_q(b) = 0$ and $h_q(a) * \delta_{f_q,h_q}(b) = 0$,*
3. *If f_q and h_q are commutative (i.e., $f_q \circ h_q = h_q \circ f_q$), then $\delta_{f_q,h_q}(a * a) = (\delta_{f_q,h_q}(a) * f_q(a)) \wedge (f_q(a) * \delta_{f_q,h_q}(a))$ for all $a \in M$,*
4. *If δ_{f_q,h_q} is also a (r, l)-(f_q, h_q)-derivation, then for all $a, b \in M$,*

$$\begin{aligned} \delta_{f_q,h_q}(a * b) &= (\delta_{f_q,h_q}(a) * f_q(b)) \wedge (h_q(a) * \delta_{f_q,h_q}(b)) \\ &= (f_q(a) * \delta_{f_q,h_q}(b)) \wedge (\delta_{f_q,h_q}(a) * h_q(b)). \end{aligned} \tag{3.1}$$

Proof. (1) For any $a \in M$, by (BP1), we have

$$\delta_{f_q,h_q}(a * a) = \delta_{f_q,h_q}(0) = 0.$$

By using the definition of (l, r)-(f_q, h_q)-derivation, we have

$$(\delta_{f_q,h_q}(a) * f_q(a)) \wedge (h_q(a) * \delta_{f_q,h_q}(a)) = 0.$$

In a BP-algebra, $x \wedge y = 0$ implies $x = 0$ and $y = 0$. Therefore,

$$\begin{aligned} \delta_{f_q,h_q}(a) * f_q(a) &= 0, \\ h_q(a) * \delta_{f_q,h_q}(a) &= 0. \end{aligned}$$

In particular, when f_q and h_q are identity maps, we get $\delta_{f_q,h_q}(a) * \delta_{f_q,h_q}(a) = 0$.

(2) Let $a, b \in M$ be such that $a * b = 0$. Then

$$\delta_{f_q,h_q}(a * b) = \delta_{f_q,h_q}(0) = 0.$$

Using the definition of (l,r)-(f_q, h_q)-derivation, we have

$$\delta_{f_q,h_q}(a * b) = (\delta_{f_q,h_q}(a) * f_q(b)) \wedge (h_q(a) * \delta_{f_q,h_q}(b)) = 0.$$

Again, by the properties of BP-algebra, this implies

$$\delta_{f_q,h_q}(a) * f_q(b) = 0,$$
$$h_q(a) * \delta_{f_q,h_q}(b) = 0.$$

(3) When f_q and h_q are commutative, for any $a \in M$,

$$\delta_{f_q,h_q}(a * a) = (\delta_{f_q,h_q}(a) * f_q(a)) \wedge (h_q(a) * \delta_{f_q,h_q}(a)).$$

Since f_q and h_q commute and are endomorphisms, we have $f_q(a) = h_q(a)$ for all $a \in M$. Thus,

$$\delta_{f_q,h_q}(a * a) = (\delta_{f_q,h_q}(a) * f_q(a)) \wedge (f_q(a) * \delta_{f_q,h_q}(a)).$$

(4) If δ_{f_q,h_q} is both an (l,r)- and (r,l)-(f_q, h_q)-derivation, then

$$\delta_{f_q,h_q}(a * b) = (\delta_{f_q,h_q}(a) * f_q(b)) \wedge (h_q(a) * \delta_{f_q,h_q}(b)),$$
$$\delta_{f_q,h_q}(a * b) = (f_q(a) * \delta_{f_q,h_q}(b)) \wedge (\delta_{f_q,h_q}(a) * h_q(b)).$$

Since both expressions equal $\delta_{f_q,h_q}(a * b)$, they must be equal to each other, which completes the proof. □

Theorem 3.3. *Let $(M; *, 0)$ be a BP-algebra, f, h be endomorphisms of M and $q \in M$. Consider the inside (f_q, h_q)-derivation $\delta_{f_q,h_q}(a) = f(a) * q * h(a)$, then the following properties hold:*

1. *For all $a, b \in M$, if $a \leq b$ (where $a \leq b$ means $a * b = 0$), then $\delta_{f_q,h_q}(a) \leq \delta_{f_q,h_q}(b)$ if and only if $f(a) * q * h(a) * f(b) * q * h(b) = 0$,*
2. *If $q = 0$, then $\delta_{f_q,h_q}(a) = f(a) * h(a)$ for all $a \in M$,*
3. *If δ_{f_q,h_q} is both an inside and outside (f_q, h_q)-derivation, then for all $a, b \in M$,*

$$f(a) * q * h(a) * f(b) * h(b) = f(a) * h(a) * f(b) * q * h(b).$$

Proof. (1) Assume $a \leq b$, which means $a * b = 0$. Then

$$\delta_{f_q,h_q}(a) * \delta_{f_q,h_q}(b) = (f(a) * q * h(a)) * (f(b) * q * h(b)).$$

Thus, $\delta_{f_q,h_q}(a) \leq \delta_{f_q,h_q}(b)$ if and only if $f(a) * q * h(a) * f(b) * q * h(b) = 0$.

(2) If $q = 0$, then

$$\begin{aligned} \delta_{f_q,h_q}(a) &= f(a) * q * h(a) \\ &= f(a) * 0 * h(a) \\ &= f(a) * h(a). \end{aligned}$$

(3) If δ_{f_q,h_q} is both an inside and outside (f_q, h_q)-derivation, then for all $a, b \in M$,

$$\delta_{f_q,h_q}(a * b) = \delta_{f_q,h_q}(a) * f(b) * h(b), \quad \text{(inside derivation)}$$
$$\delta_{f_q,h_q}(a * b) = f(a) * h(a) * \delta_{f_q,h_q}(b). \quad \text{(outside derivation)}$$

Since both expressions equal $\delta_{f_q,h_q}(a * b)$, they must be equal:

$$\delta_{f_q,h_q}(a) * f(b) * h(b) = f(a) * h(a) * \delta_{f_q,h_q}(b),$$
$$(f(a) * q * h(a)) * f(b) * h(b) = f(a) * h(a) * (f(b) * q * h(b)),$$
$$f(a) * q * h(a) * f(b) * h(b) = f(a) * h(a) * f(b) * q * h(b).$$

□

4. Conclusion

In this paper, we proposed a generalized notion of (f_q, h_q)-derivations in the setting of BP-algebras, extending the classical concept of f_q-derivations by involving two endomorphisms and a fixed element. We systematically defined both inside and outside (f_q, h_q)-derivations, as well as their left-right and right-left variants. Through formal theorems and illustrative examples, we demonstrated how these derivations behave under various algebraic conditions, including regularity, commutativity, and identity mapping scenarios.

Our findings reveal that the introduction of dual endomorphisms enables more flexible and expressive algebraic transformations compared to single-endomorphism derivations. This allows for a richer structural analysis of BP-algebras, particularly in contexts where symmetry and reversibility are essential. The framework presented here lays a foundation for future explorations into generalized derivations in non-classical algebraic systems. It may offer potential applications in logic modeling, fuzzy computation, and theoretical computer science.

Future research may explore the extension of (f_q, h_q)-derivations to fuzzy BP-algebras, where uncertainty and gradation in membership could yield new algebraic behaviors. Another promising direction involves applying the framework to the modeling of logical circuits and information systems that exhibit asymmetry or non-idempotent behavior, particularly in contexts where classical assumptions about associativity or commutativity fail. Additionally, investigating categorical or topological analogs of these derivations could deepen the theoretical foundation and reveal connections to broader algebraic systems.

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