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New Fuzzy Algebraic Structure Consists of Homomorphism via Multi-Fuzzy Set Applied to Cubic Vague Subbisemirings Over Bisemirings

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Abstract. We discussed that the concept of multi-fuzzy cubic vague subbisemiring (MFCVSBS) is a novel generalized hybrid structure of vague subbisemiring. The MFCVSBS and level sets using MFCVSBS of bisemirings are discussed. We define some simple operations on them, including intersection and Cartesian product, to discuss some of their basic properties under MFCVSBS. It is assumed that $\mathcal{Z} = \langle \coprod_i \circ \bar{\aleph}_{\mathcal{Z}}, \coprod_i \circ \beth_{\mathcal{Z}} \rangle$ is the multi-fuzzy cubic vague subset of $\Bbbk$. Assuming that any non-empty level set $\mathcal{Z}_{(\zeta,\sigma)}(\zeta, \sigma \in D[0,1])$ is an SBS, it can be demonstrated that $\mathcal{Z}$ is an MFCVSBS. It will be demonstrated that MFCVSBS is both its homomorphic image and pre-image. Examples are given to illustrate our conclusions.

1. Introduction

The subject of fuzzy sets (FS) was first established by Zadeh [10] in 1965. The membership grade (MG) of x belonging to X is $\mu(x)$. An FS on a set X is a mapping μ onto the interval $[0, 1]$ for $x \in X$. The MG does not provide information about not belonging; it just provides an estimate of belonging. Gau et al. (1993) extended the FS to the vague set (VS) [3], which has two MGs, such as true (TMG) and false (FMG). To a certain extent, VSs are more capable than FSs at processing ambiguous data. Typically, human intellect develops gradually. This raises the

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question of how to define a vague term and quantify its uncertainty, an intriguing issue that merits further research. Palanikumar et al. discussed the vague set and their extensions, such as square root vague sets [4] and neutrosophic vague set [7]. Rosenfeld was a pioneer in fuzzy algebraic structures [8]. Biswas [2] discussed the theory of vague algebra, including ambiguous groups, normal groups, and other related concepts. Interval-valued FS and FS, a more generic tool, were used to describe the idea of a cubic set. This is because FS works with a single value, whereas IVFS ranges the MG in the form of intervals. The primary benefit of the cubic set is that it includes both FS and an IVFS. In 1993, Ahsan and colleagues [1] presented the fuzzy semiring theory. Sen and colleagues presented bisemirings research in 2001 [9]. Palanikumar et al. [5,6] discussed the concept of spherical vague sets and their extension. This study aims to further develop the concept of imprecise SBS theory in relation to MFCVSBS theory and explore some of its key characteristics. The article is divided into the following five sections. The introduction is given in Section 1, and the preliminary discussion of vague sets and multi-fuzzy cubic vague sets is covered in Section 2. MFCVSBS is presented along with examples of its features in Section 3. The concept of MFCVSBS homomorphism and its characteristics are presented in Section 4. Section 5 provides a conclusion for more research. Additionally, a few numerical examples are provided to assess the multi-fuzzy cubic ambiguous subbisemirings.

2. Preliminaries

Definition 2.1. *[3] The vague set (VS) $\mathcal{Z}$ in the universe U is $[\top_{\mathcal{Z}}, F_{\mathcal{Z}}]$ it consists of two MG called as TMG $\top_{\mathcal{Z}} : U \to [0,1]$ and FMG $F_{\mathcal{Z}} : U \to [0,1]$, such that $0 \leq \top_{\mathcal{Z}}[\tau] + F_{\mathcal{Z}}[\tau] \leq 1$, for all $\tau \in U$, where $\top_{\mathcal{Z}}[\tau]$ is a lower bound of MG of u derived from the "evidence for u", and $F_{\mathcal{Z}}[\tau]$ is a lower bound of FMG of u derived from the "evidence against τ". The VS $\mathcal{Z}$ as $\mathcal{Z} = \{\tau, [\top_{\mathcal{Z}}[\tau], 1 - F_{\mathcal{Z}}[\tau]] | \tau \in U\}$, the interval $[\top_{\mathcal{Z}}[\tau], 1 - F_{\mathcal{Z}}[\tau]]$ is called the vague value of u in $\mathcal{Z}$. That is $\sqsupset_{\mathcal{Z}}[\tau] = [\top_{\mathcal{Z}}[\tau], 1 - F_{\mathcal{Z}}[\tau]]$.*

Definition 2.2. *[3] The complement of a VS $\mathcal{Z}$ is defined by $\top_{\mathcal{Z}^c} = F_{\mathcal{Z}}$ and $1 - F_{\mathcal{Z}^c} = 1 - \top_{\mathcal{Z}}$.*

Definition 2.3. *[3] Let $\mathcal{Z}$ and Γ be any two VSs in U.*

1. *A VS $\mathcal{Z}$ is contained in the other VS Γ, $\mathcal{Z} \subseteq \Gamma$ if and only if $\sqsupset_{\mathcal{Z}}[\tau] \leq \sqsupset_{\Gamma}[\tau]$, i.e., $\top_{\mathcal{Z}}[\tau] \leq \top_{\Gamma}[\tau]$ and $1 - F_{\mathcal{Z}}[\tau] \leq 1 - F_{\Gamma}[\tau]$, for all $u \in U$.*
2. *The union of two VSs $\mathcal{Z}$ and Γ, as $\Upsilon = \mathcal{Z} \cup \Gamma$, $\top_{\Upsilon} = \max\{\top_{\mathcal{Z}}, \top_{\Gamma}\}$ and $1 - F_{\Upsilon} = \max\{1 - F_{\mathcal{Z}}, 1 - F_{\Gamma}\} = 1 - \min\{F_{\mathcal{Z}}, F_{\Gamma}\}$.*
3. *The intersection of two VSs $\mathcal{Z}$ and Γ as $\Upsilon = \mathcal{Z} \cap \Gamma$, $\top_{\Upsilon} = \min\{\top_{\mathcal{Z}}, \top_{\Gamma}\}$ and $1 - F_{\Upsilon} = \min\{1 - F_{\mathcal{Z}}, 1 - F_{\Gamma}\} = 1 - \max\{F_{\mathcal{Z}}, F_{\Gamma}\}$.*

Definition 2.4. *[2] Let $\mathcal{Z}$ be a VS of U, $\zeta, \sigma \in [0,1]$ with $\zeta \leq \sigma$, the $[\zeta, \sigma]$- cut or vague cut of a VS $\mathcal{Z}$ is the crisp subset of U is given by $\mathcal{Z}_{[\zeta,\sigma]} = \{\tau \in U | \sqsupset_{\mathcal{Z}}[\tau] \geq [\zeta, \sigma]\}$.*
That is, $\mathcal{Z}_{[\zeta,\sigma]} = \{\tau \in U | \top_{\mathcal{Z}}[\tau] \geq \zeta,\ 1 - F_{\mathcal{Z}}[\tau] \geq \sigma]\}$.

Definition 2.5. *[2] Let $\mathcal{Z}$ and Γ be any two VSs in U.*

1. *$\mathcal{Z} \cap \Gamma = \left\{ \left\langle \tau, \min\{\top_{\mathcal{Z}}[\tau], \top_{\Gamma}[\tau]\}, \min\{1 - F_{\mathcal{Z}}[\tau], 1 - F_{\Gamma}[\tau]\} \right\rangle \right\}$*

(2) $\mathcal{Z} \cup \Gamma = \left\{ \left\langle \tau, \max\{\top_{\mathcal{Z}}[\tau], \top_{\Gamma}[\tau]\}, \max\{\text{ı} - F_{\mathcal{Z}}[\tau], \text{ı} - F_{\Gamma}[\tau]\} \right\rangle \right\}$

(3) $\Box \mathcal{Z} = \left\{ \left\langle \tau, \top_{\mathcal{Z}}[\tau], \text{ı} - \top_{\mathcal{Z}}[\tau] \right\rangle | \tau \in U \right\}$

(4) $\Diamond \mathcal{Z} = \left\{ \langle \tau, \text{ı} - F_{\mathcal{Z}}[\tau], F_{\mathcal{Z}}[\tau] \rangle | \tau \in U \right\}$ *for all* $\tau \in U$.

Definition 2.6. *Let X be a nonempty set. An multi-IFS $f = \{\tau, f^+[\tau], f^*[\tau] : \tau \in X, 0 \leq f^+[\tau] \boxplus f^*[\tau] \leq 1\}$, where $f^+ : X \to [0,1]^m, f^* : X \to [0,1]^m$ and $0 \in \{0\}^m$ and $1 \in \{1\}^m$. The value of f^+ and f^* are defined as $f^+[\tau] = [\pi_1 \cdot f^+][\tau], [\pi_2 \cdot f^+][\tau], ..., [\pi_m \cdot f^+][\tau]$, and $f^*[\tau] = [\pi_1 \cdot f^*][\tau], [\pi_2 \cdot f^*][\tau], ..., [\pi_m \cdot f^*][\tau]$, such that $\pi_i : [0,1]^m \to [0,1]$ is a projection i-th mapping and $0 \leq [f^+][\tau] \boxplus [f^*][\tau] \leq 1$. It means that $0 \leq [\pi \cdot f^+][\tau] + [\pi \cdot f^*][\tau] \leq 1, \forall i \in \{1, 2, ..., m\}$.*

3. Multi-fuzzy set applied to cubic vague subbisemirings

Definition 3.1. *A multi-fuzzy cubic vague set $\mathcal{Z} = \langle \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{Z}}, \coprod_i \circ \beth_{\mathcal{Z}} \rangle$ of $\Bbbk$ is called an MFCVSBS if*

$$\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_2]\}$$

and

$$\coprod_i \circ \beth_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{\coprod_i \circ \beth_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ \beth_{\mathcal{Z}}[\varsigma_2]\},$$

i.e.,

$$\coprod_i \circ \overrightarrow{\top^+} \aleph_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\top^+} \aleph_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ \overrightarrow{\top^+} \aleph_{\mathcal{Z}}[\varsigma_2]\}$$

$$\text{ı} - \coprod_i \circ \overrightarrow{F^+} \aleph_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\text{ı} - \coprod_i \circ \overrightarrow{F^+} \aleph_{\mathcal{Z}}[\varsigma_1], \text{ı} - \coprod_i \circ \overrightarrow{F^+} \aleph_{\mathcal{Z}}[\varsigma_2]\}$$

and

$$\coprod_i \circ \top^* \beth_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{\coprod_i \circ \top^* \beth_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ \top^* \beth_{\mathcal{Z}}[\varsigma_2]\}$$

$$\text{ı} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{\text{ı} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_1], \text{ı} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_2]\}$$

for all $\varsigma_1, \varsigma_2 \in \Bbbk$.

Example 3.1. *Let $\Bbbk = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be the bisemirings.*

$\uplus_1$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_1	ς_1	ς_1
ς_2	ς_1	ς_2	ς_1	ς_2
ς_3	ς_1	ς_1	ς_3	ς_3
ς_4	ς_1	ς_2	ς_3	ς_4

$\uplus_2$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_4	ς_4
ς_3	ς_3	ς_4	ς_3	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

$\uplus_3$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_1	ς_1	ς_1
ς_2	ς_1	ς_2	ς_3	ς_4
ς_3	ς_4	ς_4	ς_4	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

Table 2

$\Bbbk$	$\overrightarrow{\aleph^+}_{\mathcal{Z}}$	$\beth^+_{\mathcal{Z}}$	$\overrightarrow{\aleph^*}_{\mathcal{Z}}$	$\beth^*_{\mathcal{Z}}$
ς_1	$\langle[0.29, 0.39], [0.44, 0.49]\rangle$	$[0.30, 0.35]$	$\langle[0.25, 0.35], [0.40, 0.45]\rangle$	$[0.3, 0.35]$
ς_2	$\langle[0.24, 0.34], [0.39, 0.44]\rangle$	$[0.20, 0.40]$	$\langle[0.20, 0.30], [0.35, 0.40]\rangle$	$[0.2, 0.4]$
ς_3	$\langle[0.14, 0.19], [0.29, 0.34]\rangle$	$[0, 0.50]$	$\langle[0.10, 0.15], [0.25, 0.30]\rangle$	$[0, 0.47]$
ς_4	$\langle[0.19, 0.24], [0.34, 0.39]\rangle$	$[0.1, 0.45]$	$\langle[0.15, 0.20], [0.30, 0.35]\rangle$	$[0.1, 0.42]$

Hence, $\mathcal{Z}$ is an MFCVSBS.

Theorem 3.1. *The intersection of an infinite collection of MFCVSBSs is again an MFCVSBS.*

Proof. Let $\{\wp_c : c \in C\}$ be an infinite collection of MFCVSBSs, put $\mathcal{Z} = \bigcap_{c\in C}\wp_c$. For $\varsigma_1, \varsigma_2 \in \Bbbk$. Now,

$$\begin{aligned}
\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] &= \inf_{c\in C} \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\wp_c}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \\
&\geq \inf_{c\in C} Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\wp_c}[\varsigma_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\wp_c}[\varsigma_2]\} \\
&= Min^i\left\{\inf_{c\in C} \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\wp_c}[\varsigma_1], \inf_{c\in C} \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\wp_c}[\varsigma_2]\right\} \\
&= Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\mathcal{Z}}[\varsigma_2]\},
\end{aligned}$$

$$\begin{aligned}
\text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] &= \inf_{c\in C} [\text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\wp_c}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]] \\
&\geq \inf_{c\in C} Min^i\{\text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\wp_c}[\varsigma_1], \text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\wp_c}[\varsigma_2]\} \\
&= Min^i\left\{\inf_{c\in C} \text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\wp_c}[\varsigma_1], \inf_{c\in C} \text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\wp_c}[\varsigma_2]\right\} \\
&= Min^i\{\text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_1], \text{I} - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_2]\}.
\end{aligned}$$

Thus, $\coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_2]\}$. Now,

$$\begin{aligned}
\coprod_i \circ\mathsf{T}^*\beth_{\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] &= \inf_{c\in C} \coprod_i \circ\mathsf{T}^*\beth_{\wp_c}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \\
&\geq \inf_{c\in C} \min\{\coprod_i \circ\mathsf{T}^*\beth_{\wp_c}[\varsigma_1], \coprod_i \circ\mathsf{T}^*\beth_{\wp_c}[\varsigma_2]\} \\
&= \min\left\{\inf_{c\in C} \coprod_i \circ\mathsf{T}^*\beth_{\wp_c}[\varsigma_1], \inf_{c\in C} \coprod_i \circ\mathsf{T}^*\beth_{\wp_c}[\varsigma_2]\right\} \\
&= \min\{\coprod_i \circ\mathsf{T}^*\beth_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ\mathsf{T}^*\beth_{\mathcal{Z}}[\varsigma_2]\},
\end{aligned}$$

$$\begin{aligned}
\text{I} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] &= \inf_{c\in C} [\text{I} - \coprod_i \circ F^*_{\beth_{\wp_c}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]] \\
&\geq \inf_{c\in C} \min\{\text{I} - \coprod_i \circ F^*_{\beth_{\wp_c}}[\varsigma_1], \text{I} - \coprod_i \circ F^*_{\beth_{\wp_c}}[\varsigma_2]\} \\
&= \min\left\{\inf_{c\in C} \text{I} - \coprod_i \circ F^*_{\beth_{\wp_c}}[\varsigma_1], \inf_{c\in C} \text{I} - \coprod_i \circ F^*_{\beth_{\wp_c}}[\varsigma_2]\right\} \\
&= \min\{\text{I} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_1], \text{I} - \coprod_i \circ F^*_{\beth_{\mathcal{Z}}}[\varsigma_2]\}.
\end{aligned}$$

Hence, $\coprod_i \circ\mathcal{Z}[\varsigma_1 \uplus_{1,2,3} \varsigma_2] \geq \min\{\coprod_i \circ\mathcal{Z}[\varsigma_1], \coprod_i \circ\mathcal{Z}[\varsigma_2]\}$. Hence, $\mathcal{Z}$ is an MFCVSBS. $\square$

Theorem 3.2. *The Cartesian product of MFCVSBSs is also an MFCVSBS.*

Proof. Let Υ and Γ be two MFCVSBSs of $\Bbbk_1$ and $\Bbbk_2$, respectively. Let $\epsilon_1, \epsilon_2 \in \Bbbk_1$ and $\varsigma_1, \varsigma_2 \in \Bbbk_2$. Then $[\epsilon_1, \epsilon_2]$ and $[\varsigma_1, \varsigma_2]$ are in $\Bbbk_1 \times \Bbbk_2$. Now,

$$\begin{aligned}
&\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon\times\Gamma}[[\epsilon_1,\varsigma_1] \uplus_{[1,2,3]} [\epsilon_2,\varsigma_2]] \\
&= \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2, \varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \\
&= Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]\} \\
&\geq Min^i\{Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon}[\epsilon_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon}[\epsilon_2]\}, Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Gamma}[\varsigma_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Gamma}[\varsigma_2]\}\} \\
&= Min^i\{Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon}[\epsilon_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Gamma}[\varsigma_1]\}, Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon}[\epsilon_2], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Gamma}[\varsigma_2]\}\} \\
&= Min^i\{\coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_1,\varsigma_1], \coprod_i \circ\overrightarrow{\mathsf{T}^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_2,\varsigma_2]\}, \\
&\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon\times\Gamma}[[\epsilon_1,\varsigma_1] \uplus_{[1,2,3]} [\epsilon_2,\varsigma_2]] \\
&= \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2, \varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \\
&\geq Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]\} \\
&\geq Min^i\{Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon}[\epsilon_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon}[\epsilon_2]\}, Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Gamma}[\varsigma_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Gamma}[\varsigma_2]\}\} \\
&= Min^i\{Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon}[\epsilon_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Gamma}[\varsigma_1]\}, Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon}[\epsilon_2], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Gamma}[\varsigma_2]\}\} \\
&= Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_1,\varsigma_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\Upsilon\times\Gamma}[\epsilon_2,\varsigma_2]\}.
\end{aligned}$$

Thus, $\coprod_i \circ\overrightarrow{\aleph}_{\Upsilon\times\Gamma}[[\epsilon_1,\varsigma_1] \uplus_{[1,2,3]} [\epsilon_2,\varsigma_2]] \geq Min^i\{\coprod_i \circ\overrightarrow{\aleph}_{\Upsilon\times\Gamma}[\epsilon_1,\varsigma_1], \coprod_i \circ\overrightarrow{\aleph}_{\Upsilon\times\Gamma}[\epsilon_2,\varsigma_2]\}$. Now,

$$\begin{aligned}
&\coprod_i \circ\mathsf{T}^*\beth_{\Upsilon\times\Gamma}[[\epsilon_1,\varsigma_1] \uplus_{[1,2,3]} [\epsilon_2,\varsigma_2]] \\
&= \coprod_i \circ\mathsf{T}^*\beth_{\Upsilon\times\Gamma}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2, \varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \\
&= \min\{\coprod_i \circ\mathsf{T}^*\beth_{\Upsilon}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2], \coprod_i \circ\mathsf{T}^*\beth_{\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]\} \\
&\geq \min\{\min\{\coprod_i \circ\mathsf{T}^*\beth_{\Upsilon}[\epsilon_1], \coprod_i \circ\mathsf{T}^*\beth_{\Upsilon}[\epsilon_2]\}, \min\{\coprod_i \circ\mathsf{T}^*\beth_{\Gamma}[\varsigma_1], \coprod_i \circ\mathsf{T}^*\beth_{\Gamma}[\varsigma_2]\}\} \\
&= \min\{\min\{\coprod_i \circ\mathsf{T}^*\beth_{\Upsilon}[\epsilon_1], \coprod_i \circ\mathsf{T}^*\beth_{\Gamma}[\varsigma_1]\}, \min\{\coprod_i \circ\mathsf{T}^*\beth_{\Upsilon}[\epsilon_2], \coprod_i \circ\mathsf{T}^*\beth_{\Gamma}[\varsigma_2]\}\}
\end{aligned}$$

$$= \min\{\coprod_i \circ\top^*\beth_{\Upsilon\times\Gamma}[\epsilon_1,\varsigma_1], \coprod_i \circ\top^*\beth_{\Upsilon\times\Gamma}[\epsilon_2,\varsigma_2]\},$$

$$\imath - \coprod_i \circ F^*_{\beth_{\Upsilon\times\Gamma}}[[\epsilon_1,\varsigma_1] \uplus_{[1,2,3]} [\epsilon_2,\varsigma_2]]$$

$$= \imath - \coprod_i \circ F^*_{\beth_{\Upsilon\times\Gamma}}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2, \varsigma_1 \uplus_{[1,2,3]} \varsigma_2]$$

$$\geq \min\{\imath - \coprod_i \circ F^*_{\beth_{\Upsilon}}[\epsilon_1 \uplus_{[1,2,3]} \epsilon_2], \imath - \coprod_i \circ F^*_{\beth_{\Gamma}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$\geq \min\{\min\{\imath - \coprod_i \circ F^*_{\beth_{\Upsilon}}[\epsilon_1], \imath - \coprod_i \circ F^*_{\beth_{\Upsilon}}[\epsilon_2]\}, \min\{\imath - \coprod_i \circ F^*_{\beth_{\Gamma}}[\varsigma_1], \imath - \coprod_i \circ F^*_{\beth_{\Gamma}}[\varsigma_2]\}\}$$

$$= \min\{\min\{\imath - \coprod_i \circ F^*_{\beth_{\Upsilon}}[\epsilon_1], \imath - \coprod_i \circ F^*_{\beth_{\Gamma}}[\varsigma_1]\}, \min\{\imath - \coprod_i \circ F^*_{\beth_{\Upsilon}}[\epsilon_2], \imath - \coprod_i \circ F^*_{\beth_{\Gamma}}[\varsigma_2]\}\}$$

$$= \min\{\imath - \coprod_i \circ F^*_{\beth_{\Upsilon\times\Gamma}}[\epsilon_1,\varsigma_1], \imath - \coprod_i \circ F^*_{\beth_{\Upsilon\times\Gamma}}[\epsilon_2,\varsigma_2]\}.$$

Hence, $\coprod_i \circ\beth_{\Upsilon\times\Gamma}[[\epsilon_1,\varsigma_1] \uplus_{1,2,3} [\epsilon_2,\varsigma_2]] \geq \min\{\coprod_i \circ\beth_{\Upsilon\times\Gamma}[\epsilon_1,\varsigma_1], \coprod_i \circ\beth_{\Upsilon\times\Gamma}[\epsilon_2,\varsigma_2]\}$. Therefore, $\Upsilon\times\Gamma$ is an MFCVSBS. □

Theorem 3.3. *Let $\mathcal{Z}$ be a multi-fuzzy cubic vague subset of $\Bbbk$. Then $\mathcal{Z} = \langle \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}, \coprod_i \circ\beth_{\mathcal{Z}}\rangle$ is an MFCVSBS if and only if all non empty level sets $\mathcal{Z}_{[\zeta,\sigma]}$ $[\zeta,\sigma \in D[0,1]]$ is an SBS.*

Proof. Assume that $\mathcal{Z} = \langle \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}, \coprod_i \circ\beth_{\mathcal{Z}}\rangle$ is an MFCVSBS. For each $\zeta,\sigma \in D[0,1]$ and $\kappa_1,\kappa_2 \in \mathcal{Z}_{[\zeta,\sigma]}$. We have $\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1] \geq \overrightarrow{\zeta}$, $\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_2] \geq \overrightarrow{\zeta}$, $\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1] \geq \overrightarrow{\sigma}$ and $\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_2] \geq \overrightarrow{\sigma}$. Since $\mathcal{Z}$ is an MFCVSBS, $\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \min\{\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_2]\} \geq \overrightarrow{\zeta}$. Similarly, $\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \min\{\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_2]\} \geq \overrightarrow{\sigma}$. Now, $\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1] \geq \zeta$, $\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_2] \geq \zeta$, $\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1] \geq \sigma$ and $\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_2] \geq \sigma$. Since $\mathcal{Z}$ is an MFCVSBS, we have $\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \min\{\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_2]\} \geq \zeta$. Similarly, $\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \min\{\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_2]\} \geq \sigma$. This implies that $\kappa_1 \uplus_{[1,2,3]} \kappa_2 \in \mathcal{Z}_{[\zeta,\sigma]}$. Therefore, $\mathcal{Z}_{[\zeta,\sigma]}$ is an SBS for each $\zeta,\sigma \in D[0,1]$.

Conversely, let us assume $\mathcal{Z}_{[\zeta,\sigma]}$ is an SBS for each $\zeta,\sigma \in D[0,1]$. Let $\kappa_1,\kappa_2 \in \Bbbk$. Then $\kappa_1,\kappa_2 \in \mathcal{Z}_{[\zeta,\sigma]}$, where $\overrightarrow{\zeta} = \min\{\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_2]\}$ and $\overrightarrow{\sigma} = \min\{\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_2]\}$. Thus, $\kappa_1 \uplus_{[1,2,3]} \kappa_2 \in \mathcal{Z}_{[\zeta,\sigma]}$ imply $\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \overrightarrow{\zeta} = \min\{\coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\overrightarrow{\top}_{\aleph_{\mathcal{Z}}}[\kappa_2]\}$ and $\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \overrightarrow{\sigma} = \min\{\imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\kappa_2]\}$. Now, $\zeta = \min\{\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_2]\}$ and $\sigma = \min\{\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_2]\}$. Thus, $\kappa_1 \uplus_{[1,2,3]} \kappa_2 \in \mathcal{Z}_{[\zeta,\sigma]}$, so $\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \zeta = \min\{\coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_1], \coprod_i \circ\top_{\beth_{\mathcal{Z}}}[\kappa_2]\}$ and $\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1 \uplus_{[1,2,3]} \kappa_2] \geq \sigma = \min\{\imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_1], \imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\kappa_2]\}$. Hence, $\mathcal{Z} = \langle \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}, \coprod_i \circ\beth_{\mathcal{Z}}\rangle$ is an MFCVSBS. □

Theorem 3.4. *Let $\wp$ be the strongest vague relation of $\Bbbk$. Then $\mathcal{Z}$ is an MFCVSBS if and only if $\wp$ is an MFCVSBS of $\Bbbk\times\Bbbk$.*

Proof. For $\epsilon = [\epsilon_1, \epsilon_2]$ and $\varsigma = [\varsigma_1, \varsigma_2]$ are in $\Bbbk \times \Bbbk$. Now,

$$\begin{aligned}
&\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\epsilon \uplus_{[1,2,3]} \varsigma] \\
&= \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[[[\epsilon_1, \epsilon_2] \uplus_{[1,2,3]} [\varsigma_1, \varsigma_2]] \\
&= \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2] \\
&= Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\} \\
&\geq Min^i\{Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_1], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\varsigma_1]\}, Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_2], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\varsigma_2]\}\} \\
&= Min^i\{Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_1], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\epsilon_2]\}, Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\varsigma_1], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_Z[\varsigma_2]\}\} \\
&= Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\epsilon_1, \epsilon_2], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\varsigma_1, \varsigma_2]\} \\
&= Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\epsilon], \coprod_i \circ \overrightarrow{\mathsf{T}^+}\aleph_\wp[\varsigma]\}, \\
&\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\epsilon \uplus_{[1,2,3]} \varsigma] \\
&= \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[[[\epsilon_1, \epsilon_2] \uplus_{[1,2,3]} [\varsigma_1, \varsigma_2]] \\
&= \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2] \\
&= Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\} \\
&\geq Min^i\{Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_1], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\varsigma_1]\}, Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_2], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\varsigma_2]\}\} \\
&= Min^i\{Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_1], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\epsilon_2]\}, Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\varsigma_1], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_Z[\varsigma_2]\}\} \\
&= Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\epsilon_1, \epsilon_2], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\varsigma_1, \varsigma_2]\} \\
&= Min^i\{\mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\epsilon], \mathsf{I} - \coprod_i \circ \overrightarrow{F^+}\aleph_\wp[\varsigma]\}.
\end{aligned}$$

Thus, $\coprod_i \circ \overrightarrow{\aleph}_\wp[\epsilon \uplus_{[1,2,3]} \varsigma] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_\wp[\epsilon], \coprod_i \circ \overrightarrow{\aleph}_\wp[\varsigma]\}$. Now,

$$\begin{aligned}
&\coprod_i \circ \mathsf{T}^* \beth_\wp[\epsilon \uplus_{[1,2,3]} \varsigma] \\
&= \coprod_i \circ \mathsf{T}^* \beth_\wp[[[\epsilon_1, \epsilon_2] \uplus_{[1,2,3]} [\varsigma_1, \varsigma_2]]
\end{aligned}$$

$$= \coprod_i \circ \mathsf{T}^* \beth_\wp[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$

$$= \min\{\coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$\geq \min\{\min\{\coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_1], \coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\varsigma_1]\}, \min\{\coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_2], \coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\varsigma_2]\}\}$$

$$= \min\{\min\{\coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_1], \coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\epsilon_2]\}, \min\{\coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\varsigma_1], \coprod_i \circ \mathsf{T}^* \beth_\mathcal{Z}[\varsigma_2]\}\}$$

$$= \min\{\coprod_i \circ \mathsf{T}^* \beth_\wp[\epsilon_1, \epsilon_2], \coprod_i \circ \mathsf{T}^* \beth_\wp[\varsigma_1, \varsigma_2]\}$$

$$= \min\{\coprod_i \circ \mathsf{T}^* \beth_\wp[\epsilon], \coprod_i \circ \mathsf{T}^* \beth_\wp[\varsigma]\}$$

$$\mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\epsilon \uplus_{[1,2,3]} \varsigma]$$

$$= \mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[[[\epsilon_1, \epsilon_2] \uplus_{[1,2,3]} [\varsigma_1, \varsigma_2]]$$

$$= \mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$

$$= \min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$\geq \min\{\min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1], \mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_1]\}, \min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2], \mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_2]\}\}$$

$$= \min\{\min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1], \mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2]\}, \min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_1], \mathsf{I} - \coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_2]\}\}$$

$$= \min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\epsilon_1, \epsilon_2], \mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\varsigma_1, \varsigma_2]\}$$

$$= \min\{\mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\epsilon], \mathsf{I} - \coprod_i \circ F^*_{\beth_\wp}[\varsigma]\}.$$

Thus, $\coprod_i \circ \beth_\wp[\epsilon \uplus_{[1,2,3]} \varsigma] \geq \min\{\coprod_i \circ \beth_\wp[\epsilon], \coprod_i \circ \beth_\wp[\varsigma]\}$. Hence, $\coprod_i \circ \mathcal{Z}_\wp[\epsilon \uplus_{1,2,3} \varsigma] \geq \min\{\coprod_i \circ \mathcal{Z}_\wp[\epsilon], \coprod_i \circ \mathcal{Z}_\wp[\varsigma]\}$. Therefore, $\wp$ is an MFCVSBS of $\Bbbk \times \Bbbk$.

Conversely, assume that $\wp$ is an MFCVSBS of $\Bbbk \times \Bbbk$. Let $\epsilon = [\epsilon_1, \epsilon_2]$ and $\varsigma = [\varsigma_1, \varsigma_2]$ be in $\Bbbk \times \Bbbk$. Now,

$$Min^i\{\coprod_i \circ \overrightarrow{\mathsf{T}^+} \aleph_\mathcal{Z}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \coprod_i \circ \overrightarrow{\mathsf{T}^+} \aleph_\mathcal{Z}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$= \coprod_i \circ \overrightarrow{\mathsf{T}^+} \aleph_\wp[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$

$$= \coprod_i \circ \overrightarrow{\mathsf{T}^+} \aleph_\wp[[\epsilon_1, \epsilon_2] \uplus_{[1,2,3]} [\varsigma_1, \varsigma_2]]$$

$$= \coprod_i \circ \overrightarrow{\mathsf{T}^+} \aleph_\wp[\epsilon \uplus_{[1,2,3]} \varsigma]$$

$$\geq Min^i\{\coprod_i \circ\overrightarrow{\top^+}\aleph_\varphi[\epsilon], \coprod_i \circ\overrightarrow{\top^+}\aleph_\varphi[\varsigma]\}$$

$$= Min^i\{\coprod_i \circ\overrightarrow{\top^+}\aleph_\varphi[\epsilon_1,\epsilon_2]\}, \coprod_i \circ\overrightarrow{\top^+}\aleph_\varphi[\varsigma_1,\varsigma_2]\}$$

$$= Min^i\{Min^i\{\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_1], \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_2]\}, Min^i\{\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\varsigma_2]\}\}.$$

If $\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \leq \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$, then $\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_1] \leq \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_2]$ and $\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\varsigma_1] \leq \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\varsigma_2]$. We get $\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq Min^i\{\coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\epsilon_1], \coprod_i \circ\overrightarrow{\top^+}\aleph_{\mathcal{Z}}[\varsigma_1]\}$ for all $\epsilon_1, \varsigma_1 \in \Bbbk$, and

$$Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$= \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$

$$= \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[[\epsilon_1,\epsilon_2] \uplus_{[1,2,3]} [\varsigma_1,\varsigma_2]]$$

$$= \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\epsilon \uplus_{[1,2,3]} \varsigma]$$

$$\geq Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\epsilon], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\varsigma]\}$$

$$= Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\epsilon_1,\epsilon_2]\}, \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_\varphi[\varsigma_1,\varsigma_2]\}$$

$$= Min^i\{Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_2]\}, Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_2]\}\}.$$

If $\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \leq \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$, then $\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1] \leq \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_2]$ and $\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_1] \leq \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_2]$. We get $\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq Min^i\{\imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\epsilon_1], \imath - \coprod_i \circ\overrightarrow{F^+}\aleph_{\mathcal{Z}}[\varsigma_1]\}$ for all $\epsilon_1, \varsigma_1 \in \Bbbk$. Thus, $\coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq Min^i\{\coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\epsilon_1], \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1]\}$. Now,

$$\min\{\coprod_i \circ\top^*\beth_{\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], \coprod_i \circ\top^*\beth_{\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$

$$= \coprod_i \circ\top^*\beth_\varphi[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1, \epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$

$$= \coprod_i \circ\top^*\beth_\varphi[[\epsilon_1,\epsilon_2] \uplus_{[1,2,3]} [\varsigma_1,\varsigma_2]]$$

$$= \coprod_i \circ\top^*\beth_\varphi[\epsilon \uplus_{[1,2,3]} \varsigma]$$

$$\geq \min\{\coprod_i \circ\top^*\beth_\varphi[\epsilon], \coprod_i \circ\top^*\beth_\varphi[\varsigma]\}$$

$$= \min\{\coprod_i \circ \top^*{}_{\beth_\wp}[\epsilon_1,\epsilon_2]\},\coprod_i \circ \top^*{}_{\beth_\wp}[\varsigma_1,\varsigma_2]\}$$
$$= \min\{\min\{\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_1],\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_2]\},\min\{\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\varsigma_1],\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\varsigma_2]\}\}.$$

If $\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \leq \coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$, then $\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_1] \leq \coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_2]$ and $\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\varsigma_1] \leq \coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\varsigma_2]$. We get $\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq \min\{\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\epsilon_1],\coprod_i \circ \top^*{}_{\beth_\mathcal{Z}}[\varsigma_1]\}$ for all $\epsilon_1,\varsigma_1 \in \Bbbk$, and

$$\min\{1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1], 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]\}$$
$$=1-\coprod_i \circ F^*_{\beth_\wp}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1,\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$$
$$=1-\coprod_i \circ F^*_{\beth_\wp}[[\epsilon_1,\epsilon_2] \uplus_{[1,2,3]} [\varsigma_1,\varsigma_2]]$$
$$=1-\coprod_i \circ F^*_{\beth_\wp}[\epsilon \uplus_{[1,2,3]} \varsigma]$$
$$\geq \min\{1-\coprod_i \circ F^*_{\beth_\wp}[\epsilon], 1-\coprod_i \circ F^*_{\beth_\wp}[\varsigma]\}$$
$$= \min\{1-\coprod_i \circ F^*_{\beth_\wp}[\epsilon_1,\epsilon_2]\}, 1-\coprod_i \circ F^*_{\beth_\wp}[\varsigma_1,\varsigma_2]\}$$
$$= \min\{\min\{1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1], 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2]\},\min\{1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_1], 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_2]\}\}.$$

If $1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \leq 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2 \uplus_{[1,2,3]} \varsigma_2]$, then $1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1] \leq 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_2]$ and $1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_1] \leq 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_2]$. We get $1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq \min\{1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\epsilon_1], 1-\coprod_i \circ F^*_{\beth_\mathcal{Z}}[\varsigma_1]\}$ for all $\epsilon_1,\varsigma_1 \in \Bbbk$. Thus, $\coprod_i \circ \beth_\mathcal{Z}[\epsilon_1 \uplus_{[1,2,3]} \varsigma_1] \geq \min\{\coprod_i \circ \beth_\mathcal{Z}[\epsilon_1],\coprod_i \circ \beth_\mathcal{Z}[\varsigma_1]\}$. Hence, $\coprod_i \circ \mathcal{Z}[\epsilon_1 \uplus_{1,2,3} \varsigma_1] \geq \min\{\coprod_i \circ \mathcal{Z}[\epsilon_1],\coprod_i \circ \mathcal{Z}[\varsigma_1]\}$. Therefore, $\mathcal{Z}$ is an MFCVSBS. □

Theorem 3.5. *If $\mathcal{Z}$ is an MFCVSBS, then $\Delta\mathcal{Z}$ is an MFCVSBS.*

Proof. Let $\mathcal{Z}$ be an MFCVSBS of a bisemiring $\Bbbk$. Consider $\mathcal{Z} = \{\left\langle \varsigma_1, \coprod_i \circ \overrightarrow{\aleph}_\mathcal{Z}[\varsigma_1], \coprod_i \circ \beth_\mathcal{Z}[\varsigma_1]\right\rangle\}$, for all $\varsigma_1 \in \Bbbk$. Take $\Delta\mathcal{Z} = \Gamma = \{\left\langle \varsigma_1, \coprod_i \circ \overrightarrow{\aleph}_\Gamma[\varsigma_1], \coprod_i \circ \beth_\Gamma[\varsigma_1]\right\rangle\}$, where $\coprod_i \circ \overrightarrow{\top}_{\aleph_\Gamma}[\varsigma_1] = \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{Z}}[\varsigma_1]$, $\coprod_i \circ F_{\aleph_\Gamma}[\varsigma_1] = 1 - \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{Z}}[\varsigma_1]$, $\coprod_i \circ \top_{\beth_\Gamma}[\varsigma_1] = \coprod_i \circ \top_{\beth_\mathcal{Z}}[\varsigma_1]$ and $F_{\beth_\Gamma}[\varsigma_1] = 1 - \coprod_i \circ \top_{\beth_\mathcal{Z}}[\varsigma_1]$. Clearly, $\coprod_i \circ \overrightarrow{\top}_{\aleph_\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\top}_{\aleph_\Gamma}[\varsigma_1],\coprod_i \circ \overrightarrow{\top}_{\aleph_\Gamma}[\varsigma_2]\}$ and $\coprod_i \circ \top_{\beth_\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{\coprod_i \circ \top_{\beth_\Gamma}[\varsigma_1],\coprod_i \circ \top_{\beth_\Gamma}[\varsigma_2]\}$, for all $\varsigma_1,\varsigma_2 \in \Bbbk$. Since $\mathcal{Z}$ is an MFCVSBS. Then $\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{Z}}[\varsigma_1],\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{Z}}[\varsigma_2]\}$ and
$\coprod_i \circ \top_{\beth_\mathcal{Z}}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{\coprod_i \circ \top_{\beth_\mathcal{Z}}[\varsigma_1],\coprod_i \circ \top_{\beth_\mathcal{Z}}[\varsigma_2]\}$.
Thus, $1-\coprod_i \circ F_{\aleph_\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq Min^i\{[1-\coprod_i \circ F_{\aleph_\Gamma}[\varsigma_1]],[1-\coprod_i \circ F_{\aleph_\Gamma}[\varsigma_2]]\}$ and
$1-\coprod_i \circ F_{\beth_\Gamma}[\varsigma_1 \uplus_{[1,2,3]} \varsigma_2] \geq \min\{[1-\coprod_i \circ F_{\beth_\Gamma}[\varsigma_1]],[1-\coprod_i \circ F_{\beth_\Gamma}[\varsigma_2]]\}$. Hence, $\Delta\mathcal{Z}$ is an MFCVSBS. □

Theorem 3.6. *If $\mathcal{Z}$ is an MFCVSBS, then $\mho\mathcal{Z}$ is an MFCVSBS.*

Proof. Let $\mathcal{Z}$ be an MFCVSBS of a bisemiring $\Bbbk$. Consider $\mathcal{Z} = \{\left\langle \varsigma_1, \coprod_i \circ\overrightarrow{\aleph}_{\mathcal{Z}}[\varsigma_1], \coprod_i \circ\beth_{\mathcal{Z}}[\varsigma_1]\right\rangle\}$, for all $\varsigma_1 \in \Bbbk$. Take $\mho\mathcal{Z} = \Gamma = \{\left\langle \varsigma_1, \coprod_i \circ\overrightarrow{\aleph}_{\Gamma}[\varsigma_1], \coprod_i \circ\beth_{\Gamma}[\varsigma_1]\right\rangle\}$, where $\coprod_i \circ\overrightarrow{\top}_{\aleph_\Gamma}[\varsigma_1] = \imath - \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\varsigma_1]$, $\coprod_i \circ F_{\aleph_\Gamma}[\varsigma_1] = \coprod_i \circ F_{\aleph_{\mathcal{Z}}}[\varsigma_1]$, $\coprod_i \circ \top_{\beth_\Gamma}[\varsigma_1] = \imath - \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\varsigma_1]$ and $F_{\beth_\Gamma}[\varsigma_1] = \coprod_i \circ F_{\beth_{\mathcal{Z}}}[\varsigma_1]$ and the applying Theorem 3.5. Hence, $\mho\mathcal{Z}$ is an MFCVSBS. □

4. Homomorphism via multi-fuzzy cubic vague subbisemirings

Definition 4.1. *Let $[\Bbbk_1, \curlyvee_1, \curlyvee_2, \curlyvee_3]$ and $[\Bbbk_2, \curlywedge_1, \curlywedge_2, \curlywedge_3]$ be any two bisemirings. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be any function and $\mathcal{L}$ be an MFCVSBS in $\Bbbk_1$, $\mathcal{W}$ be an MFCVSBS in $\rho[\Bbbk_1] = \Bbbk_2$, the image of multi-fuzzy cubic vague set can be defined by*

$$\coprod_i \circ\overrightarrow{\aleph}_{\rho[\mathcal{W}]}[\kappa_2] = [\coprod_i \circ\overrightarrow{\top}_{\rho[\mathcal{W}]}[\kappa_2], \coprod_i \circ F_{\rho[\mathcal{W}]}[\kappa_2]]$$

and

$$\coprod_i \circ\beth_{\rho[\mathcal{W}]}[\kappa_2] = [\coprod_i \circ\top_{\rho[\mathcal{W}]}[\kappa_2], \coprod_i \circ F_{\rho[\mathcal{W}]}[\kappa_2]],$$

where $\coprod_i \circ\overrightarrow{\top}_{\rho[\mathcal{W}]}[\kappa_2] = \coprod_i \circ\overrightarrow{\top}_{\mathcal{W}}\rho[\kappa_2]$, $\coprod_i \circ F_{\rho[\mathcal{W}]}[\kappa_2] = \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_2]$, $\coprod_i \circ\top_{\rho[\mathcal{W}]}[\kappa_2] = \coprod_i \circ\top_{\mathcal{W}}\rho[\kappa_2]$ and $\coprod_i \circ F_{\rho[\mathcal{W}]}[\kappa_2] = \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_2]$.

Definition 4.2. *Let $[\Bbbk_1, \curlyvee_1, \curlyvee_2, \curlyvee_3]$ and $[\Bbbk_2, \curlywedge_1, \curlywedge_2, \curlywedge_3]$ be any two bisemirings. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be any function. Let $\mathcal{W}$ be a multi-fuzzy cubic vague set in $\rho[\Bbbk_1] = \Bbbk_2$. Then the inverse image of $\mathcal{W}$, ρ^{-1} is the multi-fuzzy cubic vague set in $\Bbbk_1$ by*

$$\coprod_i \circ\overrightarrow{\aleph}_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = [\coprod_i \circ\overrightarrow{\top}_{\rho^{-1}[\mathcal{W}]}[\kappa_1], \coprod_i \circ F_{\rho^{-1}[\mathcal{W}]}[\kappa_1]]$$

and

$$\coprod_i \circ\beth_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = [\coprod_i \circ\top_{\rho^{-1}[\mathcal{W}]}[\kappa_1], \coprod_i \circ F_{\rho^{-1}[\mathcal{W}]}[\kappa_1]],$$

where $\coprod_i \circ\overrightarrow{\top}_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = \coprod_i \circ\overrightarrow{\top}_{\mathcal{W}}[\rho^{-1}[\kappa_1]]$, $\coprod_i \circ F_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = \coprod_i \circ F_{\mathcal{W}}[\rho^{-1}[\kappa_1]]$, $\coprod_i \circ\top_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = \coprod_i \circ\top_{\mathcal{W}}[\rho^{-1}[\kappa_1]]$ and $F_{\rho^{-1}[\mathcal{W}]}[\kappa_1] = \coprod_i \circ F_{\mathcal{W}}[\rho^{-1}[\kappa_1]]$.

Theorem 4.1. *The homomorphic image of an MFCVSBS is also an MFCVSBS.*

Proof. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be a homomorphism. Then $\rho[\kappa_1 \curlyvee_1 \kappa_2] = \rho[\kappa_1] \curlywedge_1 \rho[\kappa_2], \rho[\kappa_1 \curlyvee_2 \kappa_2] = \rho[\kappa_1] \curlywedge_2 \rho[\kappa_2]$ and $\rho[\kappa_1 \curlyvee_3 \kappa_2] = \rho[\kappa_1] \curlywedge_3 \rho[\kappa_2]$ for all $\kappa_1, \kappa_2 \in \Bbbk_1$.
Let $\mathcal{W} = \rho[\mathcal{L}]$, where $\mathcal{L}$ is an MFCVSBS of $\Bbbk_1$. Let $\rho[\kappa_1], \rho[\kappa_2] \in \Bbbk_2$, $\coprod_i \circ\top^u_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \coprod_i \circ\top^u_{\aleph_{\mathcal{L}}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\coprod_i \circ\top^u_{\aleph_{\mathcal{L}}}[\kappa_1], \coprod_i \circ\top^u_{\aleph_{\mathcal{L}}}[\kappa_2]\} = \min\{\coprod_i \circ\top^u_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ\top^u_{\mathcal{W}}\rho[\kappa_2]\}$ and $\coprod_i \circ\top^l_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \coprod_i \circ\top^l_{\aleph_{\mathcal{L}}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\coprod_i \circ\top^l_{\aleph_{\mathcal{L}}}[\kappa_1], \coprod_i \circ\top^l_{\aleph_{\mathcal{L}}}[\kappa_2]\} = \min\{\coprod_i \circ\top^l_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ\top^l_{\mathcal{W}}\rho[\kappa_2]\}$.
Thus, $\coprod_i \circ\overrightarrow{\top}_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\coprod_i \circ\overrightarrow{\top}_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ\overrightarrow{\top}_{\mathcal{W}}\rho[\kappa_2]\}$.
Now, $\imath - \coprod_i \circ F^u_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \imath - \coprod_i \circ F^u_{\aleph_{\mathcal{L}}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\imath - \coprod_i \circ F^u_{\aleph_{\mathcal{L}}}[\kappa_1], \imath - \coprod_i \circ F^u_{\aleph_{\mathcal{L}}}[\kappa_2]\} = \min\{\imath - \coprod_i \circ F^u_{\mathcal{W}}\rho[\kappa_1], \imath - \coprod_i \circ F^u_{\mathcal{W}}\rho[\kappa_2]\}$ and $\imath - \coprod_i \circ F^l_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \imath - \coprod_i \circ F^l_{\aleph_{\mathcal{L}}}[\kappa_1 \curlyvee_1 \kappa_2] \geq$

$\min\{\imath - \coprod_i \circ F^l_{\aleph_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F^l_{\aleph_\mathcal{L}}[\kappa_2]\} = \min\{\imath - \coprod_i \circ F^l_{\mathcal{W}}\rho[\kappa_1], \imath - \coprod_i \circ F^l_{\mathcal{W}}\rho[\kappa_2]\}$.
Thus, $\imath - \coprod_i \circ F_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\imath - \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_1], \imath - \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_2]\}$.
Hence, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_2]\}$. Similarly, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_2 \rho[\kappa_2]] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_2]\}$ and $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_3 \rho[\kappa_2]] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}\rho[\kappa_2]\}$.
Now, $\coprod_i \circ \top_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_1], \coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_2]\}$
$= \min\{\coprod_i \circ \top_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \top_{\mathcal{W}}\rho[\kappa_2]\}$ and $\imath - \coprod_i \circ F_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2]$
$\geq \min\{\imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_2]\} = \min\{\imath - \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_1], \imath - \coprod_i \circ F_{\mathcal{W}}\rho[\kappa_2]\}$.
Thus, $\coprod_i \circ \beth_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_2]\}$. Similarly, $\coprod_i \circ \beth_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_2 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_2]\}$ and $\coprod_i \circ \beth_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_3 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_1], \coprod_i \circ \beth_{\mathcal{W}}\rho[\kappa_2]\}$. Hence, $\mathcal{W}$ is an MFCVSBS of $\Bbbk_2$. □

Theorem 4.2. *The homomorphic pre-image of an MFCVSBS is also an MFCVSBS.*

Proof. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be a homomorphism. Then $\rho[\kappa_1 \curlyvee_1 \kappa_2] = \rho[\kappa_1] \curlywedge_1 \rho[\kappa_2], \rho[\kappa_1 \curlyvee_2 \kappa_2] = \rho[\kappa_1] \curlywedge_2 \rho[\kappa_2]$ and $\rho[\kappa_1 \curlyvee_3 \kappa_2] = \rho[\kappa_1] \curlywedge_3 \rho[\kappa_2]$ for all $\kappa_1, \kappa_2 \in \Bbbk_1$.
Let $\mathcal{W} = \rho[\mathcal{L}]$, where $\mathcal{W}$ is an MFCVSBS of $\Bbbk_2$. Now, $\coprod_i \circ \top^l_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \coprod_i \circ \top^l_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \top^l_{\mathcal{W}}[\rho[\kappa_1]], \coprod_i \circ \top^l_{\mathcal{W}}[\rho[\kappa_2]]\} = \min\{\coprod_i \circ \top^l_{\aleph_\mathcal{L}}[\kappa_1], \coprod_i \circ \top^l_{\aleph_\mathcal{L}}[\kappa_2]\}$ and
$\coprod_i \circ \top^u_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \coprod_i \circ \top^u_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \top^u_{\mathcal{W}}[\rho[\kappa_1]], \coprod_i \circ \top^u_{\mathcal{W}}[\rho[\kappa_2]]\}$
$= \min\{\coprod_i \circ \top^u_{\aleph_\mathcal{L}}[\kappa_1], \coprod_i \circ \top^u_{\aleph_\mathcal{L}}[\kappa_2]\}$. Thus, $\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_1], \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_2]\}$.
Now, $\imath - \coprod_i \circ F^l_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \imath - \coprod_i \circ F^l_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\imath - [F^l_{\mathcal{W}}\rho[\kappa_1]], \imath - [F^l_{\mathcal{W}}\rho[\kappa_2]]\} = \min\{\imath - \coprod_i \circ F^l_{\aleph_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F^l_{\aleph_\mathcal{L}}[\kappa_2]\}$ and $\imath - \coprod_i \circ F^u_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \imath - \coprod_i \circ F^u_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\imath - [F^u_{\mathcal{W}}\rho[\kappa_1]], \imath - [F^u_{\mathcal{W}}\rho[\kappa_2]]\} = \min\{\imath - \coprod_i \circ F^u_{\aleph_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F^u_{\aleph_\mathcal{L}}[\kappa_2]\}$. Thus, $\imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \min\{\imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_2]\}$. Hence, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_2]\}$. Similarly, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \curlyvee_2 \kappa_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_2]\}$ and $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \curlyvee_3 \kappa_2] \geq Min^i\{\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1], \coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_2]\}$.
Now, $\coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \coprod_i \circ \top_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\coprod_i \circ \top_{\mathcal{W}}[\rho[\kappa_1]], \coprod_i \circ \top_{\mathcal{W}}[\rho[\kappa_2]]\}$
$= \min\{\coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_1], \coprod_i \circ \top_{\beth_\mathcal{L}}[\kappa_2]\}$ and $\imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] = \imath - \coprod_i \circ F_{\mathcal{W}}[\rho[\kappa_1] \curlywedge_1 \rho[\kappa_2]] \geq \min\{\imath - [F_{\mathcal{W}}\rho[\kappa_1]], \imath - [F_{\mathcal{W}}\rho[\kappa_2]]\} = \min\{\imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_1], \imath - \coprod_i \circ F_{\beth_\mathcal{L}}[\kappa_2]\}$.
Hence, $\coprod_i \circ \beth_{\mathcal{L}}[\kappa_1 \curlyvee_{1,2,3} \kappa_2] \geq \min\{\coprod_i \circ \beth_{\mathcal{L}}[\kappa_1], \coprod_i \circ \beth_{\mathcal{L}}[\kappa_2]\}$. Therefore, $\mathcal{L}$ is an MFCVSBS of $\Bbbk_1$. □

Theorem 4.3. *If ρ is a homomorphism, then $\mathcal{L}_{[\zeta,\sigma]}$ is a level SBS of an MFCVSBS.*

Proof. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be a homomorphism. Then $\rho[\kappa_1 \curlyvee_1 \kappa_2] = \rho[\kappa_1] \curlywedge_1 \rho[\kappa_2], \rho[\kappa_1 \curlyvee_2 \kappa_2] = \rho[\kappa_1] \curlywedge_2 \rho[\kappa_2]$ and $\rho[\kappa_1 \curlyvee_3 \kappa_2] = \rho[\kappa_1] \curlywedge_3 \rho[\kappa_2]$ for all $\kappa_1, \kappa_2 \in \Bbbk_1$. Let $\mathcal{W} = \rho[\mathcal{L}]$, where $\mathcal{W}$ is an MFCVSBS of $\Bbbk_2$. By Theorem 4.2, $\mathcal{L}$ is an MFCVSBS of $\Bbbk_1$. Let $\rho[\mathcal{L}_{[\zeta,\sigma]}]$ be a level SBS of $\mathcal{W}$. Suppose $\rho[\kappa_1], \rho[\kappa_2] \in \rho[\mathcal{L}_{[\zeta,\sigma]}]$. Then $\rho[\kappa_1 \curlyvee_1 \kappa_2], \rho[\kappa_1 \curlyvee_2 \kappa_2]$ and $\rho[\kappa_1 \curlyvee_3 \kappa_2] \in \rho[\mathcal{L}_{[\zeta,\sigma]}]$. Now, $\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_1] = \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{W}}[\rho[\kappa_1]] \geq \overrightarrow{\zeta}, \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_2] = \coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{W}}[\rho[\kappa_2]] \geq \overrightarrow{\zeta}$. Then $\coprod_i \circ \overrightarrow{\top}_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \overrightarrow{\zeta}$ and $\imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_1] = \imath - \coprod_i \circ F_{\aleph_\mathcal{W}}[\rho[\kappa_1]] \geq \overrightarrow{\sigma}$, $\imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_2] = \imath - \coprod_i \circ F_{\aleph_\mathcal{W}}[\rho[\kappa_2]] \geq \overrightarrow{\sigma}$. Then $\imath - \coprod_i \circ F_{\aleph_\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq \overrightarrow{\sigma}$. Thus, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \curlyvee_1 \kappa_2] \geq [\overrightarrow{\zeta}, \overrightarrow{\sigma}]$. Similarly, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \curlyvee_2 \kappa_2] \geq$

$[\overrightarrow{\zeta}, \overrightarrow{\sigma}]$ and $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{L}}[\kappa_1 \vee_3 \kappa_2] \geq [\overrightarrow{\zeta}, \overrightarrow{\sigma}]$. Now, $\coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_1] = \coprod_i \circ \top_{\beth_{\mathcal{W}}}[\rho[\kappa_1]] \geq \zeta, \coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_2] = \coprod_i \circ \top_{\beth_{\mathcal{W}}}[\rho[\kappa_2]] \geq \zeta$. Then $\coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \zeta$ and $\imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_1] = \imath - \coprod_i \circ F_{\beth_{\mathcal{W}}}[\rho[\kappa_1]] \geq \sigma$, $\imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_2] = \imath - \coprod_i \circ F_{\beth_{\mathcal{W}}}[\rho[\kappa_2]] \geq \sigma$. Then $\imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \sigma$. Thus, $\coprod_i \circ \beth_{\mathcal{L}}[\kappa_1 \vee_{1,2,3} \kappa_2] \geq [\zeta, \sigma]$. Hence, $\mathcal{L}_{[\zeta,\sigma]}$ is a level SBS of an MFCVSBS $\mathcal{L}$ of $\Bbbk_1$. □

Theorem 4.4. *If ρ is a homomorphism, then $\rho[\mathcal{L}_{[\zeta,\sigma]}]$ is a level SBS of an MFCVSBS.*

Proof. Let $\rho : \Bbbk_1 \to \Bbbk_2$ be a homomorphism. Then $\rho[\kappa_1 \vee_1 \kappa_2] = \rho[\kappa_1] \wedge_1 \rho[\kappa_2], \rho[\kappa_1 \vee_2 \kappa_2] = \rho[\kappa_1] \wedge_2 \rho[\kappa_2]$ and $\rho[\kappa_1 \vee_3 \kappa_2] = \rho[\kappa_1] \wedge_3 \rho[\kappa_2]$ for all $\kappa_1, \kappa_2 \in \Bbbk_1$. Let $\mathcal{W} = \rho[\mathcal{L}]$, where $\mathcal{L}$ is an MFCVSBS of $\Bbbk_1$. By Theorem 4.1, $\mathcal{W}$ is an MFCVSBS of $\Bbbk_2$. Let $\mathcal{L}_{[\zeta,\sigma]}$ be a level SBS of $\mathcal{L}$. Suppose $\kappa_1, \kappa_2 \in \mathcal{L}_{[\zeta,\sigma]}$. Then $\rho[\kappa_1 \vee_1 \kappa_2], \rho[\kappa_1 \vee_2 \kappa_2]$ and $\rho[\kappa_1 \vee_3 \kappa_2] \in \mathcal{L}_{[\zeta,\sigma]}$. Now, $\coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{W}}}[\rho[\kappa_1]] \geq \coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{L}}}[\kappa_1] \geq \overrightarrow{\zeta}, \coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{W}}}[\rho[\kappa_2]] \geq \coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{L}}}[\kappa_2] \geq \overrightarrow{\zeta}$. Then $\coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{W}}}[\rho[\kappa_1] \wedge_1 \rho[\kappa_2]] \geq \coprod_i \circ \overrightarrow{\top}_{\aleph_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \overrightarrow{\zeta}$, for all $\rho[\kappa_1], \rho[\kappa_2] \in \Bbbk_2$. And $\imath - \coprod_i \circ F_{\aleph_{\mathcal{W}}}[\rho[\kappa_1]] \geq \imath - \coprod_i \circ F_{\aleph_{\mathcal{L}}}[\kappa_1] \geq \overrightarrow{\sigma}, \imath - \coprod_i \circ F_{\aleph_{\mathcal{W}}}[\rho[\kappa_2]] \geq \imath - \coprod_i \circ F_{\aleph_{\mathcal{L}}}[\kappa_2] \geq \overrightarrow{\sigma}$. Then $\imath - \coprod_i \circ F_{\aleph_{\mathcal{W}}}[\rho[\kappa_1] \wedge_1 \rho[\kappa_2]] \geq \imath - \coprod_i \circ F_{\aleph_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \overrightarrow{\sigma}$, for all $\rho[\kappa_1], \rho[\kappa_2] \in \Bbbk_2$. Thus, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \wedge_1 \rho[\kappa_2]] \geq [\overrightarrow{\zeta}, \overrightarrow{\sigma}]$. Similarly, $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \wedge_2 \rho[\kappa_2]] \geq [\overrightarrow{\zeta}, \overrightarrow{\sigma}]$ and $\coprod_i \circ \overrightarrow{\aleph}_{\mathcal{W}}[\rho[\kappa_1] \wedge_3 \rho[\kappa_2]] \geq [\overrightarrow{\zeta}, \overrightarrow{\sigma}]$. Now, $\coprod_i \circ \top_{\beth_{\mathcal{W}}}[\rho[\kappa_1]] \geq \coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_1] \geq \zeta, \coprod_i \circ \top_{\beth_{\mathcal{W}}}[\rho[\kappa_2]] \geq \coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_2] \geq \zeta$. Then $\coprod_i \circ \top_{\beth_{\mathcal{W}}}[\rho[\kappa_1] \wedge_1 \rho[\kappa_2]] \geq \coprod_i \circ \top_{\beth_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \zeta$, for all $\rho[\kappa_1], \rho[\kappa_2] \in \Bbbk_2$. And $\imath - \coprod_i \circ F_{\beth_{\mathcal{W}}}[\rho[\kappa_1]] \geq \imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_1] \geq \sigma, \imath - \coprod_i \circ F_{\beth_{\mathcal{W}}}[\rho[\kappa_2]] \geq \imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_2] \geq \sigma$. Then $\imath - \coprod_i \circ F_{\beth_{\mathcal{W}}}[\rho[\kappa_1] \wedge_1 \rho[\kappa_2]] \geq \imath - \coprod_i \circ F_{\beth_{\mathcal{L}}}[\kappa_1 \vee_1 \kappa_2] \geq \sigma$, for all $\rho[\kappa_1], \rho[\kappa_2] \in \Bbbk_2$. Thus, $\coprod_i \circ \beth_{\mathcal{W}}[\rho[\kappa_1] \wedge_{1,2,3} \rho[\kappa_2]] \geq [\zeta, \sigma]$. Hence, $\rho[\mathcal{L}_{[\zeta,\sigma]}]$ is a level SBS of an MFCVSBS $\mathcal{W}$ of $\Bbbk_2$. □

5. Conclusion

Presenting a vague subsemiring of semirings to MFCVSBS of bisemirings is the primary objective of this effort. We determine that $\mathcal{Z}$ is a multi-fuzzy cubic vague fuzzy set if and only if $\wp$ is an MFCVSBS of $\Bbbk \times \Bbbk$. The homomorphic image and pre-image are also defined. We discovered that MFCVSBS is an MFCVSBS homomorphic image and pre-image. Therefore, the ordered MFCVSBS, cubic soft subbisemirings, and multi-fuzzy cubic vague soft subbisemirings of bisemirings and their applications should be taken into consideration in the future.

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