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Application of the Fourth-Order Differential Subordination and Superordination Results for Analytic Functions Associated With an Operator

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Abstract. This study focuses on differential subordination using arithmetic and geometric approaches when the dominant function is linear. In addition to the results of differential subordination of arithmetic and geometric means in which a convex function was dominant, one can study such differential subordination for a selected convex function. We investigate several results of the differential subordinations of analytic functions are associated with an operator built using arithmetic and geometric means.

1. Introduction and Preliminaries

Let $\mathcal{K}(\mathcal{U})$ be the class of analytic functions are in the open unit disk $\mathcal{U} = \{z : |z| < 1; z \in \mathbb{C}\}$. For n being a positive integer and $a \in \mathbb{C}$, we let $\mathcal{K}[a,l] = \{f \in \mathcal{K}(\mathcal{U}) \;:\; f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$ and $\mathcal{K}[1,n] = \mathcal{K}_n$. Let $\mathcal{A}$ be the class of all analytic functions in $\mathcal{U}$ and usually normalized by

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in \mathcal{U}). \tag{1.1}$$

For functions f_1 and f_2 belonging to the class $\mathcal{H}(\mathcal{U})$, we say that f_1 is said to be subordinate to f_2, and write $f_1 \prec f_2$, if there exists a Schwarz function $\psi(z)$, which, by definition, is analytic in $\mathcal{U}$ with $\psi(0) = 0$ and $|\psi(z)| < 1 \quad (z \in \mathcal{U})$, such that $f_1 = f_2(\psi(z)) \quad (z \in \mathcal{U})$. It is known

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that $f_1 \prec f_2 \quad (z \in \mathcal{U}) \Longrightarrow f_1(0) = f_2(0)$ and $f_1(\mathcal{U}) \subset f_2(\mathcal{U})$. Furthermore, if the function f_2 is univalent in $\mathcal{U}$, then $f_1 \prec f_2 \Longleftrightarrow f_1(0) = f_2(0)$ and $f_1(\mathcal{U}) \subset f_2(\mathcal{U})$ (see, for details, [20,22]).

We remember that, the generalized Bessel function $\varphi_{\gamma,k}(z)$ of the first kind of order p was introduced by Deniz [15] and Deniz *et al.* [16] (see, also [9–11]) by

$$\varphi_{\gamma,k}(z) = z + \sum_{n=1}^{\infty} \frac{(-\gamma)^n}{4^n (k)_n} \frac{z^{n+1}}{n!} \quad \left(k = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where $\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$ and $(\delta)_m$ is the Pochhammer symbol:

$$(\delta)_m = \frac{\Gamma(\delta + m)}{\Gamma(\delta)} = \begin{cases} 1, & if\ m = 0, \\ \delta(\delta+1)\cdots(\delta+m-1), & if\ m \in \mathbb{N}. \end{cases}$$

Also, according to [12], the definition of the Carlson-Shaffer Operator $\mathcal{L}(\alpha,\beta)f(z)$ is given by

$$\mathcal{L}(\alpha,\beta)f(z) = \theta(\alpha,\beta;z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\beta)_n} a_{n+1} z^{n+1},$$

where $\theta(\alpha,\beta;z)$ is the incomplete beta function

$$\theta(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^{n+1} \quad (\beta \neq 0, -1, -2, \cdots \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ z \in \mathcal{U}).$$

Now, we defined the operator as follows:

Definition 1.1. *[19] For $f \in \mathcal{A}$. The operator $\mathcal{D}_{\alpha,\beta}^{\gamma,k} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined for each nonnegative integer n and for any $z \in \mathcal{U}$ as the Hadamard product between the Carlson-Shaffer Operator $\mathcal{L}(\alpha,\beta)$ and the generalized Bessel functions $\varphi_{\gamma,k}(z)$:*

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z) = \varphi_{\gamma,k}(z) * \mathcal{L}(\alpha,\beta)f(z) = z + \sum_{n=1}^{\infty} \frac{(-\gamma)^n (\alpha)_n a_{n+1}}{4^n (k)_n (\beta)_n} \frac{z^{n+1}}{n!}, \tag{1.2}$$

where $\beta \neq 0, -1, -2, \cdots \in \mathbb{C} \setminus \mathbb{Z}_0^-; \gamma, \alpha \in \mathbb{C}; k = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$.

On simple computation of (1.2) the identity relation gives,

$$z\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z)\right)' = k\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z) - (k-1)\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z). \tag{1.3}$$

Presently, we offer the background of widely recognized symbols and definitions utilized in deriving the primary findings.

Definition 1.2. *[3] Let $\mathcal{Q}$ be the set of all functions ϱ that are analytic and univalent on $\overline{\mathcal{U}} \setminus E(\varrho)$ where*

$$E(\varrho) = \left\{\zeta \in \partial\mathcal{U} : \lim_{\omega \longrightarrow \zeta} \varrho(\omega) = \infty\right\},$$

and are such that $\min |\varrho'(\zeta)| = \rho > 0$ for $\zeta \in \partial\mathcal{U} \setminus E(\varrho)$. Further, let $\mathcal{Q}(a)$ denote the subclass of $\mathcal{Q}$ consisting of functions ϱ for which $\varrho(0) = a$ and $\mathcal{Q}(1) = \mathcal{Q}_1 = \{\varrho(z) \in \mathcal{Q} : \varrho(0) = 1\}$.

Definition 1.3. *[6] Assume that h is univalent in $\mathcal{U}$ and $\psi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$. If the analytic function g fulfills the fourth-order differential subordination*

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) \prec h(z) \quad (z \in \mathcal{U}), \tag{1.4}$$

then the function g is called a solution of the differential subordination (1.4). A univalent function ϱ is called a dominant of the solutions of the differential subordination if $g \prec \varrho$ for all g satisfying (1.4). A dominant $\tilde{\varrho}(z)$ that fulfils $\tilde{\varrho} \prec \varrho$ for all dominants $\tilde{\varrho}$ of (1.4) is called the best dominant.

Definition 1.4. *[6] If $\Omega \subseteq \mathbb{C}$, $\varrho \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{2\}$. Let $\Psi_j[\Omega, \varrho]$ be the family of admissible functions consisting of functions $\psi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$, which fulfill the admissibility condition:*

$$\psi(r, s, t, u, v; z) \notin \Omega$$

whenever

$$r = \varrho(\zeta), \quad s = m\zeta\varrho'(\zeta), \quad \Re\left\{\frac{t}{s} + 1\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{u}{s}\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \text{ and } \Re\left\{\frac{v}{s}\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(\varrho)$ and $m \geq n$.

Lemma 1.1. *[6] Let $g \in \mathcal{H}[a, n]$ with $n \geq 3$. Furthermore, let $\varrho \in \mathcal{Q}$ and fulfill the following conditions:*

$$\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0 \text{ and } \left|\frac{z^2g''(z)}{\varrho'(\zeta)}\right| \leq m^2,$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(\varrho)$ and $m \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_j[\Omega, \varrho]$ and

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) \in \Omega,$$

then

$$g(z) \prec \varrho(z) \quad (z \in \mathcal{U}).$$

Definition 1.5. *[7] Assume $\psi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ and h be analytic in $\mathcal{U}$. If $p(z)$ and*

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z\right),$$

are univalent in $\mathcal{U}$ and satisfy the following fourth-order differential superordination

$$h(z) \prec \psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z\right), \tag{1.5}$$

then $p(z)$ is called a solution of the differential superordination. The analytic function $\varrho(z)$ is called a subordinate of the solutions of the differential superordination or more simply a subordinate if $\varrho(z) \prec p(z)$ for all $p(z)$ satisfying (1.5). A univalent subordinate $\tilde{\varrho}(z)$ that satisfies the condition $\varrho(z) \prec \tilde{\varrho}(z)$ for all subordinates $\varrho(z)$ of (1.5) is said to be the best subordinate. We note that the best subordinate is unique up to a rotation of $\mathcal{U}$.

Definition 1.6. *[7] Assume $g \in \mathcal{H}[a, j]$, $\varrho'(z) \neq 0$ and Ω is a set in $\mathbb{C}$. The class of admissible functions $\Phi'_n[\Omega, \varrho]$ consists of those functions: $\psi : \mathbb{C}^5 \times \overline{\mathcal{U}} \longrightarrow \mathbb{C}$, that satisfy the following admissible condition:*

$$\psi(r, s, t, u, v; z) \notin \Omega$$

whenever

$$r = \varrho(\zeta), s = \frac{1}{\lambda}\zeta\varrho'(\zeta), \Re\left\{\frac{t}{s} + 1\right\} \geq \frac{1}{\lambda}\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{u}{s}\right\} \geq \frac{1}{\lambda^2}\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\}, \Re\left\{\frac{v}{s}\right\} \geq \frac{1}{\lambda^3}\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$ and $\lambda \geq n \geq 3$.

Lemma 1.2. *[7] Assume that $\psi \in \Phi'_n[\Omega, \varrho]$ and $\varrho(z) \in \mathcal{H}[a, j]$. If*

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z\right)$$

is univalent in $\mathcal{U}$ and $p(z) \in \mathcal{Q}(a)$ satisfy the conditions

$$\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0, \left|\frac{z^2\varrho''(z)}{\varrho'(\zeta)}\right| \leq \frac{1}{\lambda^2},$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$ and $k \geq n \geq 3$, then

$$\Omega \subset \left\{\left(\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z\right), z \in \mathcal{U}\right)\right\}$$

thus, $\varrho(z) \prec p(z) \quad (z \in \mathcal{U})$.

This paper's investigation utilizes the established principles of differential subordination and differential superordination. Miller and Mocanu offer a thorough explanation of the theory of differential subordination in their publication in 2000 [21]. Additionally, they introduced the concept of differential superordination as a dual concept to that of subordination. This concept was also unveiled by the authors in 2003 [22]. In 1992, Ponnusamy and Juneja [23] presented the ideas of third-order differential inequalities within the complex plane, followed by the introduction of third-order differential subordination theory by Antonino and Miller [3] in 2011. In 2020, Atshan et al. [6] presented and explored the ideas behind fourth-order differential subordination, which extends the findings of third-order differential subordination discovered by Antonino and Miller [3]. Within this area, the number of publications focusing on fourth-order differential subordination issues is quite limited (see, for examples, [6, 7, 17]). Through the application of third-order differential subordination and superordination theories (referenced in [1, 2, 8, 13, 14, 18, 24, 26, 27]), numerous researchers have discovered a range of intriguing outcomes related to both linear and nonlinear operators, as well as the sandwich-type results for analytic functions (referenced in [4,5,25]). Their contributions have spurred and inspired additional progress in this area.

The field of complex analysis has significantly advanced through the study of differential subordination and superordination, particularly in geometric function theory. While second-order cases have been extensively explored, higher-order cases, such as fourth-order differential subordination and superordination, remain relatively uncharted. This paper bridges this gap by investigating the applications of fourth-order differential subordination and superordination results for analytic functions associated with a specific operator, focusing on particular admissible functions within the unit disk $\mathcal{U}$.

This document aims to explore various outcomes related to fourth-order differential subordination and superordination, focusing on particular acceptable types of admissible functions linked to a new operator introduced in (1.2) of an analytic functions within $\mathcal{U}$.

The main body of the paper is organized as follows. In Section 2, we devoted to providing some preliminaries related to notations and lemmas that will be used in this paper. In Section 3, we discuss the main results of fourth-order differential subordination associated with the new operator by employing a suitable admissible class. Also, we investigated the results of fourth-order differential superordination associated with the new operator and combined them with the results of Section 3 to get sandwich-type results in Section 4. Finally, Section 5 provides the conclusion of the work along with the future study to do in this field.

2. Fourth-order subordination results

We need the class of admissible functions to prove the differential subordination theorems using the operator $\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)$ defined by (1.2).

Definition 2.1. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in \mathcal{Q}_1 \cap \mathcal{K}_n$. Let $\Phi_1[\Omega, \varrho]$ be the family of admissible functions which consists of functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a, b, x, y, w; z) \notin \Omega,$$

whenever $a = \varrho(\zeta), \quad b = \frac{m\zeta\varrho'(\zeta)+(k+2)\varrho(z)}{(k+3)}$,

$$\Re\left\{\frac{(k+3)[b+(k+2)x]-(k+2)^2 a}{(k+3)b-(k+2)a} - 2(k+2)\right\} \geq m\Re\left\{1+\frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{(k+2)(k+3)[(k+1)y-3(k+2)x+2(k+1)a]}{(k+3)b-(k+2)a} + 3(k+2)(k+3)\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\} \text{ and}$$

$$\Re\left\{\frac{k(k+1)(k+2)(k+3)w-4(k+1)(k+2)^2(k+3)y+6(k+2)^2(k+3)^2 x}{(k+3)b-(k+2)a} - \frac{3(k+1)(k+2)(k+3)(k+4)a}{(k+3)b-(k+2)a} - 4(k+2)(k+3)(k+4)\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $z \in \mathcal{U}, \zeta \in \partial\mathcal{U} \setminus E(\varrho)$ and $m \geq 3$.

Theorem 2.1. *Suppose that $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_1[\Omega, \varrho]$. Let $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right) \geq 0, \quad \left|\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{\varrho'(\zeta)}\right)\right| \leq m^2 \tag{2.1}$$

and

$$\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z);z\right)\subset\Omega, \tag{2.2}$$

then

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)\prec\varrho(z)\quad(z\in\mathcal{U}).$$

Proof. Let $g(z)$ be the function in $\mathcal{U}$ defined by

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)=g(z)\quad(z\in\mathcal{U}). \tag{2.3}$$

Differentiating (2.3) on both sides with respect to z and by using the recurrence relation (1.3), we get

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)=\frac{zg'(z)+(k+2)g(z)}{(k+3)}. \tag{2.4}$$

Again, differentiating (2.4) on both sides with respect to z and using the relation (1.3), we have

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)=\frac{z^2g''(z)+2(k+2)zg'(z)+(k+1)(k+2)g(z)}{(k+2)(k+3)}. \tag{2.5}$$

Similarly, further computations we obtain

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)=\frac{z^3g'''(z)+3(k+2)z^2g''(z)+3(k+1)(k+2)zg'(z)+k(k+1)(k+2)g(z)}{(k+1)(k+2)(k+3)} \tag{2.6}$$

and

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)=\frac{z^4g''''(z)+4(k+2)z^3g'''(z)+6(k+1)(k+2)z^2g''(z)}{k(k+1)(k+2)(k+3)}+\frac{4k(k+1)(k+2)zg'(z)+k(k^2-1)(k+2)g(z)}{k(k+1)(k+2)(k+3)}. \tag{2.7}$$

Let $\mathtt{a}=\mathtt{r}$, $\mathtt{b}=\frac{\mathtt{s}+(k+2)\mathtt{r}}{(k+3)}$, $\mathtt{x}=\frac{\mathtt{t}+(k+2)[2\mathtt{s}+(k+1)\mathtt{r}]}{(k+2)(k+3)}$, $\mathtt{y}=\frac{\mathtt{u}+(k+2)[3\mathtt{t}+3(k+1)\mathtt{s}+k(k+1)\mathtt{r}]}{(k+1)(k+2)(k+3)}$,

and $\mathtt{w}=\dfrac{\mathtt{v}+(k+2)[4\mathtt{u}+6(k+1)\mathtt{t}+k(k+1)\mathtt{s}+k(k^2-1)\mathtt{r}]}{k(k+1)(k+2)(k+3)}$.

Let us define the transformation $\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z):\mathbb{C}^5\times\mathcal{U}\longrightarrow\mathbb{C}$ by $\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z)=$

$$\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z)=\phi\left(\mathtt{r},\frac{\mathtt{s}+(k+2)\mathtt{r}}{(k+3)},\frac{\mathtt{t}+(k+2)[2\mathtt{s}+(k+1)\mathtt{r}]}{(k+2)(k+3)},\frac{\mathtt{u}+3(k+2)\mathtt{t}}{(k+1)(k+2)(k+3)}\right.$$
$$\left.\frac{(k+2)(k+1)[3\mathtt{s}+k\mathtt{r}]}{(k+1)(k+2)(k+3)},\frac{\mathtt{v}+(k+2)[4\mathtt{u}+(k+1)\{6\mathtt{t}+k\mathtt{s}+k(k-1)\mathtt{r}\}]}{k(k+1)(k+2)(k+3)};z\right). \tag{2.8}$$

Using the above equations (2.3) to (2.7), we get from (2.8) that

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)$$
$$=\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z);z\right). \tag{2.9}$$

Consequently, (2.2) convert into

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)\in\Omega.$$

We notice that

$$\frac{\mathtt{t}}{\mathtt{s}}+1=\frac{(k+3)[\mathtt{b}+(k+2)\mathtt{x}]-(k+2)^2\mathtt{a}}{(k+3)\mathtt{b}-(k+2)\mathtt{a}}-2(k+2),$$

$$\frac{\mathtt{u}}{\mathtt{s}}=\frac{(k+2)(k+3)[(k+1)\mathtt{y}-3(k+2)\mathtt{x}+2(k+1)\mathtt{a}]}{(k+3)\mathtt{b}-(k+2)\mathtt{a}}+3(k+2)(k+3)$$

and

$$\frac{\mathtt{v}}{\mathtt{s}}=\frac{k(k+1)(k+2)(k+3)\mathtt{w}-4(k+1)(k+2)^2(k+3)\mathtt{y}}{(k+3)\mathtt{b}-(k+2)\mathtt{a}} + \frac{6(k+2)^2(k+3)^2\mathtt{x}-3(k+1)(k+2)(k+3)(k+4)\mathtt{a}}{(k+3)\mathtt{b}-(k+2)\mathtt{a}}-4(k+2)(k+3)(k+4).$$

As the admissibility condition for the function $\phi\in\Phi_1[\Omega,\varrho]$ of Definition 2.1 and the admissibility condition for the function $\psi\in\Psi_j[\Omega,\varrho]$ are equivalent, hence by using Lemma 1.1, we get $g(z)\prec\varrho(z)$ or

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)\prec\varrho(z)\quad(z\in\mathcal{U}).$$

□

If the behavior of the function $\varrho(\omega)$ on $\partial\mathcal{U}$ is unknown, then the extension of Theorem 2.1 can be obtained by the following result.

Corollary 2.1. *If $\Omega\subseteq\mathbb{C}$ and ϱ is univalent in $\mathcal{U}$ with $\varrho\in\mathcal{Q}_1$. Let $\phi\in\Phi_1[\Omega,\varrho_\rho]$ for some $\rho\in(0,1)$,where $\varrho_\rho(z)=\varrho(\rho z)$. If $f\in\mathcal{A}$ and $\varrho_\rho(z)$ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta^2\varrho_\rho'''(\zeta)}{\varrho_\rho'(\zeta)}\right\}\geq 0\quad\text{and}\quad\left|\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{\varrho_\rho'(\zeta)}\right)\right|\leq m^2\quad(z\in\Delta)\tag{2.10}$$

and

$$\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z),\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z);z\right)\subset\Omega,$$

then

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)\prec\varrho(z)\quad(z\in\mathcal{U}).$$

Proof. From Theorem 2.1, we notice that $\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)\prec\varrho_\rho(z)\quad(z\in\mathcal{U})$. The conclusion assumed by Corollary 2.1 is now implied from the following subordination relationship:

$$\varrho_\rho(z)\prec\varrho(z)\quad(z\in\mathcal{U}).$$

□

Suppose Ω is a simply connected domain such that $\Omega\neq\mathbb{C}$, then for some conformal mapping h from $\mathcal{U}$ into the domain Ω, we have $\Omega=h(\mathcal{U})$. We give the notation for the class $\Phi[h(\Delta),\varrho]$ by $\Phi[h,\varrho]$. The following two outcomes are the actual consequences of Theorem 2.1 and Corollary 2.1.

Theorem 2.2. *Let $\phi \in \Phi[h, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in Q_1$ satisfies (2.1) and*

$$\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z); z\right) \prec h(z), \tag{2.11}$$

then

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \prec \varrho(z) \quad (z \in \mathcal{U}).$$

Corollary 2.2. *If ϱ is univalent function in $\mathcal{U}$ with $\varrho \in Q_1$ and $\phi \in \Phi[h, \varrho_\rho]$ for some $\varrho \in (0, 1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If $f \in \mathcal{A}$ and ϱ_ρ satisfies (2.11) and*

$$\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z); z\right) \prec h(z),$$

then

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \prec \varrho(z) \quad (z \in \mathcal{U}).$$

Now, the next theorem given below will give the best dominant of the differential subordination (2.11).

Theorem 2.3. *Assume that h is univalent in $\mathcal{U}$. Again let $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ and the differential equation*

$$\begin{aligned} \phi\Bigg(\varrho(z), \frac{z\varrho'(z) + (k+2)\varrho(z)}{(k+3)}, \frac{z^2\varrho''(z) + 2(k+2)z\varrho'(z) + (k+1)(k+2)\varrho(z)}{(k+2)(k+3)}, \\ \frac{z^3\varrho'''(z) + (k+2)[3z^2\varrho''(z) + (k+1)\{3z\varrho'(z) + k\varrho(z)\}]}{(k+1)(k+2)(k+3)}, \frac{z^4\varrho''''(z) + 4(k+2)z^3\varrho'''(z)}{k(k+1)(k+2)(k+3)} \\ + \frac{(k+2)[6(k+1)z^2\varrho''(z) + 4k(k+1)z\varrho'(z) + k(k^2-1)\varrho(z)]}{k(k+1)(k+2)(k+3)}; z\Bigg) = h(z), \end{aligned} \tag{2.12}$$

has a solution $\varrho(z)$ with $\varrho(0) = 1$ and $\varrho(z)$ satisfies the condition (2.1). If $f \in \mathcal{A}$, $\phi \in \Phi[h, \varrho_\rho]$ and

$$\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z); z\right)$$

is analytic in $\mathcal{U}$, then (2.11) implies that

$$\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \prec \varrho(z) \quad (z \in \mathcal{U})$$

and $\varrho(z)$ is the best dominant.

Proof. Since, we know that $\varrho(z)$ is a solution of (2.11) so, $\varrho(z)$ satisfies (2.12). Now, it can be prove that $\varrho(z)$ is a dominant of equation (2.11) by applying Theorem 2.2 and hence $\varrho(z)$ will be dominated by all dominants. Therefore $\varrho(z)$ is the best dominant. □

Now, putting $\varrho(z) = \mathtt{M}z, \mathtt{M} > 0$, and using the Definition 2.1, we denote the class of admissible function $\phi[\Omega, \varrho]$, by $\phi[\Omega, \mathtt{M}]$, is decorated below.

Definition 2.2. *Let $\Omega \subseteq \mathbb{C}$ and $\mathtt{M} > 0$. The family of admissible functions $\Phi[\Omega, \mathtt{M}]$ consists of the functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$, which satisfy the following admissibility condition*

$$\phi\Bigg(\mathtt{M}e^{i\theta}, \frac{[\mathtt{n}+(k+2)]}{(k+3)}\mathtt{M}e^{i\theta}, \frac{\mathtt{L}+(k+2)[2\mathtt{n}+(k+1)]\mathtt{M}e^{i\theta}}{(k+2)(k+3)}, \frac{\mathtt{N}+(k+2)\{3\mathtt{L}+(k+1)[3\mathtt{n}+k]\}\mathtt{M}e^{i\theta}}{(k+1)(k+2)(k+3)}, \frac{\mathtt{X}+4(k+2)\mathtt{N}+6(k+1)(k+2)\mathtt{L}+k(k+1)(k+2)[4\mathtt{n}+(k-1)]\mathtt{M}e^{i\theta}}{k(k+1)(k+2)(k+3)}; z\Bigg) \notin \Omega$$

whenever $\omega \in \mathcal{U}$, $\Re\left\{\mathtt{L}e^{-i\theta}\right\} \geq (\mathtt{n}-1)\mathtt{n}\mathtt{M}$, $\Re\left\{\mathtt{N}e^{-i\theta}\right\} \geq 0$ and $\Re\left\{\mathtt{X}e^{-i\theta}\right\} \geq 0$ for every $\theta \in \mathbb{R}$ and $\mathtt{n} \geq 3$.

The following results are obtained by applying the definition of the family of admissible functions and from the outcomes in Theorem 2.1.

Theorem 2.4. *Suppose that $\phi \in \Phi[\Omega, \mathtt{M}]$. Let $f \in \mathcal{A}$ satisfy the conditions:*

$$\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| \leq n^2\mathtt{M}, n \geq 3, \mathtt{M} > 0,$$

and

$$\phi\left(\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+3}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+2}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+1}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k}_{\alpha,\beta} f(z); z\right) \in \Omega,$$

then

$$\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| < \mathtt{M} \quad (z \in \mathcal{U}).$$

Now, by assuming $\Omega = \varrho(\mathcal{U}) = \{w : |w| < \mathtt{M}\}$, the class $\Phi[\Omega, \mathtt{M}]$ is directly denoted by $\Phi[\mathtt{M}]$.

Corollary 2.3. *Suppose $\phi \in \Phi[\mathtt{M}]$. Let $f \in \mathcal{A}$ verifies $\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| \leq n^2 M \quad (n \geq 3;\ \mathtt{M} > 0)$ and*

$$\left|\phi\left(\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+3}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+2}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+1}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k}_{\alpha,\beta} f(z); z\right)\right| < \mathtt{M},$$

then

$$\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| < \mathtt{M} \quad (z \in \mathcal{U}).$$

Corollary 2.4. *Let $n \geq 3$ and $\mathtt{M} > 0$ If $f \in \mathcal{A}$ satisfies $\left|\mathcal{D}^{\gamma,k+3}_{\alpha,\beta} f(z)\right| < \mathtt{M}$, then*

$$\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| < \mathtt{M} \quad (z \in \mathcal{U}).$$

Proof. This follows from Corollary 2.3 by taking $\phi(\mathtt{a}, \mathtt{b}, \mathtt{x}, \mathtt{y}, \mathtt{w}; z) = \mathtt{b} = \frac{\mathtt{n}+(k+2)}{(k+3)}\mathtt{M}e^{i\theta}$. □

Theorem 2.5. *Let us assume that $n \geq 3$, $\mathtt{M} > 0$. Suppose $f \in \mathcal{A}$ satisfies the conditions $\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| \leq n^2 M$ and*

$$\left|k(k+1)(k+2)(k+3)\mathcal{D}^{\gamma,k}_{\alpha,\beta} f(z) - (k^2-1)(k+2)(k+3)\mathcal{D}^{\gamma,k+1}_{\alpha,\beta} f(z)\right| < h(z),$$

then

$$\left|\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)\right| < \mathtt{M} \quad (z \in \mathcal{U}).$$

Proof. Let us take $\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z) = k(k+1)(k+2)(k+3)\mathtt{w} - (k^2-1)(k+2)(k+3)\mathtt{y}$, $\Omega = h(\mathcal{U})$ such that

$$h(z) = \Big(|6+11k+6k^2+k^3| + 3|(6+5k+k^2)|\Big)3\mathtt{M}z.$$

Now, by applying the Theorem 2.4, we able to show that $\phi \in \Phi[\Omega,\mathtt{M}]$. Because

$$\begin{aligned}
&\left|\phi\left(\mathtt{M}e^{i\theta}, \frac{[\mathtt{n}+(k+2)]}{(k+3)}\mathtt{M}e^{i\theta}, \frac{\mathtt{L}+(k+2)[2\mathtt{n}+(k+1)]\mathtt{M}e^{i\theta}}{(k+2)(k+3)}, \frac{\mathtt{N}+(k+2)\{3\mathtt{L}+(k+1)[3\mathtt{n}+k]\}\mathtt{M}e^{i\theta}}{(k+1)(k+2)(k+3)},\right.\right.\\
&\quad\left.\left.\frac{\mathtt{X}+4(k+2)\mathtt{N}+6(k+1)(k+2)\mathtt{L}+k(k+1)(k+2)[4\mathtt{n}+(k-1)]\mathtt{M}e^{i\theta}}{k(k+1)(k+2)(k+3)};z\right)\right|\\
&\quad= |\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z)|.
\end{aligned}$$

Subsequently,

$$\begin{aligned}
|\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z)| &= k(k+1)(k+2)(k+3)\mathtt{w} - (k^2-1)(k+2)(k+3)\mathtt{y}\\
&= |\mathtt{X}+(9+3k)\mathtt{N}+(3k^2+15k+18)\mathtt{L}+(k^3+6k^2+11k+6)n\mathtt{M}e^{i\theta}|\\
&= \left|\mathtt{X}e^{-i\theta}+(9+3k)\mathtt{N}e^{-i\theta}+(3k^2+15k+18)\mathtt{L}e^{-i\theta}+(k^3+6k^2+11k+6)n\mathtt{M}\right|\\
&\geq \Re(\mathtt{X}e^{-i\theta})+|(9+3k)|\Re(\mathtt{N}e^{-i\theta})+|(3k^2+15k+18)|\Re(\mathtt{L}e^{-i\theta})+|(k^3+6k^2+11k+6)|n\mathtt{M}\\
&\geq |(k^3+6k^2+11k+6)|n\mathtt{M}+|(3k^2+15k+18)|n(n-1)\mathtt{M}\\
&\geq \Big(|(k^3+6k^2+11k+6)|+3|(3k^2+15k+18)|\Big)3\mathtt{M},
\end{aligned}$$

such that $\Re(\mathtt{A}e^{-i\theta}) \geq 0, \Re(\mathtt{N}e^{-i\theta}) \geq 0$ and $\Re(\mathtt{L}e^{-i\theta}) \geq (n-1)n\mathtt{M}$ for all $\theta \in \mathbb{R}, z \in \mathcal{U}$ and $n \geq 3$. The proof is complete. □

Definition 2.3. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in \mathcal{Q}_1 \cap \mathcal{H}_1$. Let $\Phi_1[\Omega,\varrho]$ be the family of admissible functions which consists of functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a,b,x,y,w;z) \notin \Omega,$$

whenever $a = \varrho(\zeta)$, $b = \frac{m\zeta\varrho'(\zeta)+(k+3)\varrho(z)}{(k+3)}$,

$$\begin{aligned}
&\Re\left\{\frac{(k+2)x+b-(k+3)a}{b-a} - 2(k+3)\right\} \geq m\Re\left\{1+\frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},\\
&\Re\left\{\frac{(k+2)[(k+1)y-3(k+3)x+2(k+4)a]}{b-a} + 3(k+3)(k+4)\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\},\\
&\Re\left\{\frac{(k+2)[k(k+1)w-4(k+1)(k+3)y+6(k+3)(k+4)x-3(k+4)(k+5)a]}{b-a}\right.\\
&\qquad\qquad\left.\frac{-4(k+3)(k+4)(k+5)(b-a)}{b-a}\right\} \geq m^3\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},
\end{aligned}$$

where $z \in \mathcal{U}, \zeta \in \partial\mathcal{U} \setminus E(\varrho)$ and $m \geq 3$.

Theorem 2.6. *Let us consider* $\Omega \subseteq \mathbb{C}$ *and* $\phi \in \Phi_1[\Omega, \varrho]$. *Let* $f \in \mathcal{A}$ *and* $\varrho \in \mathcal{Q}_1$ *fulfil the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right) \geq 0, \quad \left\|\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z\varrho'(z)}\right)\right\| \leq m^2 \tag{2.13}$$

and

$$\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z}; z\right) \subset \Omega, \tag{2.14}$$

then

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z} \prec \varrho(z) \quad (z \in \mathcal{U}).$$

Proof. Suppose the function $g(z)$ is defined in $\mathcal{U}$ by

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z} = g(z) \quad (z \in \mathcal{U}). \tag{2.15}$$

Differentiating the equation (2.15) with respect to z and making use of the recurrence relation (1.3), we get

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z} = \frac{zg'(z) + (k+3)g(z)}{(k+3)}. \tag{2.16}$$

Again, differentiating (2.16) on both sides with respect to z and using the identity relation (1.3), we have

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z} = \frac{z^2g''(z) + 2(k+3)zg'(z) + (k+2)(k+3)g(z)}{(k+2)(k+3)}. \tag{2.17}$$

Similarly, Further computations, we have

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z} = \frac{z^3g'''(z) + 3(k+3)z^2g''(z)}{(k+1)(k+2)(k+3)} + \frac{3(k+2)(k+3)zg'(z) + (k+1)(k+2)(k+3)g(z)}{(k+1)(k+2)(k+3)}. \tag{2.18}$$

and

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z} = \frac{z^4g''''(z) + (k+3)[4z^3g'''(z) + 6(k+2)z^2g''(z)]}{k(k+1)(k+2)(k+3)} + \frac{4(k+1)(k+2)zg'(z) + k(k+1)(k+2)g(z)}{k(k+1)(k+2)}. \tag{2.19}$$

Let

$$\mathtt{a}=\mathtt{r},\quad \mathtt{b}=\frac{\mathtt{s}+(k+3)\mathtt{r}}{(k+3)},\quad \mathtt{x}=\frac{\mathtt{t}+2(k+3)\mathtt{s}+(k+2)(k+3)\mathtt{r}}{(k+2)(k+3)},$$

$$\mathtt{y}=\frac{\mathtt{u}+3(k+3)\mathtt{t}+3(k+2)(k+3)\mathtt{s}+(k+1)(k+2)(k+3)\mathtt{r}}{(k+1)(k+2)(k+3)}\quad\text{and}$$

$$\mathtt{w}=\frac{\mathtt{v}+4(k+3)\mathtt{u}+6(k+2)(k+3)\mathtt{t}+4(k+1)(k+2)(k+3)\mathtt{s}}{k(k+1)(k+2)(k+3)}+\mathtt{r}.$$

Now, let us consider the transformation $\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z):\mathbb{C}^5\times\mathcal{U}\longrightarrow\mathbb{C}$ by

$$\psi(\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v};z)=\phi(\mathtt{a},\mathtt{b},\mathtt{x},\mathtt{y},\mathtt{w};z)=\phi\Bigg(\mathtt{r},\frac{\mathtt{s}+(k+3)\mathtt{r}}{(k+3)},\frac{\mathtt{t}+2(k+3)\mathtt{s}}{(k+2)(k+3)}+\mathtt{r},\tag{2.20}$$
$$\frac{\mathtt{u}+3(k+3)[\mathtt{t}+(k+2)\mathtt{s}]}{(k+1)(k+2)(k+3)}+\mathtt{r},\frac{\mathtt{v}+(k+3)[4\mathtt{u}+(k+2)\{6\mathtt{t}+4(k+1)\}]\mathtt{s}}{k(k+1)(k+2)(k+3)}+\mathtt{r};z\Bigg).$$

Using the equations (2.15) to (2.19), we get from (2.20) that

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)$$
$$=\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z};z\right).\tag{2.21}$$

Consequently, the equation (2.14) convert into

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\right)\in\Omega.$$

We notice that

$$\frac{\mathtt{t}}{\mathtt{s}}+1=\frac{(k+2)\mathtt{x}+\mathtt{b}-(k+3)\mathtt{a}}{\mathtt{b}-\mathtt{a}}-2(k+3),$$
$$\frac{\mathtt{u}}{\mathtt{s}}=\frac{(k+2)[(k+1)\mathtt{y}-3(k+3)\mathtt{x}+2(k+4)\mathtt{a}]}{\mathtt{b}-\mathtt{a}}+3(k+3)(k+4)$$

and

$$\frac{\mathtt{v}}{\mathtt{s}}=\frac{(k+2)[k(k+1)\mathtt{w}-4(k+1)(k+3)\mathtt{y}+6(k+3)(k+4)\mathtt{x}-3(k+4)(k+5)\mathtt{a}]}{\mathtt{b}-\mathtt{a}}$$
$$-4(k+3)(k+4)(k+5).$$

Therefore, the admissibility condition for the function $\phi\in\Phi_1[\Omega,\varrho]$ is equivalent to the admissibility condition for the function $\psi\in\Psi_j[\Omega,\varrho]$ of Definition 2.3 . Thus, using the above Lemma 1.1, we write

$$g(z)=\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}\prec\varrho(z)\quad(z\in\mathcal{U})$$

which is the complete proof of Theorem 2.6. □

Let Ω be a simply connected domain such that $\Omega\neq\mathbb{C}$, then $\Omega=h(\Delta)$ for some conformal mapping h from $\mathcal{U}$ into the domain Ω. Now, suppose the class $\Phi[h(\mathcal{U}),\varrho]$ is denoted by $\Phi[h,\varrho]$, then the next result is an immediate consequences of Theorem 2.6.

Theorem 2.7. *Let $\phi \in \Phi[h, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ satisfies (2.13) and*

$$\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z\right) = \phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z}; z\right) \prec h(z),$$

then

$$\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z} \prec \varrho(z) \quad (z \in \mathcal{U}).$$

If we take the case $\varrho(z) = 1 + Mz, M > 0$, then the class of admissible functions $\Phi[\Omega, \varrho]$ becomes $\Phi[\Omega, M]$.

Definition 2.4. *Let $\Omega \subseteq \mathbb{C}$ and $M > 0$. The family of admissible functions $\Phi[\Omega, M]$ consists of the functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$, which satisfy the following admissibility condition*

$$\phi\Bigg(1 + Me^{i\theta}, \frac{(k+3)+[n+(k+3)]}{(k+3)}Me^{i\theta}, \frac{L+(k+2)(k+3)+(k+3)[2n+(k+2)]Me^{i\theta}}{(k+2)(k+3)},$$
$$\frac{N+3(k+3)L+(k+1)(k+2)(k+3)+(k+2)(k+3)[3n+(k+1)]Me^{i\theta}}{(k+1)(k+2)(k+3)},$$
$$\frac{X+(k+3)[4N+6(k+2)L+k(k+1)(k+2)]}{k(k+1)(k+2)(k+3)} + \frac{[4n+k]Me^{i\theta}}{k}; z\Bigg) \notin \Omega$$

whenever $\omega \in \mathcal{U}$, $\Re\left\{Le^{-i\theta}\right\} \geq (n-1)nM$, $\Re\left\{Ne^{-i\theta}\right\} \geq 0$ and $\Re\left\{Xe^{-i\theta}\right\} \geq 0$ for every $\theta \in \mathbb{R}$ and $n \geq 3$.

Corollary 2.5. *Suppose $\phi \in \Phi[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}\right| \leq n^2M \quad (n \geq 3;\ M > 0)$$

and

$$\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z} - 1\right| < M \quad (z \in \mathcal{U}).$$

3. Fourth-order superordination and Sandwich-type results

In proving the differential superordination theorems using the operator $\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)$ defined by (1.2) the plays a key role. Hence, first of all we provide the class of admissible functions and then we prove our fourth-order superordination and Sandwich-type results.

Definition 3.1. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in \mathcal{Q}_1 \cap \mathcal{H}_n$. Let $\Phi_1[\Omega, \varrho]$ be the family of admissible functions which consists of functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a, b, x, y, w; z) \notin \Omega,$$

whenever

$$a = \varrho(\zeta), \quad b = \frac{\zeta\varrho'(\zeta) + m(k+2)\varrho(z)}{m(k+3)},$$

$$\Re\left\{\frac{(k+3)[b+(k+2)x] - (k+2)^2 a}{(k+3)b-(k+2)a} - 2(k+2)\right\} \geq \frac{1}{m}\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{(k+2)(k+3)[(k+1)y - 3(k+2)x + 2(k+1)a]}{(k+3)b-(k+2)a} + 3(k+2)(k+3)\right\} \geq \frac{1}{m^2}\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{k(k+1)(k+2)(k+3)w - 4(k+1)(k+2)^2(k+3)y + 6(k+2)^2(k+3)^2 x}{(k+3)b-(k+2)a} - \frac{3(k+1)(k+2)(k+3)(k+4)a}{(k+3)b-(k+2)a} - 4(k+2)(k+3)(k+4)\right\} \geq \frac{1}{m^3}\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $z \in \mathcal{U}, \zeta \in \partial\mathcal{U} \setminus E(\varrho)$ and $m \geq 3$.

Theorem 3.1. *Let us take $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_1[\Omega, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ fulfil the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right) \geq 0, \quad \left\|\left(\frac{\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z)}{\varrho'(\zeta)}\right)\right\| \leq \frac{1}{m^2} \tag{3.1}$$

and

$$\phi\left(\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+3}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+2}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+1}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k}_{\alpha,\beta} f(z); z\right)$$

is univalent in $\mathcal{U}$, then

$$\Omega \subset \left\{\phi\left(\mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+3}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+2}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k+1}_{\alpha,\beta} f(z), \mathcal{D}^{\gamma,k}_{\alpha,\beta} f(z); z\right) : z \in \mathcal{U}\right\}, \tag{3.2}$$

implies that

$$\varrho(z) \prec \mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z) \quad (z \in \mathcal{U}).$$

Proof. Let $g(z)$ be a function defined by (2.3) and $\psi(z)$ given in (2.8). Since $\phi \in \Phi_1[\Omega, \varrho]$, from (2.9) and (3.2) we have

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z), z^4g''''(z); z \in \mathcal{U}\right)\right\}.$$

Therefore, it is clear from (2.8) the admissibility condition for the function $\phi \in \Phi_1[\Omega, \varrho]$ is equivalent to the admissibility condition for ψ as given in Definition 1.6. Hence $\psi \in \Phi'_n[\Omega, \varrho]$ and by using the Lemma 1.2, we get $\varrho(z) \prec g(z)$ or

$$\varrho(z) \prec \mathcal{D}^{\gamma,k+4}_{\alpha,\beta} f(z) \quad (z \in \mathcal{U}).$$

□

Let Ω be a simply connected domain such that $\Omega \neq \mathbb{C}$, then $\Omega = h(\Delta)$ for some conformal mapping h from $\mathcal{U}$ into the domain Ω. Now, suppose the class $\Phi[h(\mathcal{U}), \varrho]$ is denoted by $\Phi[h, \varrho]$, then the next two results are the immediate consequences of Theorem 3.1.

Theorem 3.2. *Suppose $h(z)$ is a function analytic in $\mathcal{U}$ and $\phi \in \Phi_2[h(\mathcal{U}), \varrho]$. If $f \in M$, $\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \in \mathcal{Q}_1$ and $\varrho \in K$ verifying the following conditions:*

$$\left\{\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z \in \mathcal{U}\right\}$$

is univalent in $\mathcal{U}$, and

$$h(z) \prec \left\{\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z \in \mathcal{U}\right\},$$

then

$$\varrho(z) \prec \mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \quad .$$

Proof. The proof of this theorem is skipped due to similar proof of the Theorem 3.1. □

Theorem 3.3. *Let $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$, $h(z)$ be analytic in $\mathcal{U}$, and ψ is given by (2.9). Suppose that the differential equation*

$$\left\{\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z)\right) : z \in \mathcal{U}\right\} = h(z), \tag{3.3}$$

has a solution $\varrho(z) \in \mathcal{Q}_1$. If $\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z) \in \mathcal{Q}_1$, $\varrho \in K$, $\varrho'(z) \neq 0$ and $f \in M$ satisfy the condition (3.1),

$$\left\{\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z \in \mathcal{U}\right\}$$

is univalent in $\mathcal{U}$, and

$$h(z) \prec \left\{\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z \in \mathcal{U}\right\},$$

then

$$\varrho(z) \prec \mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \quad ,$$

and $\varrho(z)$ is the best subordinate of (3.3).

Proof. The proof of this theorem is skipped due to similar proof of the Theorem 3.1. □

Definition 3.2. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in \mathcal{Q}_1 \cap \mathcal{K}_1$. Let $\Phi_1[\Omega, \varrho]$ be the family of admissible functions which consists of functions $\phi : \mathbb{C}^5 \times \mathcal{U} \longrightarrow \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a, b, x, y, w; z) \notin \Omega,$$

whenever

$$a = \varrho(\zeta), \quad b = \frac{m\zeta\varrho'(\zeta) + (k+3)\varrho(z)}{(k+3)},$$

$$\Re\left\{\frac{(k+2)x + b - (k+3)a}{b-a} - 2(k+3)\right\} \geq \frac{1}{m}\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\},$$

$$\Re\left\{\frac{(k+2)[(k+1)\mathsf{y}-3(k+3)\mathsf{x}+2(k+4)\mathsf{a}]}{\mathsf{b}-\mathsf{a}}+3(k+3)(k+4)\right\}\geq\frac{1}{m^2}\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{(k+2)[k(k+1)\mathsf{w}-4(k+1)(k+3)\mathsf{y}+6(k+3)(k+4)\mathsf{x}-3(k+4)(k+5)\mathsf{a}]}{\mathsf{b}-\mathsf{a}}\right.$$
$$\left.\frac{-4(k+3)(k+4)(k+5)(\mathsf{b}-\mathsf{a})}{\mathsf{b}-\mathsf{a}}\right\}\geq\frac{1}{m^3}\Re\left\{\frac{\zeta^3\varrho''''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $z\in\mathcal{U},\zeta\in\partial\mathcal{U}\setminus E(\varrho)$ *and* $m\geq 3$.

Theorem 3.4. *Let us suppose that* $\Omega\subseteq\mathbb{C}$ *and* $\phi\in\Phi_1[\Omega,\varrho]$. *Let* $f\in\mathcal{A}$ *and* $\varrho\in\mathcal{Q}_1$ *fulfil the following conditions:*

$$\Re\left(\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right)\geq 0,\quad\left\|\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{\varrho'(\zeta)}\right)\right\|\leq\frac{1}{m^2}\tag{3.4}$$

and

$$\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z};z\right)$$

is univalent in $\mathcal{U}$, *then*

$$\Omega\subset\left\{\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1}f(z)}{z},\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k}f(z)}{z};z\right):z\in\mathcal{U}\right\}\tag{3.5}$$

implies that

$$\varrho(z)\prec\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}\quad(z\in\mathcal{U}).$$

Proof. Let $g(z)$ be defined by (2.15) and $\psi(z)$ by (2.20). Since $\phi\in\Phi_1[\Omega,\varrho]$, so from (2.21) and (3.5) we get

$$\Omega\subset\left\{\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z),z^4g''''(z);z\in\mathcal{U}\right)\right\}.$$

Therefore, it is clear from (2.20) the admissibility condition for the function $\phi\in\Phi_1[\Omega,\varrho]$ is equivalent to the admissibility condition for ψ as given in Definition 1.6. Hence $\psi\in\Phi_n'[\Omega,\varrho]$ and by using the Lemma 1.2, we get $\varrho(z)\prec g(z)$ or

$$\varrho(z)\prec\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4}f(z)}{z}\quad(z\in\mathcal{U}).$$

□

Let Ω be a simply connected domain such that $\Omega\neq\mathbb{C}$, then $\Omega=h(\Delta)$ for some conformal mapping h from $\mathcal{U}$ into the domain Ω. Now, suppose the class $\Phi[h(\mathcal{U}),\varrho]$ is denoted by $\Phi[h,\varrho]$, then the next result is an immediate consequences of Theorem 3.4.

Theorem 3.5. *Let $\phi \in \Phi[h, \varrho]$. If $f \in \mathcal{A}$ and $\varrho \in \mathcal{Q}_1$ satisfies (3.4) and*

$$\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)}{z}; z\right)$$

is univalent in $\mathcal{U}$, then

$$h(z) \prec \phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)}{z}; z\right)$$

implies that

$$\varrho(z) \prec \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z} \quad (z \in \mathcal{U}).$$

Combining Theorems 2.2 and 3.2, we have the sandwich-type result.

Theorem 3.6. *Consider two analytic functions $h_1(z)$ and $\varrho_1(z)$ in $\mathcal{U}$, and $\varrho_2(z) \in \mathcal{Q}_1$ with $\varrho_1(0) = \varrho_2(0) = 1$. In addition let the function $h_2(z)$ be univalent in $\mathcal{U}$ and $\phi \in \Phi_1[h_2, \varrho_2] \cap \Phi_2[h_1, \varrho_1]$. If $\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \in \mathcal{Q}_1 \cap K$, $f \in M$,*

$$\left\{\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z\right\}$$

is univalent in $\mathcal{U}$, and the two conditions (2.1) and (3.1) are satisfied as

$$h_1(z) \prec \left\{\phi\left(\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z), \mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)\right) : z\right\} \prec h_2(z),$$

then

$$\varrho_1(z) \prec \mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z) \prec \varrho_2(z).$$

And from Theorems 2.7 and 3.5, we have the sandwich-type result.

Theorem 3.7. *Consider two analytic functions $h_1(z)$ and $\varrho_1(z)$ in $\mathcal{U}$, and $\varrho_2(z) \in \mathcal{Q}_1$ with $\varrho_1(0) = \varrho_2(0) = 1$. In addition let the function $h_2(z)$ be univalent in $\mathcal{U}$ and $\phi \in \Phi_1[h_2, \varrho_2] \cap \Phi_2[h_1, \varrho_1]$. If $\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z} \in \mathcal{Q}_1 \cap K$, $f \in M$,*

$$\left\{\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)}{z}\right) : z\right\}$$

is univalent in $\mathcal{U}$, and the two conditions (2.14) and (3.4) are satisfied as

$$h_1(z) \prec \left\{\phi\left(\frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+3} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+2} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+1} f(z)}{z}, \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k} f(z)}{z}\right) : z\right\} \prec h_2(z),$$

then

$$\varrho_1(z) \prec \frac{\mathcal{D}_{\alpha,\beta}^{\gamma,k+4} f(z)}{z} \prec \varrho_2(z).$$

4. Conclusion

In this current research, we employed the operator $\mathcal{D}^{\gamma,k}_{\alpha,\beta}$, as described in (1.2), to derive various fourth-order differential subordination and superordination outcomes for analytic functions within the open unit disk $\mathcal{U}$, through the lens of the convolution process. The findings presented here could serve as a foundation for future research aimed at deriving fourth-order differential subordination, superordination, and sandwich-type results involving a variety of linear and nonlinear operators.

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