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Extending Certified Domination: Bondage Numbers in Generalized Petersen Graphs

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Abstract. The paper extends the concept of bondage numbers to certified domination, introducing the certified bondage number of a graph. A *certified dominating set* R is a dominating set of a graph H, if every vertex in R has either zero or at least two neighbours in $V\backslash R$, where V is the vertex set of H. The minimum cardinality of certified dominating set of H is the *certified domination number* of H denoted by $\gamma_{cer}(H)$. The *bondage number* $b(H)$ is defined to be the cardinality of least number of edges $F \subset E(H)$ such that $\gamma(H-F) > \gamma(H)$. Motivated by this parameter, we extended this concept on certified domination number and defined *certified bondage number* of a graph H, $b^{+}_{cer}(H)$ $[b^{-}_{cer}(H)]$ to be the cardinality of the least number of edges $F \subset E(H)$ such that $\gamma_{cer}(H-F) > \gamma_{cer}(H)$ $[\gamma_{cer}(H-F) < \gamma_{cer}(H)]$ that is minimum number of edges to be removed to increase (or decrease) the certified domination number of H. In this paper, we establish the values of certified bondage number for generalised Petersen graphs $P(n,k)$, where $k = 1,2$, as well as for certain classes of graphs.

1. Introduction

Haynes [6] introduced one of graph theory's most essential and widely researched concepts, called *domination* in graphs. A *dominating set* of a graph H is a set $U \subseteq V$ with the property that for each vertex $u \in V\backslash U$ there exists at least a vertex $x \in U$ adjacent to u. The least cardinality

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amongst all dominating sets of H is the *domination number* $\gamma(H)$ and U is called γ -set of H, if U is minimum.

A wide range of extensive research is being conducted on various concepts related to dominance terminology [7, 8, 18, 29–31]. One of the most recent concepts in this field is *certified domination*, which was introduced by Dettlaff et al. [12]. A *certified dominating set* is defined as $R \subseteq V$, where R is a dominating set of a graph H, and each vertex in R has either no neighbours or at least two neighbours in $V \backslash R$. The *certified domination number* $\gamma_{cer}(H)$ is the least cardinality of a certified dominating set of H, and R is the γ_{cer}-set of H if R is minimal. Further results on this parameter can be found in [5, 10, 11, 23] . The bondage number of a graph is a key concept in domination theory that quantifies the stability of a graph's domination number. *Bondage number* $b(H)$ is defined as the minimum number of edges $F \subset E(H)$ for which $\gamma(H - F) > \gamma(H)$. Additional findings on this parameter are discussed in [15–17, 19]. Motivated by this concept, we apply the bondage number to the certified domination number of a graph and defined the *certified bondage number* of a graph H denoted by $b^{+}_{cer}(H)$ $(b^{-}_{cer}(H))$ to be the size of the least number of edges $F \subset E(H)$ such that γ_{cer} $(H - F) > \gamma_{cer}(H)$ $(\gamma_{cer}(H - F) < \gamma_{cer}(H))$ that is least number of edges to be removed to enhance or (diminish) the certified domination number of H. In this paper, we establish the values of certified bondage number for generalised Petersen graphs $P(n, k)$, where $k = 1, 2$, as well as for certain classes of graphs.

Domination in graph theory has many applications in network design, resource optimization, and fault-tolerant systems [24]. Certified bondage numbers-expanding domination parameters analyze the minimum edge removals that change the certified domination number, thereby providing more measures of resilience [25]. The certified bondage numbers may be used to analyze a network's vulnerability to edge failures in communication. In such a design, edge criticality is used to find edges whose removal would affect the network's connectivity or coverage . In wireless sensor networks, uninterrupted functioning requires that nodes are left with sufficient connectivity even when some edges fail [26]. Verified bondage numbers can help design and maintain transportation or utility networks were disruptions from edge failures, such as road closures or power line failures, must be kept as low as possible. The certified bondage numbers of such networks in biological systems, such as a metabolic or neural network, could model the effect of edge failures on the functional properties of those networks. Removals of connections in a neural network might model synaptic damage and allow for an analysis of the system's resilience or a recovery strategy [27]. Bondage numbers might identify influential relationships in social or organizational networks. The elimination of sure edges can correspond to a failure of communication or collaboration between important persons, which affects the network's overall structure and functioning . Certified bondage numbers can count the effects of edge removals in supply chain networks, which may include removing transportation routes or supplier links. These insights are useful for building stronger supply chains by targeting such vulnerabilities that may exist in connection [28] . Researchers and practitioners could develop more robust systems by including certified bondage numbers in the study of real-world networks, constraining both local and global connectivity. This generalization of classical graph parameters thus points out the practical relevance with regard to the theoretical development of graph theory [22].

1.1 Motivation

Domination in graphs is widely applied in various fields, particularly in optimizing facility location problems. The objective is to identify the most efficient sites for essential services like healthcare centers, schools, fire stations, grocery stores etc, minimizing the average distance or travel time for the population served. The concept of the certified domination number introduces additional constraints to traditional domination problems. By incorporating a bondage number into a graph's certified domination number, we can assess the robustness of a network against potential connectivity failures. This method is especially useful for analyzing networks with modems that operate independently while also being capable of connecting to multiple computers, ensuring both resilience and efficient network connectivity, even after failure of some connectivity.

1.2 Novelty

The novelty of this work lies in the introduction and exploration of the certified bondage number $b^{+}_{cer}(H)$ and $b^{-}_{cer}(H)$, which extends the traditional concept of bondage numbers to the domain of certified domination in graphs. While the standard bondage number focuses on the minimum number of edge removals that increase the domination number, this research generalizes this idea to certified domination, a more restrictive form of domination. Additionally, the certified bondage number is examined in both increasing and decreasing the certified domination number, offering a new perspective on how edge removal affects the robustness and vulnerability of graph structures. Determining these parameters for Petersen graphs $P(n,k)$, $k = 1,2$, provides a significant contribution by highlighting their unique properties under the certified domination framework, expanding the understanding of graph dynamics in certified settings.

2. Preliminaries

Let $H = (V, E)$ be a connected, simple graph of order $|V| = n$. Graph theoretic terminology is based on Harary [9] . For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup\limits_{v \in S} N(v)$, the closed neighbourhood of S is $N[S] = N(S) \cup S$, $N_X(v)$ denotes the neighbours of v in X where X is a subset of V and the *private neighbourhood* $pn(v,S)$ of a vertex $v \in S$ is defined by $pn(v,S) = \{u \in V - S : N(u) \cap S = \{v\}\}$. A *path* is a walk with no repeated vertices. A nontrivial closed path is called a *cycle*. A *tree* is a connected, acyclic graph where there is exactly one unique simple path between any pair of vertices. A *complete binary tree* is a binary tree where each leaves are at the same depth, and every internal vertices have a degree of three. If T represents such a tree with a root vertex v, the collection of all vertices at depth k is referred to as the vertices at level k. A graph H is said to be *connected graph* if there is a path between every pair of vertices in the graph. The generalized Petersen graph $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 1 \leq i \leq n-1\}$ and $E(Pn,k)) = \{v_i, v_{i+1,}, v_i, u_i, u_i u_{i+k} : 1 \leq i \leq n\}$ subscipts taken modulo $n+1$. The edges $u_i v_i$ for $1 \leq i \leq n-1$ are called the *spokes* of $P(n,k)$.

3. Main results

In this division we determine the value of $b^-_{cer}(H)$ for connected graph and $b^+_{cer}(H)$ for Petersen graphs $P(n,k), k = 1,2$.

Theorem 3.1. *Let H be a connected graph with order $n \geq 3$, $b^-_{cer}(H) = 1$ if and only if $\gamma(H) < \gamma_{cer}(H)$.*

Proof. Let H be a connected graph with order $n \geq 3$. J and J' be a γ-set and γ_{cer} -set of H respectively. Suppose $\gamma(H) < \gamma_{cer}(H)$.Then $|N_{V-J}(v)| \geq 1$ for every $v \in J$. If $|N_{V-J}(v)| > 1$ for all $v \in J$, then $\gamma(J)$ = $\gamma_{cer}(J)$ is a contradiction to the hypothesis. Hence $|N_{V-J}(t)| = 1$ for at least one vertex $t \in J$ and let $u \in N_{V-J}(t)$, clearly $u \in pn(t, J)$, if not $J - \{t\}$ is a γ-set of H a contradiction to the minimality of γ-set of H. Now $J' = J \cup \{u\}$ is a γ_{cer}-set of H and $u, v \in J'$. Let $J' = J - \{uv\}$ be a subgraph. Clearly either $J_1 - \{u\}$ or $J_1 - \{v\}$ is a γ_{cer} -set of H'. Hence $b^-_{cer}(H) = 1$.
Suppose $b^-_{cer}(H) = 1$.Then there exist at least an edge $v_1 v_2 \in E(H)$ so that $v_1 v_2 \in J_1$. This is possible only if there exists exactly one vertex either $v_1 \in J$ with $v_2 \in N_{V-J}(v_1)$ or $v_2 \in J$ with $v_1 \in N_{V-J}(v_2)$ so that $J_1 = J \cup \{v_1\}$ is a γ_{cer} -set of H in former case and $J_1 = J \cup \{v_2\}$ in the lateral case. Hence $\gamma(H) < \gamma_{cer}(H)$. □

Corollary 3.1.1 For a path P_n, $n \geq 3$ and $n = 4$, $b^-_{cer}(H) = 1$.

Proof. Since $\gamma(P_n) = 2 < \gamma_{cer}(P_n) = 4$ for $n = 4$. Hence by Theorem 3.1, $b^-_{cer}(P_n) = 1$. □

Theorem 3.2. *[12] Let H be a connected graph with at least three vertices. Then $\gamma(H) = \gamma_{cer}(H)$ if and only if there exists a γ-set R in H such that every vertex in R has at least two neighbors in the set $V(H) \setminus R$.*

Theorem 3.3. *[22] For $n \geq 3$,*

$$\gamma(P(n,1)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,1,3 \ (mod\ 4) \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2 \ (mod\ 4) \end{cases}$$

Theorem 3.4. *For any Petersen graph $H \cong P(n,1)$, $n \geq 4$,*

$$b^+_{cer}(H) = \begin{cases} 2 & \text{if } n \equiv 0,1 \ (mod\ 4); \\ 3 & \text{if } n \equiv 3 \ (mod\ 4); \\ 4 & \text{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

Proof. Let $H \cong P(n,1)$. Let B' and B'' be the outer and inner cycles of H respectively. Let $V(B') = \{u_1, u_2, \ldots u_n\}$,$V(B'') = \{v_1, v_2, \ldots, v_n\}$ and $E(H) = E_1(H) \cup E_2(H) \cup E_3(H)$ be the edge sets of H. Where $E_1(H)$ and $E_2(H)$ be the edge sets of B' and B'' respectively and $E_3(H)$ be the edge set in the spokes of H, where $E_1(H) = \{u_1u_2, u_2u_3, u_3u_4, \ldots u_nu_1\}$, $E_2(H) = \{v_1v_2, v_2v_3, \ldots v_nv_1\}$ and $E_3(H) = \{u_1v_1, u_2v_2, \ldots u_nv_n\}$.

TABLE 1. γ_{cer}-set of H'' illustrating case (ii) for $n \equiv 2 \pmod 4$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0, 2	R is a γ_{cer}-set.
	1	$R' = \{u_3\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-2}{4}\}$
		$R'' = \{v_1\} \cup \{v_{4k} : 1 \le k \le \frac{n-2}{4}\}$
$e_1, e_2 \in E_2$	1	$R' = \{u_3\} \cup \{u_{4k+4} : 1 \le k \le \frac{n-6}{4}\}$
		$R'' = \{v_1, v_5\} \cup \{v_{4k+2} : 1 \le v \le \frac{n-2}{4}\}$
	2	R is a γ_{cer}-set.
	0	$R' = \{u_3\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
		$R'' = \{v_1, v_4\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
$e_1 \in E_1$ & $e_2 \in E_3$	0, 2	R is a γ_{cer}-set.
	1	$R' = \{u_3\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-2}{4}\}$
		$R'' = \{v_1\} \cup \{v_{4k} : 1 \le k \le \frac{n-2}{4}\}$
$e_1 \in E_2$ & $e_2 \in E_3$	0	$R' = \{u_1\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
		$R'' = \{v_3\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-3}{4}\}$
	1	$R' = \{u_3\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
		$R'' = \{v_1, v_3\} \cup \{v_{4k+2} : 1 \le k \le \frac{n-6}{4}\}$
$e_1 \in E_1$ & $e_2 \in E_2$	1, 2	R is a γ_{cer}-set.

Let $R = R' \cup R''$ be a γ-set of G where $R' = R \cap V(B')$ and $R'' = R \cap V(B'')$.

$$R' = \begin{cases} u_{4k+3} : 0 \le k \le \left[\frac{n}{4}\right]\} \cup \{u_n\}, & \text{for } n \equiv 2 \pmod 4; \\ u_{4k+3} : 0 \le k \le \left\lfloor\frac{n}{4}\right\rfloor, & \text{otherwise.} \end{cases}$$

$$R'' = v_{4k+1} : 0 \le k \le \left\lfloor\frac{n}{4}\right\rfloor, \ for\ all\ n;$$

and we observe that each vertex in $v \in R$, $N(v) \ge 2$. Hence by theorem 3.1 and theorem 3.2, we have $\gamma(H) = \gamma_{cer}(H)$. Let $H' = H - \{e\}$ and $e = u_1u_2$ [or v_nv_{n-1} or u_2v_2]. In all the above choice of e it is evident that R is a γ_{cer}-set of H' for all n. Therefore, $b^+_{cer}(H) > 1$. Now, consider the following cases:

Case (i): $n \equiv 0, 1 \pmod 4$ Let $H'' = H - \{e_1, e_2\}$, where $e_1 = u_4v_4$ and $e_2 = u_5v_5$. Here $u_4 \in N(u_3)$, $v_4 \in N(v_5)$ and $u_5 \notin N(R)$. In order to dominate u_5, the configuration of the set R can be modified as $R_1 = R'_1 \cup R''_1$, where

$$R'_1 = \{u_3\} \cup \left\{u_{4k+2} : 1 \le k \le \frac{n-4}{4}\right\} \text{ and}$$
$$R''_1 = \{v_1, v_5\} \cup \left\{v_{4k+4} : 1 \le k \le \frac{n-8}{4}\right\} \text{ for } n \equiv 0 \pmod 4 \text{ and}$$
$$R'_1 = \{u_3\} \cup \left\{u_{4k+2} : 1 \le k \le \frac{n-5}{4}\right\}$$
$$R''_1 = \{v_1, v_5\} \cup \left\{v_{4k+4} : 1 \le k \le \frac{n-5}{4}\right\} \text{ for } n \equiv 1 \pmod 4.$$

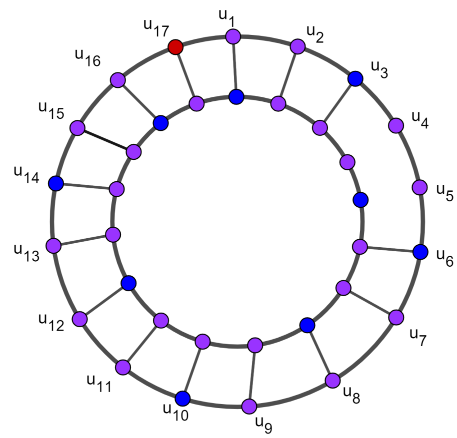

Figure 1. **Certified Dominating sets of H'' illustrating case (i) for $n \equiv 1 \pmod 4$**

Table 2. γ_{cer}-set of H'' illustrating case (ii) for $n \equiv 3 \pmod 4$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1(H)$	0 3	R is a γ_{cer}-set
	1	$R' = \{u_3\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-4}{4}\}$
		$R'' = \{v_1, v_5\} \cup \{v_{4k+4} : 1 \le k \le \frac{n-7}{4}\}$
	2	$R' = \{u_3, u_7\} \cup \{u_{4k+6} : 1 \le k \le \frac{n-7}{4}\}$
		$R'' = \{v_1, v_5\} \cup \{v_{4k+4} : 1 \le k \le \frac{n-7}{4}\}$
$e_1, e_2 \in E_2(H)$	0 3	R is a γ_{cer}-set
	1	$R' = \{u_3\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-3}{4}\}$
		$R'' = \{v_1\} \cup \{v_{4k} : 1 \le k \le \frac{n-3}{4}\}$
	2	$R' = \{u_2, u_3\} \cup \{u_{4k+5} : 1 \le k \le \frac{n-7}{4}\}$
		$R'' = \{v_5\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-3}{4}\}$
$e_1 \in E_1(H)$ & $e_2 \in E_2(H)$	0, 1, 2 & 3	R is a γ_{cer}-set
$e_1 \in E_2(H)$ & $e_2 \in E_3(H)$	0 2	R is a γ_{cer}-set
	1 3	$R' = \{u_2\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-3}{4}\}$ $R'' = \{v_1\} \cup \{v_{4k} : 1 \le k \le \frac{n-3}{4}\}$

Again $u_n, v_{n-1} \notin N_{H''}(R_1)$, hence $R_1 \cup \{v_n\}$ is a γ_{cer}-set of H'' for $n \equiv 0 \pmod 4$. Similarly $u_n \notin N(R_1)$, hence $R_1 \cup \{u_n\}$ is a γ_{cer}-set of H'' for $n \equiv 1 \pmod 4$, refer Figure 1. Therefore, $|R_1| > |R|$. Hence $b^+_{cer}(H) = 2$ for $n \equiv 1 \pmod 4$.

Case (ii): $n \equiv 2, 3 \pmod 4$ Let $H'' = H - \{e_1, e_2\}$. In tables 1 and 2 gives the γ_{cer}-sets based on the pairwise distances between the edges e_1 and e_2 which are removed from H. Clearly R_1 is the new configuration of γ_{cer}-set of H''. In all the above position of the edges e_1 and e_2, we see that $|R| = |R_1|$. Similarly, removing any two edges e_1 and e_2, regardless of the distance between them, yields the same domination number for $n \equiv 2, 3 \pmod 4$. Hence $b^+_{cer}(H'') > 2$.

TABLE 3. γ_{cer}-set of H''' illustrating case (ii) for $n \equiv 2 \pmod 4$

Edge set	$d(e_1, e_2)$	$d(e_2, e_3)$	γ_{cer}-set
$e_1, e_2, e_3 \in E_1(H)$	0	0	$R' = \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\} \cup \{u_n\}$
			$R'' = \{v_2, v_3\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
	0	1	$R' = \{u_4\} \cup \{u_{4k+3} : 1 \le k \le \frac{n-3}{4}\}$
			$R'' = \{v_1, v_2\} \cup \{v_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
	1	2	$R' = \{u_3, u_6\} \cup \{u_{4k+5} : 1 \le k \le \frac{n-6}{4}\}$
			$R'' = \{v_1, v_4\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
$e_1, e_2, e_3 \in E_2(H)$	0	0	$R' = \{u_2, u_3\} \cup \{u_{4k+3} : 1 \le k \le \frac{n-2}{4}\}$
			$R'' = \{v_{4k+1} : 1 \le k \le \frac{n-2}{4}\} \cup \{v_n\}$
	0	1	$R' = \{u_2\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
			$R'' = \{v_1, v_4\} \cup \{v_{4k+1} : 1 \le k \le \frac{n-6}{4}\}$
	2	2	$R' = \{u_2, u_5\} \cup \{u_{4k+4} : 1 \le k \le \frac{n-6}{4}\}$
			$R'' = \{v_4, v_7\} \cup \{v_{4k+6} : 1 \le k \le \frac{n-6}{4}\}$
$e_1, e_2 \in E_1(H), e_3 \in E_3(H)$	1	1	$R' = \{u_3\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
			$R'' = \{v_1, v_5\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
	1	2	$R' = \{u_3, u_6\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-6}{4}\}$
			$R'' = \{v_1, v_4\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
	2	3	R is a γ_{cer}-set.
$e_1, e_2 \in E_2(H), e_3 \in E_3(H)$	1	3	$R' = \{u_3\} \cup \{u_{4k+1} : 1 \le k \le \frac{n-2}{4}\}$
			$R'' = \{v_1, v_4\} \cup \{v_{4k+3} : 1 \le k \le \frac{n-6}{4}\}$
	2	3	$R' = \{u_3\} \cup \{u_{4k+2} : 1 \le k \le \frac{n-2}{4}\}$
			$R'' = \{v_1, v_5\} \cup \{v_{4k+4} : 1 \le k \le \frac{n-6}{4}\}$
$e_1 \in E_1(H), e_2 \in E_2, e_3 \in E_3$	0 1 2	0 1 2	R is a γ_{cer}-set.

For $n \equiv 3 \pmod 4$, let $H''' = H - \{e_1, e_2, e_3\}$, without loss of generality, let $e_1 = u_4u_5$, $e_2 = u_5u_6$ and $e_3 = u_6u_7$. Here $u_4 \in N_{G'''}(u_3)$, $u_5 \in N_{H'''}(v_5)$, $u_3, v_5 \in R$ and $u_6 \notin N(R)$. In order to dominate the vertex u_6 the configuration of the set R is modified to R_1, $R_1' \cup R_1''$, where

$$R_1' = \{u_3\} \cup \left\{u_{4k+4} : 1 \le k \le \frac{n-4}{4}\right\}$$

$$R_1'' = \{v_1, v_5\} \cup \{v_{4k+2} : 1 \le k \le \frac{n-3}{4}\}.$$

the vertex $u_n \notin N(R)$. Hence either $R \cup \{u_6\}$ or $R_1 \cup \{u_n\}$ is the γ_{cer}-set of H'''. Therefore $b^+_{cer}(H) = 3$. For $n \equiv 2 \pmod 4$, let $H''' = H - \{e_1, e_2, e_3\}$. In this case the following table 3 gives the γ_{cer}-sets based on the pairwise distances between the edges e_1, e_2 and e_3 which are removed from H. From the table 3 we see that R_1 is the new configuration of γ_{cer}-set of H'''. In the above choice of edges, we observe that $|R| = |R_1|$. Hence $b^+_{cer}(H) > 3$. Let $H^{iv} = H - \{e_1, e_2, e_3, e_4\}$, where $e_1 = u_1u_2$, $e_2 = u_2u_3$, $e_3 = u_3u_4$ and $e_4 = u_4u_5$. Here $u_1 \in N(v_1) \cap N(u_n)$, $u_2 \notin N(R)$. To dominate u_2, the set $R_1 = R_1' \cup R_1''$, where $R_1' = \{u_{4k+2} : 1 \le k \le \frac{n-2}{4}\}$ and $R_1'' = \{v_2, v_3\} \cup \{v_{4k} : 1 \le k \le \frac{n-2}{4}\}$ the new configuration of R is constructed. Here $v_3 \in R_1$ dominates exactly one vertex $u_3 \in V/R_1$, contradicts the definition of certified domination number. Hence $R_2 = R_1 \cup \{u_3\}$ is a γ_{cer}-set of H^{iv}. Therefore $b^+_{cer}(H^{iv}) = 4$. □

Theorem 3.5. *[22] For $n \geq 5$ we have $\gamma(P(n,2)) = \left\lceil \frac{3n}{5} \right\rceil$.*

Theorem 3.6. *For any petersen graph $H \cong P(n,2)$, $n \geq 5$*

$$b^{+}_{cer}(H) = \begin{cases} 3 & \text{if } n \equiv 0,3,4 \ (mod\ 5) \\ 5 & \text{if } n \equiv 1,2 \ (mod\ 5) \end{cases}$$

Table 4: γ_{cer}-set of H'' illustrating for $n \equiv 0$ (mod 5)

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0	R is a γ_{cer}-set
	1	$R' = \{u_{5k} : 1 \leq k \leq \frac{n}{5}\}$ $R'' = \{v_{5k-3} : 1 \leq k \leq \frac{n}{5}\} \cup$ $\{v_{5k-2} : 1 \leq k \leq \frac{n}{5}\}$
	2	$R' = \{u_{5k-2} : 1 \leq k \leq \frac{n}{5}\}$ $R'' = \{v_{5k-4} : 1 \leq k \leq \frac{n}{5}\} \cup \{v_{5k} : 1 \leq k \leq \frac{n}{5}\}$
$e_1, e_2 \in E_2$	0	R is a γ_{cer}-set
	1	$R' = \{u_{5k-2} : 1 \leq k \leq \frac{n}{5}\}$ $R'' = \{v_{5k-4} : 1 \leq k \leq \frac{n}{5}\} \cup \{v_{5k} : 1 \leq k \leq \frac{n}{5}\}$
$e_1 \in E_1$ & $e_2 \in E_2$	1 2	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_3$	0 1 2	R is a γ_{cer}-set
$e_1 \in E_2$ & $e_2 \in E_3$	0	R is a γ_{cer}-set
	2	$R' = \{u_{5k-2} : 1 \leq k \leq \frac{n}{5}\}$ $R'' = \{v_{5k-4} : 1 \leq k \leq \frac{n}{5}\} \cup \{v_{5k} : 1 \leq k \leq \frac{n}{5}\}$

Proof. Let $H \cong P(n,2)$, B' and B'' be the outer and inner cycles of H. Let $V(B') = \{u_1, u_2, \ldots, u_n\}$ and $V(B'') = \{v_1, v_2, \ldots, v_n\}$. By theorem 3.1 and theorem 3.4 we have $\gamma(P(n,2)) = \gamma_{cer}(P(n,2))$. Let $E(H) = E_1(H) \cup E_2(H) \cup E_3(H)$ be the edge sets of H and $E_1(H)$ and $E_2(H)$ be the edge sets of outer and inner cycles of H respectively and $E_3(H)$ be the edge set in the spokes of H, where $E_1(H) = \{u_1u_2, u_2u_3, \ldots, u_nu_1\}$ and $E_2(H) = \{v_1v_3, v_2v_4, v_3v_5, v_4v_6, \ldots, v_{n-1}v_1\}$ and $E_3(H) = \{u_1v_1, u_2v_3, \ldots, u_nv_n\}$. Now consider $n \not\equiv 3$ (mod 5).
Let $R = R' \cup R''$ be a γ-set of H where $R' = R \cap V(B')$ and $R'' = R \cap V(B'')$. Let

$$R' = \begin{cases} \{u_{5k-1} : 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\} & \text{if } n \equiv 0 \text{ (mod 5)} \\ \{u_{5k-1} : 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\} & \text{if } n \equiv 1 \text{ (mod 5)} \\ \{u_{5k-1} : 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\} & \text{if } n \equiv 2 \text{ (mod 5)} \\ \{u_{5k-1} : 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\} & \text{if } n \equiv 4 \text{ (mod 5)} \end{cases}$$

$$R'' = \begin{cases} \{v_{5k-4} : 1 \le k \le \left\lfloor \frac{n}{5} \right\rfloor\} \cup \{v_{5k-3} : 1 \le k \le \left\lfloor \frac{n}{5} \right\rfloor\} & \text{if } n \equiv 0 \pmod 5 \\ \{v_{5k-4} : 1 \le k \le \left\lfloor \frac{n}{5} \right\rfloor\} \cup \{v_{5k-3} : 1 \le k \le \left\lfloor \frac{n}{5} \right\rfloor\} & \text{if } n \equiv 1 \pmod 5 \\ \{v_{5k-4} : 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\} \cup \{v_{5k-3} : 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\} & \text{if } n \equiv 2 \pmod 5 \\ \{v_{5k-4} : 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\} \cup \{v_{5k-3} : 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\} & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

Let $H' = H - \{e\}$, without loss of generality, let $e = u_1u_2$ [or v_2v_4 or u_4v_4]. Clearly, we note that R is a γ_{cer}-set of H' for every n, except $n \equiv 3 \pmod 5$. Therefore $b^+_{cer}(H) > 1$. For $n \equiv 3 \pmod 5$. Let $R' = \{v_{5i-3} : 1 \le i \le \left\lceil \frac{n}{5} \right\rceil\} \cup \{u_{5j} : 1 \le j \le \left\lfloor \frac{n}{5} \right\rfloor\}$ and $R'' = \{u_{5k-4} : 1 \le k \le \left\lceil \frac{n}{5} \right\rceil\}$. Let $H' = H - \{e\}$, and $e = u_1u_2$[or u_2v_2 or v_1v_3].

Table 5: γ_{cer}-set of H'' illustrating for $n \equiv 1 \pmod 5$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0	R is a γ_{cer}-set
	1	$R' = \{u_3\} \cup \{u_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_1, v_2\} \cup \{v_{5k+2} : 1 \le k \le \frac{n-6}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$
$e_1, e_2 \in E_2$	0 2	R is a γ_{cer}-set
$e_1, e_2 \in E_3$	1 2	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_2$	1 2	R is a γ_{cer}-set
	3	$R' = \{u_1\} \cup \{u_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_{5k-1} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k} : 1 \le k \le \frac{n-1}{5}\}$
$e_1 \in E_1$ & $e_2 \in E_3$	0	R is a γ_{cer}-set
	3	$R' = \{u_4, u_7\} \cup \{u_{5k+6} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_2\} \cup \{v_{5k-2} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+4} : 1 \le k \le \frac{n-6}{5}\}$
$e_1 \in E_2$ & $e_2 \in E_3$	0	$R' = \{u_2, u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+1} : 1 \le k \le \frac{n-1}{5}\}$
	2	$R' = \{u_2, u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+1} : 1 \le k \le \frac{n-1}{5}\}$

Clearly, we notice that R is a γ_{cer}-set of H'. Therefore $b^+_{cer}(H) > 1$. Let $H'' = H - \{e_1, e_2\}$. In all the above choices of e_1 and e_2 the following tables 4, 5, 6, 7 and 8 give the γ_{cer}-sets of H'' for

$n \equiv 0 \pmod 5$, $n \equiv 1 \pmod 5$, $n \equiv 2 \pmod 5$, $n \equiv 3 \pmod 5$, and $n \equiv 4 \pmod 5$ respectively, based on the pairwise distances between the edges e_1 and e_2 which are deleted from H.
From the tables 4, 5, 6, 7 and 8 we note that R_1 is a new configuration of γ_{cer}-set of H''. In all the above position of the edges e_1 and e_2, we see that $|R| = |R_1|$. Hence $b^+_{cer}(H) > 2$. Similarly, removing any two edges e_1 and e_2, regardless of the distance between them, yields the same domination number. Hence $b^+_{cer}(H) > 2$. Consider the following cases for the removal of 3 edges that is $H''' = H - \{e_1, e_2, e_3\}$. **Case (i):** $n \equiv 0, 3, 4 \pmod 5$.
Subcase (i): For $n \equiv 0 \pmod 5$
Let $e_1 = v_n v_2$, $e_2 = v_1 v_n$ and $e_3 = v_1 v_3$. Here $u_3 \in N(u_4)$ and the vertices $v_3, v_n \notin N(R)$. To dominate the vertices v_3 and v_n, $R'_1 = \{u_1\} \cup \{u_{5k-3} : 1 \leq k \leq \frac{n}{5}\}$ and $R''_1 = \{v_{5k-1} : 1 \leq k \leq \frac{n}{5}\} \cup \{v_{5k} : 1 \leq k \leq \frac{n-1}{5}\}$ a new configuration of γ_{cer} - set of H''' is constructed. Again $v_n \notin N(R_1)$. Hence $R_2 = R_1 \cup \{v_n\}$ a γ_{cer}-set of H'''. Therefore $b^+_{cer}(H) = 3$.

Table 6: γ_{cer}-set of H'' illustrating for $n \equiv 2 \pmod 5$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0	R is a γ_{cer}-set
	2	$R' = \{u_4\} \cup \{u_{5k+1} : 1 \leq k \leq \frac{n-2}{5}\}$ $R'' = \{v_1, v_2, v_5\} \cup \{v_{5k+3} : 1 \leq k \leq \frac{n-7}{5}\}$ $\cup\{v_{5k+4} : 1 \leq k \leq \frac{n-7}{5}\}$
$e_1, e_2 \in E_2$	1	$R' = \{u_3\} \cup \{u_{5k} : 1 \leq k \leq \frac{n-2}{5}\}$ $R'' = \{v_1, v_4\} \cup \{v_{5k+2} : 1 \leq k \leq \frac{n-2}{5}\}$ $\cup\{v_{5k+3} : 1 \leq k \leq \frac{n-7}{5}\}$
	3	$R' = \{u_1, u_4\} \cup \{u_{5k+3} : 1 \leq k \leq \frac{n-7}{5}\}$ $R'' = \{v_2\} \cup \{v_{5k} : 1 \leq k \leq \frac{n-2}{5}\}$ $\cup\{v_{5k+1} : 1 \leq k \leq \frac{n-2}{5}\}$
$e_1, e_2 \in E_3$ & $e_1 \in E_1$ & $e_2 \in E_3$	1 3	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_2$	2 3	R is a γ_{cer}-set
$e_1 \in E_2$ & $e_2 \in E_3$	2	$R' = \{u_1, u_4\} \cup \{u_{5k+3} : 1 \leq k \leq \frac{n-7}{5}\}$ $R'' = \{v_2\} \cup \{v_{5k} : 1 \leq k \leq \frac{n-2}{5}\}$ $\cup\{v_{5k+1} : 1 \leq k \leq \frac{n-2}{5}\}$

Table 7: γ_{cer}-set of H'' illustrating For $n \equiv 3 \pmod 5$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0	R is a γ_{cer}-set
	1	$R' = \{u_{5k-3} : 1 \leq k \leq \frac{n+2}{5}\}$ $\cup\{u_{5k} : 1 \leq k \leq \frac{n-3}{5}\}$ $R'' = \{v_{5k-4} : 1 \leq k \leq \frac{n+2}{5}\}$
$e_1, e_2 \in E_2$	0	R is a γ_{cer}-set
	2	$R' = \{u_{5k-4} : 1 \leq k \leq \frac{n+2}{5}\}$ $\cup\{u_3\} \cup \{u_{5k+4} : 1 \leq k \leq \frac{n-8}{5}\}$ $R'' = \{v_2, v_7\} \cup \{v_{5k+5} : 1 \leq k \leq \frac{n-8}{5}\}$
$e_1, e_2 \in E_3$	1 3	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_2$	1	$R' = \{u_{5k-4} : 1 \leq k \leq \frac{n+2}{5}\}$ $\cup\{u_{5k-1} : 1 \leq k \leq \frac{n-3}{5}\}$ $R'' = \{v_2\} \cup \{v_{5k} : 1 \leq k \leq \frac{n-3}{5}\}$
$e_1 \in E_1$ & $e_2 \in E_3$	0 2	R is a γ_{cer}-set
$e_1 \in E_2$ & $e_2 \in E_3$	1 3	R is a γ_{cer}-set

Subcase (ii): For $n \equiv 3 \pmod 5$

Let $e_1 = u_3u_4$, $e_2 = u_4u_5$ and $e_3 = u_5u_6$. Here $u_3 \in N(u_2)$, $u_6 \in N(v_6) \cap N(u_7)$, and $u_4 \notin N(R)$ to dominate u_4, Let $R_1' = \{u_1, u_3\} \cup \{u_{5k+2} : 1 \leq k \leq \frac{n-3}{5}\} \cup \{u_{5k+4} : 1 \leq k \leq \frac{n-8}{5}\}$ and $R_1'' = \{v_4, v_5\} \cup \{v_{5k+3} : 1 \leq k \leq \frac{n-6}{5}\}$ be a new configuration of γ_{cer} set of H'''. Here the vertices $v_n, v_{n-2} \notin N(R_1)$. Hence $R_1 \cup \{v_n\}$ is a γ_{cer}-set of H'''. Therefore $b^+_{cer}(H) = 3$, refer Figure 2.

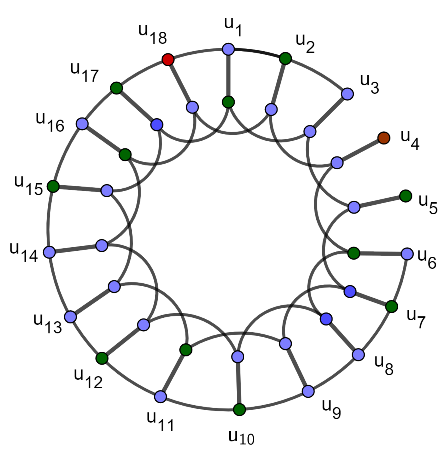

FIGURE 2. **Certified Dominating sets of H''' illustrating subcase (ii) for $n \equiv 3$ (mod 5)**

Table 8: γ_{cer}-set of H'' illustrating for $n \equiv 4 \pmod 5$

Edge set	$d(e_1, e_2)$	γ_{cer}-set
$e_1, e_2 \in E_1$	0	R is a γ_{cer}-set
	2	$R' = \{u_{5k+1} : 1 \le k \le \frac{n-4}{5}\}$ $R'' = \{v_{5k-2} : 1 \le k \le \frac{n+1}{5}\} \cup$ $\{v_{5k+4} : 1 \le k \le \frac{n-4}{5}\}$
$e_1, e_2 \in E_2$	0	$R' = \{u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-4}{5}\}$ $R'' = \{v_{5k-4} : 1 \le k \le \frac{n+1}{5}\} \cup$ $\{v_2\} \cup \{v_{5k} : 1 \le k \le \frac{n-4}{5}\}$
$e_1, e_2 \in E_3$	1 2	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_2$	1 2	R is a γ_{cer}-set
$e_1 \in E_1$ & $e_2 \in E_3$ e	1 2	R is a γ_{cer}-set
$e_1 \in E_2$ & $e_2 \in E_3$	0	R is a γ_{cer}-set
	1	$R' = \{u_1, u_4, u_6\} \cup \{u_{5k+5} : 1 \le k \le \frac{n-9}{5}\}$ $R'' = \{v_2, v_5\} \cup \{v_{5k+3} : 1 \le k \le \frac{n-x}{5}\}$ $\cup \{v_{5k+7} : 1 \le k \le \frac{n-9}{5}\}$

Subcase (iii): For $n \equiv 4 \pmod 5$

Let $e_1 = u_2v_2$, $e_2 = u_3v_3$ and $e_3 = u_4v_4$. Here $u_1 \in N(v_1)$, $u_3 \in N(v_4)$, $u_2 \notin N(R)$. To dominate u_2, the set $R_1 = R_1' \cup R_1''$, where $R_1' = \{u_1, u_2, u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-9}{5}\}$, $R_1'' = \{v_{5k} : 1 \le k \le \frac{n-4}{5}\} \cup \{v_{5k+1} : 1 \le k \le \frac{n-4}{5}\}$ a new configuration of γ_{cer}- set of H''' is constructed. Here the vertex $u_{n-1} \notin N(R_1)$. Hence $R_1 \cup \{u_{n-1}\}$ is a γ_{cer}-set of H'''. Therefore $b^+_{cer}(H) = 3$. Based on the aforementioned cases, it can be observed that $b^+_{cer}(H) = 3$ for $n \equiv 0, 3, 4 \pmod 5$.

Case (ii): $n \equiv 1, 2 \pmod 5$. Let $H''' = H - \{e_1, e_2, e_3\}$. From tables 9 and 10, we notice that the γ_{cer}-sets of H'' based on the pairwise distances between the edges e_1, e_2 and e_3 which are removed from H and the new configuration of γ_{cer}-set R_1 of H''' is constructed. In all the above position of the edges e_1, e_2 and e_3 we get $|R| = |R_1|$. Hence $b^+_{cer}(H) > 3$. Likewise, removing any three edges e_1, e_2, and e_3, regardless of the distance between them, results in the same certified domination number. Hence $b^+_{cer}(H) > 3$.

Table 9: γ_{cer}-set of H''' illustrating case (ii) for $n \equiv 1 \pmod 5$

Edge set	$d(e_1,e_2)$	$d(e_2,e_3)$	γ_{cer}-set
$e_1,e_2,e_3 \in E_1$ $\&\ e_1,e_2,e_3 \in E_2$	0	0	$R' = \{u_3\} \cup \{u_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_1\} \cup \{v_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$
	1	2	R is a γ_{cer}-set
$e_1,e_2,e_3 \in E_1$	2	1	$R' = \{u_4,u_6\} \cup \{u_{5k+5} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_1\} \cup \{v_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$
$e_1,e_2 \in E_1\ \&\ e_3 \in E_2$	2 3	0 2	R is a γ_{cer}-set
$e_1,e_2 \in E_2\ \&\ e_3 \in E_3$	1	1	$R' = \{u_3\} \cup \{u_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_1,v_2\} \cup \{v_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$
$e_1,e_2 \in E_1\ \&\ e_3 \in E_3$	2	2	$R' = \{u_4,u_6\} \cup \{u_{5k+5} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_1\} \cup \{v_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$
$e_1 \in E_1, e_2 \in E_2$ $e_2 \in E_2$ $\&\ e_3 \in E_3$	1	1	$R' = \{u_4,u_6\} \cup \{u_{5k+5} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_1,v_2\} \cup \{v_{5k+2} : 1 \le k \le \frac{n-6}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$

Table 10: γ_{cer}-set of H^{iv} illustrating case (ii) for $n \equiv 1 \pmod 5$

Edge set	$d(e_1,e_2)$	$d(e_2,e_3)$	$d(e_3,e_4)$	γ_{cer}-set
$e_1,e_2,e_3,e_4 \in E_1$	0	0	0	$R' = \{u_4\} \cup \{u_{5k+1} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_1,v_2\} \cup \{v_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$ $\cup\{v_{5k+4} : 1 \le k \le \frac{n-6}{5}\}$
	1	2	1	$R' = \{u_2,u_5,u_7\} \cup \{u_{5k+4} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k-4} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+7} : 1 \le k \le \frac{n-11}{5}\}$
$e_1,e_2,e_3,e_4 \in E_2$	0	0	0	$R' = \{u_1,u_3,u_5\} \cup \{u_{5k+2} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k-1} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+5} : 1 \le k \le \frac{n-6}{5}\}$
	1	1	1	$R' = \{u_1\} \cup \{u_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $R'' = \{v_{5k-3} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k-2} : 1 \le k \le \frac{n-1}{5}\}$
$e_1,e_2,e_3,e_4 \in E_3$	2	2	2	$R' = \{u_2,u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k+1} : 1 \le k \le \frac{n-1}{5}\}$
	3	2	2	R is a γ_{cer}-set
$e_1,e_2 \in E_1, e_3 \in E_2,$ $e_4 \in E_3$	1 3	2 2	0 3	$R' = \{u_1,u_3\} \cup \{u_{5k+2} : 1 \le k \le \frac{n-6}{5}\}$ $R'' = \{v_{5k-1} : 1 \le k \le \frac{n-1}{5}\}$ $\cup\{v_{5k} : 1 \le k \le \frac{n-1}{5}\}$

Let $H^{iv} = G - \{e_1, e_2, e_3, e_4\}$. Then the following tables 10 and 11 gives the new configuration of R_1 of H^{iv} based on the pairwise distances between the edges e_1, e_2, e_3ande_4 which are removed from H. In all the above position of the edges e_1, e_2 , e_3 *and* e_4 we see that $|R| = |R_1|$. Hence $b^+_{cer}(H) > 4$. likewise the removal of four edges e_1, e_2, e_3, and e_4, at any distance between them, yields the same certified domination number. Hence $b^+_{cer}(H) > 4$.
Now for $n \equiv 1 \pmod 5$. Let $H^v = H - \{e_1, e_2, e_3, e_4, e_5\}$, where $e_1 = u_1u_2$, $e_2 = u_2u_3$, $e_3 = u_3u_4$, $e_4 = u_4u_5$ and $e_5 = u_5u_6$. Here $u_1 \in N(v_1)$, $u_2 \in N(v_2)$, $u_3, u_5 \notin N(R)$. To dominate the vertices u_3 and u_5 the set $R_1 = R'_1 \cup R''_1$, where $R'_1 = \{u_{5k+2} : 1 \le k \le \frac{n-6}{5}\}$ and $R''_1 = \{v_2, v_3, v_4, v_8\} \cup \{v_{5k} : 1 \le k \le \frac{n-1}{5}\} \cup \{v_{5k+9} : 1 \le k \le \frac{n-11}{5}\}$ a new configuration of γ_{cer}-set of H^v is constructed. Now the vertex $u_n \notin N(R_1)$. Hence $R_1 \cup \{u_n\}$ is a R of H^v. Therefore $b^+_{cer}(H) = 5$.

Table 11: γ_{cer}-set of H^{iv} illustrating case (ii) for $n \equiv 2 \pmod 5$

Edge set	$d(e_1,e_2)$	$d(e_2,e_3)$	$d(e_3,e_4)$	γ_{cer}-set
$e_1, e_2, e_3, e_4 \in E_1$	0	0	0	$R' = \{u_{5k+1} : 1 \le k \le \frac{n-2}{5}\}$ $R'' = \{v_1, v_2\} \cup \{v_{5k-2} : 1 \le k \le \frac{n-2}{5}\}$ $\cup\{v_{5k-1} : 1 \le k \le \frac{n-2}{5}\}$
	3	1	2	R is a γ_{cer}-set
$e_1, e_2, e_3, e_4 \in E_2$	0	0	0	$R' = \{u_2, u_3, u_5\} \cup \{u_{5k+4} : 1 \le k \le \frac{n-7}{5}\}$ $R'' = \{v_{5k+1} : 1 \le k \le \frac{n-2}{5}\}$ $\cup\{v_{5k+2} : 1 \le k \le \frac{n-2}{5}\}$
$e_1, e_2, e_3, e_4 \in E_3$	1	1	1	$R' = \{u_2\} \cup \{u_{5k} : 1 \le k \le \frac{n-2}{5}\}$ $R'' = \{v_1, v_4\} \cup \{v_{5k+2} : 1 \le k \le \frac{n-7}{5}\}$ $\cup\{v_{5k+3} : 1 \le k \le \frac{n-7}{5}\}$
$e_1, e_2 \in E_1, e_3 \in E_2$, $e_4 \in E_3$	2	3	2	$R' = \{u_1, u_3, u_6, u_9\}$ $\cup\{u_{5k+8} : 1 \le k \le \frac{n-12}{5}\}$

For $n \equiv 2 \pmod 5$. Let $H^v = H - \{e_1, e_2, e_3, e_4, e_5\}$, where $e_1 = v_nv_2$, $e_2 = v_nv_{n-2}$, $e_3 = v_{n-1}v_{n-3}$, $e_4 = v_{n-1}v_1$, $e_5 = v_1v_3$. Here $v_{n-3} \in N(u_{n-3})$, $v_{n-2}, v_3 \notin N(R)$. To dominate the vertices v_3 and v_{n-2}, the set $R_1 = R'_1 \cup R''_1$, where $R'_1 = \{u_1, u_4\} \cup \{u_{5k+3} : 1 \le k \le \frac{n-7}{5}\}$, $R''_1 = \{v_2\} \cup \{v_{5k} : 1 \le k \le \frac{n-2}{5}\} \cup \{v_{5k+1} : 1 \le k \le \frac{n-2}{5}\}$ a new configuration of R of H^v is constructed. Now the vertex $v_n \notin N(R_1)$. Hence $R_1 \cup \{u_n\}$ is a γ_{cer}-set of H^v. Therefore $b^+_{cer}(H) = 5$. □

These observations are drawn from the previously mentioned theorems.
Observation 3.7 For a wheel graph W_n, $n \ge 5$, $b^+_{cer}(H) = 1$.
Observation 3.8 For a windmill graph $H = Wd(m,p)$, *where* $m \ge 3, p \ge 2$,by joining p copies of the complete graph K_m at a shared universal vertex, $b^+_{cer}(H) = 1$.
Observation 3.9 For a binary tree T, $b^+_{cer}(T) = 1$.

4. Implications of the study

The concept of certified bondage numbers for generalized Petersen graphs is defined and then analysed, thus extending theoretical knowledge on graph domination. The empirical results give an insight into the stability and resistance of graph structures toward edge modifications under very stringent rules of domination. This work identifies critical edges that determine the behaviour of certified domination parameters after edge removal. Such knowledge has direct implications for optimizing network design and fault-tolerant systems in which one would like to maintain performance under connectivity constraints on the resource allocation front. From the application viewpoint, it introduces a novel analytical tool for problems that deal with networks with relatively strong connectivity requirements. For example, in the communication network, certified bondage numbers can measure the robustness of the system for possible link failures at the cost of not losing its essential connectivity properties. In supply chain logistics or transportation networks, analogous parameters can guide the design of more robust systems and identify critical connections to strengthen first. The new approach taken here expands the area of theoretical study into graph domination. Further, it gives a direction toward many real-life applications, including network analysis, infrastructure planning, and modelling of biological systems.

5. Conclusion

This paper extends the classical notion of bondage numbers to the setting of certified domination by introducing the certified bondage number of a graph. We defined and examined the parameters $b^{+}_{\text{cer}}(H)$ and $b^{-}_{\text{cer}}(H)$, representing the smallest number of edges whose removal increases or decreases the certified domination number, respectively. Our results include exact values of certified bondage numbers for generalized Petersen graphs $P(n,k)$, for $k = 1,2$, and for certain classes of connected graphs. These findings deepen the understanding of structural graph properties related to domination under edge removal and open avenues for further research in more complex or specialized graph families.

6. Limitations

While the results of this work indicate bondage numbers for generalized Petersen graphs, its scope somewhat limits it to specific classes and parameters of graphs. The findings cannot be generalized to many other complex graph structures or many real-world networks with irregular or dynamic topologies. Further, computation involved in having bonded numbers through certified bondage is considerably a calculation of the number of edge configurations that is even more computationally intensive with increasing size and complexities in the graph.

7. Future scope

More extensive analysis of certified bondage numbers on other classes of graphs, like bipartite graphs, planar graphs, dynamic networks, etc., would give more insight into their behavior and utility. Developing algorithms to compute certified bondage numbers efficiently for larger graphs could also be a crucial application. Another way to understand the certified bondage numbers is to relate them to real-world applications, such as designing resilient networks or optimizing supply chains and biological systems, so the theoretical background is understood. So, future work may also present the role of certified bondage numbers in dynamically evolving networks where edges are added or removed, noticing what kind of adaptation those systems create and remain robust under changing conditions.

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