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Hyers-Ulam Stability of Fractional Differential Equations Using West Nile Virus

S. Dhivya[1], **Vediyappan Govindan**[1,*], **Siriluk Donganont**[2,*]

[1]*Department of Mathematics, Hindustan Institute of Technology and Science, Chennai - 603103, Tamil Nadu, India*

[2]*School of Science, University of Phayao, Phayao 56000, Thailand*

[*]*Corresponding authors:* govindoviya@gmail.com, siriluk.pa@up.ac.th

Abstract. In this paper, we use the Atangana-Baleanu Caputo derivative to design and assess a fractional-order infection model for West Nile virus. The model offers insights into the evolution of the virus and takes into account its intricate dynamics of transmission. We investigate the system's qualitative behavior and prove existence and uniqueness findings using fixed-point theory. Additionally, we examine the suggested model's stability in terms of Hyers-Ulam stability. Euler's approach for the Atangana-Baleanu integral is used to numerically simulate the fractional-order model in order to visualize the effects of different parameters. The theoretical findings are verified, and the impact of fractional-order derivatives on the dynamics of the system is demonstrated using MATLAB. The study emphasizes the use of fractional calculus in epidemiological modeling, which offers a more realistic depiction of the spread of illness in the actual world.

1. Introduction

The West Nile virus (WNV), a single-stranded virus that belongs to the Flavivirus in the family Flaviviridae, is the cause of West Nile disease (WND), an emergent vector-borne illness that was initially discovered in the West Nile area of Uganda in 1937 (Smithburn et al., 1940). The virus has spread throughout Africa, Europe, the Middle East, West and Central Asia, and most recently, North America. It is spread by female mosquitoes that have fed on the blood of infected birds and can infect humans and other animals (Campbell et al., 2002; Centers of Disease control and prevention, 2002 a; Chowers et al., 2001; Nash et al., 2001; Petersen and Marfin, 2002). Few studies have fully analyzed the viraemic responses of various bird species, and the distribution of various WNV strains along with the diversity of local avian community composition make extrapolating results to regions outside of those tested potentially erroneous [14, 17, 26, 30].

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In many disciplines, like engineering, chemistry, and biology, where differential equations are usually used to simulate real-world processes, time delays are crucial. The time lag between an activity and its impact on a system is represented by these delays [7–10, 18, 19, 25, 28, 29].

In recent years, researchers have explored different fractional derivative formulations, with the Atangana-Baleanu Caputo (ABC) [1–3] derivative gaining significant attention due to its non-locality, singular kernel, and ability to incorporate memory effects more effectively than classical Caputo or Riemann-Liouville derivatives. The ABC derivative [2] enhances the modeling of disease dynamics by accounting for long-term interactions between infected and susceptible populations, thus offering a more realistic representation of WNV transmission [12, 13].

Another crucial aspect of studying fractional models is analyzing their stability properties, which provide insight into the robustness of solutions under small perturbations. The Hyers-Ulam stability concept plays a fundamental role in determining whether small deviations in initial conditions lead to bounded or unbounded solutions over time. Establishing Hyers-Ulam stability ensures the reliability of the fractional WNV model by confirming that minor errors in population estimates do not result in unrealistic predictions. This stability analysis is particularly important for epidemiological models, where precise forecasting of infection trends is critical for implementing public health interventions.

In this study, we formulate a fractional-order West Nile virus model using the Atangana-Baleanu Caputo derivative and investigate its Hyers-Ulam stability. Our primary objectives are:

(1) To develop a fractional mathematical model incorporating bird and mosquito vectors with memory-dependent transmission rates.
(2) To analyze the Hyers-Ulam stability of the proposed model and establish conditions under which the system remains stable.
(3) To validate the effectiveness of fractional derivatives in capturing the realistic dynamics of WNV transmission.

The remainder of the document is organized as follows: Basic preliminary information is presented in Section 2, and the fractional model's mathematical formulation is presented in Section 3. WNV model using the Atangana-Baleanu Caputo derivative section 5 explores the stability analysis with a focus on Hyers - Ulam stability criteria. Section 6 provides numerical simulations to support theoretical findings and section 7 concludes the study with key insights and future research directions.

Section 4 explores the existence and uniqueness of the proposed model.

2. Preliminaries

Definition 2.1: [22]

Let $f \in [1, \infty), \Omega \subseteq \mathbb{R}$ be open subsets. Next, the following describes the Sobolev space $H^{+}(\Omega)$:

$$H^{+}(\Omega) = \left\{ g \in L^2(\Omega);\ D^{\beta} g \in L^2(\Omega),\ \forall |\beta| \leq f \right\}$$

Definition 2.2: [23]

For a function $f \in H'(0,1)$ and $0 < \lambda < 1$, and the Atangana-Baleanu fractional derivative in Caputo sense is defined as:

$$^{ABC}{}_0D_x^{\lambda} f(x) = \frac{\mu(\lambda)}{1-\lambda}\int_0^x f'(x)E_{\lambda}\left[-\lambda\frac{(x-y)^{\lambda}}{1-\gamma}\right]dy$$

where $\mu(\lambda)$ is a normalization function that satisfies $\mu(0) = 1 = \mu(1)$. The Mittag-Leffler function E_λ is represented as:

$$E_{\lambda} = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\lambda k+1)}$$

Additionally, the fractional integral of Atangana-Baleanu integral of order λ is given by:

$$L_x^{\lambda} f(x) = \frac{1-\lambda}{\lambda}f(x) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^x f(y)(x-y)^{\lambda-1}dy.$$

Theorem 2.1:

The Atangana-Baleanu fractional derivative, both in the Caputo and Riemann-Liouville senses, satisfy the Lipschitzcondition for any two functions $f, g \in H^1(l,m)$, where $m > l$:

$$||^{ABC}{}_0D_x^{\lambda} f(x) - {}^{ABC}{}_0D_x^{\lambda} g(x)|| \leq \pi||f(x) - g(x)||$$

and

$$||^{ABR}{}_0D_x^{\lambda} f(x) - {}^{ABR}{}_0D_x^{\lambda} g(x)|| \leq \pi||f(x) - g(x)||$$

3. Mathematical Formulation

3.1. **Single Resident Bird Model.** The system of ODEs describing the dynamics of mosquito vectors and bird hosts is formulated as follows:

$$\begin{aligned}
\dot{S}_v &= \psi_v N_v - \lambda_v S_v - \mu_v S_v \\
\dot{E}_v &= \lambda_v S_v \nu_v E_v - \mu_v E_v \\
\dot{I}_v &= \nu_v E_v - \mu_v I_v \\
\dot{S}_b &= \psi_b H + \rho_b R_b - \lambda_b S_b - \mu_b S_b \\
\dot{E}_b &= \lambda_b S_b - \nu_b E_b - \mu_b E_b \\
\dot{I}_b &= \nu_b E_b - \gamma_b I_b - \mu_b I_b - \delta_b I_b \\
\dot{R}_b &= \gamma_b I_b - \rho_b R_b - \mu_b R_b
\end{aligned} \tag{3.1}$$

The derivatives are taken with respect to time, measured in days. The subscript ν denotes the mosquito vector, while b represents the resident bird host.

Both birds and mosquitoes enter the susceptible population at a per-capita birth rate μ_i and exit all compartments at the same rate μ_i. Mosquitoes transition from exposed to infectious stage at a constant per-capita rate ν_v, which corresponds to the inverse of the incubation period. Infected birds progress to the infectious state at rate ν_b and recover at a constant rate γ_b. The recruitment rate of birds is expressed as:

$$H = \frac{\mu_b}{\psi_b} K$$

where K represents the carrying capacity. The mosquitoes biting rate is determined as a function of the total number of bites per day.

$$b = \frac{\sigma_v N_v . \sigma_b N_b}{\sigma_v N_v . \sigma_b N_b} \tag{3.2}$$

The parameter σ_v is the number of per day by mosquito and σ_b is the number of bites per day by an arian hoset. The force of infection λ_v and λ_b are

$$\lambda_v = b_{v_b} \beta_{v_b} \frac{I_b}{N_b}, \lambda_b = b_{b_v} \beta_{b_v} \frac{I_v}{N_v} \tag{3.3}$$

are the rates individuals are infected and move from the susceptible to the exposed class.

Parameters	Descriptions
S_v	Susceptible mosquitoes
S_b	Susceptible resident birds
E_v	Exposed vectors
E_b	Exposed birds
I_v	Infected vectors
I_b	Infected birds
R_b	Recovered birds
N_v	Total vectors
N_b	Total birds
ψ_v	Recruitment rate of vectors
ψ_b	Recruitment rate of birds
μ_i	Death rate of species i
ν_i	Incubation rate for species i
γ_i	Loss of infectivity rate of species i
ρ_i	Loss of immunity rate of species i
σ_i	Disease induced death rate of species i
b	Total number of bites per day
b_{bv}	Numbers of bites from vector per bird per day
b_{vb}	Numbers of bites from bird per vector per day
β_{ij}	Probability of transmission per bite j to i
ρ_i	Max hosts
λ_i	inoculate rate of species i
N_v	Total vectors
N_b	Total birds
ψ_v	Recruitment rate of vectors
ψ_b	Recruitment rate of birds
μ_i	Death rate of species i
ν_i	Incubation rate for species i
γ_i	Loss of infectivity rate of species i
ρ_i	Loss of immunity rate of species i
σ_i	Disease induced death rate of species i
b	Total number of bites per day
b_{bv}	Numbers of bites from vector per bird per day
b_{vb}	Numbers of bites from bird per vector per day
β_{ij}	Probability of transmission per bite j to i
ρ_i	Max hosts
λ_i	inoculate rate of species i

4. Existence and Uniqueness of Solution

This section examines if a solution exists for the model (3.1)'s, fractional order. So, we have

$$S_v(t) = S_v(0) + \frac{1-\gamma_1}{B(\gamma_1)}(\psi_v N_v^{(s)} - \lambda_v S_v^{(s)} - \mu_v S_v^{(s)}) + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1 - 1}((\psi_v N_v^{(s)} - \lambda_v S_v^{(s)} - \mu_v S_v^{(s)})dq \tag{4.1}$$

$$E_v(t) = E_v(0) + \frac{1-\gamma_2}{B(\gamma_2)}(\lambda_v S_v(s) - V_v E_v(s) - \mu_v(s)) + \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)}\int_0^s (s-q)^{\gamma_2 - 1}((\lambda_v S_v(s) - V_v E_v(s) - \mu_v(s))dq \tag{4.2}$$

$$I_v(t) = I_v(0) + \frac{1-\gamma_3}{B(\gamma_3)}(V_v E_v(s) - \mu_v I_v(s)) + \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)}\int_0^s (s-q)^{\gamma_3 - 1}((V_v E_v(s) - \mu_v I_v(s))dq \tag{4.3}$$

$$S_b(t) = S_b(0) + \frac{1-\gamma_4}{B(\gamma_4)}(\psi_b \mu(s) + \rho_b R_b(s) + \lambda_b S_b(s) - \mu_b S_b(s)) + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)}\int_0^s (s-q)^{\gamma_4 - 1}(\psi_b \mu(s) + \rho_b R_b(s) + \lambda_b S_b(s) - \mu_b S_b(s))dq \tag{4.4}$$

$$E_b(t) = E_b(0) + \frac{1-\gamma_5}{B(\gamma_5)}(\lambda_b S_b(s) - v_b E_b(s) - \mu_b E_b(s)) + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)}\int_0^s (s-q)^{\gamma_5 - 1}(\lambda_b S_b(s) - v_b E_b(s) - \mu_b E_b(s))dq \tag{4.5}$$

$$I_b(t) = I_b(0) + \frac{1-\gamma_6}{B(\gamma_6)}(V_b E_b(s) - \gamma_b I_b(s) - \mu_b I_b(s) - \delta_b E_b(s)) + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)}\int_0^s (s-q)^{\gamma_6 - 1}(V_b E_b(s) - \gamma_b I_b(s) - \mu_b I_b(s) - \delta_b E_b(s))dq \tag{4.6}$$

$$R_b(t) = R_b(0) + \frac{1-\gamma_7}{B(\gamma_7)}(\gamma_b I_b(s) - \rho_b R_b(s) - \mu_b R_b(s)) + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)}\int_0^s (s-q)^{\gamma_7 - 1}(\gamma_b I_b(s) - \rho_b R_b(s) - \mu_b R_b(s))dq \tag{4.7}$$

Now we can write as follows:

$$\begin{aligned} Z_1(s, S_v) &= \psi_v - N_v(s) - \lambda_v S_v(s) - \mu_v S_v(s) \\ Z_2(s, E_v) &= \lambda_v S_v(s) - V_v E_v(s) - \mu_v E_v(s) \\ Z_3(s, I_v) &= V_v E_v(s) - \mu_v I_v(s) \end{aligned}$$

$$
\begin{aligned}
Z_4(s, S_b) &= \psi_b - \mu(s) + \rho_b R_b(s) - \lambda_b S_b(s) - \mu_b S_b(s) \\
Z_5(s, E_b) &= \lambda_b S_b(s) - V_v E_b(s) - \mu_b E_b(s) \\
Z_6(s, I_b) &= V_b E_b(s) - \gamma_b I_b(s) - \mu_b I_b(s) - \delta_b I_b(s) \\
Z_7(s, R_b) &= \gamma_b I_b(s) - \rho_b R_b(s) - \mu_b R_b(s)
\end{aligned}
$$

And define Q_i as follows:

$$
\begin{aligned}
Q_1 &= \psi_v - N_v(s) - \lambda_v - \mu_v \\
Q_2 &= \lambda_v S_v(s) - V_v - \mu_v \\
Q_3 &= V_v E_v(s) - \mu_v \\
Q_4 &= \psi_b \mu(s) - \rho_b R_b(s) - \lambda_b - \mu_b \\
Q_5 &= \lambda_b - S_b(s) - V_b - \mu_b \\
Q_6 &= \gamma_b E_b(s) - \gamma_b - \mu_b - \delta_b \\
Q_7 &= \gamma_b I_b(s) - \rho_b - \mu_b.
\end{aligned}
\tag{4.8}
$$

To prove our findings, we use the following assumptions:

For the $S_v(s)$, $\hat{S}_v(s)$, $E_u(s)$, $\hat{E}_v(s)$, $I_v(s)$, $\hat{I}_v(s)$, $S_b(s)$, $\hat{S}_b(s)$, $E_b(s)$, $\hat{E}_b(s)$, $I_b(s)$, $\hat{I}_b(s)$, $R_b(s)$, $\hat{R}_b(s)$ $\in L0,1$ be continuous functions such that: $|S_v(s)| \leq L_1$, $|E_v(s)| \leq L_2$, $|I_v(s)| \leq L_3$, $|S_b(s)| \leq L_4$, $|E_b(s)| \leq L_5$, $|I_b(s)| \leq L_6$, $|R_b(s)| \leq L_7$

Theorem 4.1: The Lipschitz property is satisfy the Z_i for $i = 1, 2, \ldots, 7$ if $0 \leq Q_i < 1$, if the assumption (B) is hold true and fulfills $Q_i < 1$ for $i = 1, 2, \ldots, 7$.

Proof. First, we prove that $Z_1(s, S_v)$ satisfies the Lipschitz property. Now,

$$
\begin{aligned}
\|Z_1(s, S_v) - Z_1(s, \hat{S}_v)\| &= \|\psi_v N_v(s) - \lambda_v S_v(s) - \mu_v S_v(S) - (\psi_v N_v(s) - \lambda_v \hat{S}_v(s) - \mu_v \hat{S}_v(s))\| \\
&\leq (\psi_v N_v(s) - \lambda_v \hat{S}_v(s) - \mu_v \hat{S}_v(s))\|S_v - \hat{S}_v\| \\
&\leq Q\|S_v - \hat{S}\|.
\end{aligned}
\tag{4.9}
$$

where $Q_i = \psi_v N_v(s) - \lambda_v \hat{S}_v(s) - \mu_v \hat{S}_v(s)$.

Similarly, the other kernels satisfies the Lipschitz property:

$$
\begin{aligned}
\|Z_2(S, E_v) - Z_2(S, \hat{E}_v)\| &\leq Q_2\|E_v - \hat{E}_v\| \\
\|Z_3(S, I_v) - Z_3(S, \hat{I}_v)\| &\leq Q_3\|I_v - \hat{I}_v\| \\
\|Z_4(S, S_b) - Z_4(S, \hat{S}_b)\| &\leq Q_4\|S_b - \hat{S}_b\| \\
\|Z_5(S, E_b) - Z_5(S, \hat{E}_b)\| &\leq Q_5\|E_b - \hat{E}_b\| \\
\|Z_6(S, I_b) - Z_6(S, \hat{I}_b)\| &\leq Q_6\|I_b - \hat{I}_b\| \\
\|Z_7(S, R_b) - Z_7(S, \hat{R}_b)\| &\leq Q_7\|R_b - \hat{R}_b\|
\end{aligned}
$$

Thus, if all the kernels Z_i satisfy the condition $0 \leq Q_i < 1, i = 1,2,\ldots,7$, they fulfill the Lipschitz propertyand consequently form a contraction for Z_i, for all $i = 1,2,\ldots,7$.

Now, Equation (4.2) to (4.7) can be rewrite as

$$S_v(s) - S_v(0) = \frac{1-\gamma_1}{B(\gamma_1)} z_1(s, S_v(s)) + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1 - 1} Z_1(q, S_v(q))dq. \tag{4.10}$$

$$E_v(s) - E_v(0) = \frac{1-\gamma_2}{B(\gamma_2)} z_2(s, E_v(s)) + \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)} \int_0^s (s-q)^{\gamma_2 - 1} Z_2(q, E_v(q))dq. \tag{4.11}$$

$$I_v(s) - I_v(0) = \frac{1-\gamma_3}{B(\gamma_3)} z_3(s, I_v(s)) + \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)} \int_0^s (s-q)^{\gamma_3 - 1} Z_3(q, I_v(q))dq. \tag{4.12}$$

$$S_b(s) - S_b(0) = \frac{1-\gamma_4}{B(\gamma_4)} z_4(s, S_b(s)) + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)} \int_0^s (s-q)^{\gamma_4 - 1} Z_4(q, S_b(q))dq. \tag{4.13}$$

$$E_b(s) - E_b(0) = \frac{1-\gamma_5}{B(\gamma_5)} z_5(s, E_b(s)) + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)} \int_0^s (s-q)^{\gamma_5 - 1} Z_5(q, E_b(q))dq. \tag{4.14}$$

$$I_b(s) - I_b(0) = \frac{1-\gamma_6}{B(\gamma_6)} z_6(s, I_b(s)) + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)} \int_0^s (s-q)^{\gamma_6 - 1} Z_6(q, I_b(q))dq. \tag{4.15}$$

$$R_b(s) - R_b(0) = \frac{1-\gamma_7}{B(\gamma_7)} z_7(s, R_b(s)) + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)} \int_0^s (s-q)^{\gamma_7 - 1} Z_7(q, R_b(q))dq. \tag{4.16}$$

with the initial conditions are given as $S_{v_0}(s) = S_v(0)$, $E_{v_0}(s) = E_v(0)$, $I_{v_0}(s) = I_v(0)$, $S_{b_0}(s) = S_b(0)$, $E_{b_0}(s) = E_b(0)$, $I_{b_0}(s) = I_b(0)$ and $R_{b_0}(s) = R_b(0)$. Apply the recursive relation, we get:

$$\begin{aligned} S_{v_n}(s) - S_v(0) &= \frac{1-\gamma_1}{B(\gamma_1)} Z_1(s, S_{v_{n-1}}(s)) \\ &+ \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1 - 1} Z_1(q, S_{v_{n-1}}(q)) dq, \end{aligned} \tag{4.17}$$

$$\begin{aligned} E_{v_n}(s) - E_v(0) &= \frac{1-\gamma_2}{B(\gamma_2)} Z_2(s, E_{v_{n-1}}(s)) \\ &+ \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)} \int_0^s (s-q)^{\gamma_2 - 1} Z_2(q, E_{v_{n-1}}(q)) dq, \end{aligned} \tag{4.18}$$

$$\begin{aligned} I_{v_n}(s) - I_v(0) &= \frac{1-\gamma_3}{B(\gamma_3)} Z_3(s, I_{v_{n-1}}(s)) \\ &+ \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)} \int_0^s (s-q)^{\gamma_3 - 1} Z_3(q, I_{v_{n-1}}(q)) dq, \end{aligned} \tag{4.19}$$

$$\begin{aligned} S_{b_n}(s) - S_b(0) &= \frac{1-\gamma_4}{B(\gamma_4)} Z_4(s, S_{b_{n-1}}(s)) \\ &+ \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)} \int_0^s (s-q)^{\gamma_4 - 1} Z_4(q, S_{b_{n-1}}(q)) dq, \end{aligned} \tag{4.20}$$

$$\begin{aligned} E_{b_n}(s) - E_b(0) &= \frac{1-\gamma_5}{B(\gamma_5)} Z_5(s, E_{b_{n-1}}(s)) \\ &+ \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)} \int_0^s (s-q)^{\gamma_5 - 1} Z_5(q, E_{b_{n-1}}(q)) dq, \end{aligned} \tag{4.21}$$

$$\begin{aligned} I_{b_n}(s) - I_b(0) &= \frac{1-\gamma_6}{B(\gamma_6)} Z_6(s, I_{b_{n-1}}(s)) \\ &+ \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)} \int_0^s (s-q)^{\gamma_6 - 1} Z_6(q, I_{b_{n-1}}(q)) dq, \end{aligned} \tag{4.22}$$

$$\begin{aligned} R_{b_n}(s) - R_b(0) &= \frac{1-\gamma_7}{B(\gamma_7)} Z_7(s, R_{b_{n-1}}(s)) \\ &+ \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)} \int_0^s (s-q)^{\gamma_7 - 1} Z_7(q, R_{b_{n-1}}(q)) dq, \end{aligned} \tag{4.23}$$

Now, we look at the differences between the succeeding words as follows:

$$\begin{aligned} K_{1n}(s) = (S_{v_n} - S_{v_{n-1}})(s) &= \frac{1-\gamma_1}{B(\gamma_1)} (Z_1(s, S_{v_n}(s))) - (Z_1(s, S_{v_n-1}(s))) \\ &+ \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1 - 1} (Z_1(q, S_{v_n}(q)) - Z_1(q, S_{v_{n-1}}(q))) dq, \end{aligned} \tag{4.24}$$

$$K_{2n}(s) = (E_{v_n} - E_{v_{n-1}})(s) = \frac{1-\gamma_2}{B(\gamma_2)}(Z_2(s, E_{v_n}(s))) - (Z_2(s, E_{v_n-1}(s)))$$
$$+ \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)}\int_0^s (s-q)^{\gamma_2-1}(Z_2(q, E_{v_n}(q)) - Z_2(q, E_{v_{n-1}}(q)))dq, \tag{4.25}$$

$$K_{3n}(s) = (I_{v_n} - I_{v_{n-1}})(s) = \frac{1-\gamma_3}{B(\gamma_3)}(Z_3(s, I_{v_n}(s))) - (Z_3(s, I_{v_n-1}(s)))$$
$$+ \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)}\int_0^s (s-q)^{\gamma_3-1}(Z_3(q, I_{v_n}(q)) - Z_3(q, I_{v_{n-1}}(q)))dq, \tag{4.26}$$

$$K_{4n}(s) = (S_{b_n} - S_{b_{n-1}})(s) = \frac{1-\gamma_4}{B(\gamma_4)}(Z_4(s, S_{b_n}(s))) - (Z_4(s, S_{b_n-1}(s)))$$
$$+ \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)}\int_0^s (s-q)^{\gamma_4-1}(Z_4(q, S_{b_n}(q)) - Z_4(q, S_{b_{n-1}}(q)))dq, \tag{4.27}$$

$$K_{5n}(s) = (E_{b_n} - E_{b_{n-1}})(s) = \frac{1-\gamma_5}{B(\gamma_5)}(Z_5(s, E_{b_n}(s))) - (Z_5(s, E_{b_n-1}(s)))$$
$$+ \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)}\int_0^s (s-q)^{\gamma_5-1}(Z_5(q, E_{b_n}(q)) - Z_5(q, E_{b_{n-1}}(q)))dq, \tag{4.28}$$

$$K_{6n}(s) = (I_{b_n} - I_{b_{n-1}})(s) = \frac{1-\gamma_6}{B(\gamma_6)}(Z_6(s, I_{b_n}(s))) - (Z_6(s, I_{b_n-1}(s)))$$
$$+ \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)}\int_0^s (s-q)^{\gamma_6-1}(Z_6(q, I_{b_n}(q)) - Z_6(q, I_{b_{n-1}}(q)))dq, \tag{4.29}$$

$$K_{7n}(s) = (R_{b_n} - R_{b_{n-1}})(s) = \frac{1-\gamma_7}{B(\gamma_7)}(Z_7(s, R_{b_n}(s))) - (Z_7(s, R_{b_n-1}(s)))$$
$$+ \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)}\int_0^s (s-q)^{\gamma_7-1}(Z_7(q, R_{b_n}(q)) - Z_7(q, R_{b_{n-1}}(q)))dq, \tag{4.30}$$

For equations (4.24) to (4.30) taking the norm for both sides

$$\begin{aligned}
\|K_{1n}(s)\| &= \|S_{v_n} - S_{v_{n-1}}\|(s)\\
&= \frac{1-\gamma_1}{B(\gamma_1)}\|Z_1(s, S_{v_n}(s)) - Z_1(s, S_{v_{n-1}}(s))\|\\
&+ \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1-1}\|Z_1(s, S_{v_n}(q)) - Z_1(s, S_{v_{n-1}}(q))\|dq,\\
&\leq \frac{1-\gamma_1}{B(\gamma_1)}\|Z_1(s, S_{v_n}(s)) - Z_1(s, S_{v_{n-1}}(s))\|\\
&+ \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1-1}\|Z_1(s, S_{v_n}(q)) - Z_1(s, S_{v_{n-1}}(q))\|dq,
\end{aligned} \tag{4.31}$$

with Lipchitz condition we have,

$$\begin{aligned}\|K_{2n}(s)\| &= \|E_{v_n} - E_{v_{n-1}}\|(s)\\ &= \frac{1-\gamma_2}{B(\gamma_2)}\|Z_2(s, E_{v_n}(s)) - Z_2(s, E_{v_{n-1}}(s))\|\\ &+ \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)}\int_0^s (s-q)^{\gamma_2-1}\|Z_2(s, E_{v_n}(q)) - Z_2(s, E_{v_{n-1}}(q))\|dq.\end{aligned} \tag{4.32}$$

$$\begin{aligned}\|K_{3n}(s)\| &= \|I_{v_n} - I_{v_{n-1}}\|(s)\\ &= \frac{1-\gamma_3}{B(\gamma_3)}\|Z_3(s, I_{v_n}(s)) - Z_3(s, I_{v_{n-1}}(s))\|\\ &+ \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)}\int_0^s (s-q)^{\gamma_3-1}\|Z_3(s, I_{v_n}(q)) - Z_3(s, I_{v_{n-1}}(q))\|dq.\end{aligned} \tag{4.33}$$

$$\begin{aligned}\|K_{4n}(s)\| &= \|S_{b_n} - S_{b_{n-1}}\|(s)\\ &= \frac{1-\gamma_4}{B(\gamma_4)}\|Z_4(s, S_{b_n}(s)) - Z_4(s, S_{b_{n-1}}(s))\|\\ &+ \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)}\int_0^s (s-q)^{\gamma_4-1}\|Z_4(s, S_{b_n}(q)) - Z_4(s, S_{b_{n-1}}(q))\|dq.\end{aligned} \tag{4.34}$$

$$\begin{aligned}\|K_{5n}(s)\| &= \|E_{b_n} - E_{b_{n-1}}\|(s)\\ &= \frac{1-\gamma_5}{B(\gamma_5)}\|Z_5(s, E_{b_n}(s)) - Z_5(s, E_{b_{n-1}}(s))\|\\ &+ \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)}\int_0^s (s-q)^{\gamma_5-1}\|Z_5(s, E_{b_n}(q)) - Z_5(s, E_{b_{n-1}}(q))\|dq.\end{aligned} \tag{4.35}$$

$$\begin{aligned}\|K_{6n}(s)\| &= \|I_{b_n} - I_{b_{n-1}}\|(s)\\ &= \frac{1-\gamma_6}{B(\gamma_6)}\|Z_6(s, I_{b_n}(s)) - Z_6(s, I_{b_{n-1}}(s))\|\\ &+ \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)}\int_0^s (s-q)^{\gamma_6-1}\|Z_6(s, I_{b_n}(q)) - Z_6(s, I_{b_{n-1}}(q))\|dq.\end{aligned} \tag{4.36}$$

$$\begin{aligned}\|K_{7n}(s)\| &= \|R_{b_n} - R_{b_{n-1}}\|(s)\\ &= \frac{1-\gamma_7}{B(\gamma_7)}\|Z_7(s, R_{b_n}(s)) - Z_7(s, R_{b_{n-1}}(s))\|\\ &+ \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)}\int_0^s (s-q)^{\gamma_7-1}\|Z_7(s, R_{b_n}(q)) - Z_7(s, R_{b_{n-1}}(q))\|dq.\end{aligned} \tag{4.37}$$

□

Theorem 4.2: The model (3.1) has a solution provided that the following are holds true $\phi = maxQ_i < 1, i \in N_1^7$.

Proof. Using Equations (4.24) to (4.30)

$$
\begin{aligned}
K_{1n}(s) &= \frac{1-\gamma_1}{B(\gamma_1)}(Z_1(s, S_{v_n}(s))) - (Z_1(s, S_{v_n-1}(s))) \\
&\quad + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1-1}(Z_1(q, S_{v_n}(q)) - Z_1(q, S_{v_{n-1}}(q)))dq \\
&\leq \left[\frac{1-\gamma_1}{B(\gamma_1)} + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\right] Q_1 \|S_{v_n} - S_v\| \\
&\leq \left[\frac{1-\gamma_1}{B(\gamma_1)} + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\right]^n \phi^n \|S_v - S_{v_1}\| \qquad (4.38)
\end{aligned}
$$

Similarly we have,

$$K_{2n}(s) \leq \left[\frac{1-\gamma_2}{B(\gamma_2)} + \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)}\right]^n \phi^n \|E_v - E_{v_1}\| \qquad (4.39)$$

$$K_{3n}(s) \leq \left[\frac{1-\gamma_3}{B(\gamma_3)} + \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)}\right]^n \phi^n \|I_v - I_{v_1}\| \qquad (4.40)$$

$$K_{4n}(s) \leq \left[\frac{1-\gamma_4}{B(\gamma_4)} + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)}\right]^n \phi^n \|S_b - S_{b_1}\| \qquad (4.41)$$

$$K_{5n}(s) \leq \left[\frac{1-\gamma_5}{B(\gamma_5)} + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)}\right]^n \phi^n \|E_b - E_{b_1}\| \qquad (4.42)$$

$$K_{6n}(s) \leq \left[\frac{1-\gamma_6}{B(\gamma_6)} + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)}\right]^n \phi^n \|I_b - I_{b_1}\| \qquad (4.43)$$

$$K_{7n}(s) \leq \left[\frac{1-\gamma_7}{B(\gamma_7)} + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)}\right]^n \phi^n \|R_b - R_{b_1}\| \qquad (4.44)$$

$\therefore$ The functions $K_{i,n}(s) \to 0$, for $i = 1, 2, \ldots, 7$ as $n \to \infty$ for $\phi < 1$. □

Theorem 4.3: Under assumption B the model (3.1) has a unique solution if the following condition holds:

$$\left[\frac{1-\gamma_i}{B(\gamma_i)} + \frac{\gamma_i}{B(\gamma_i)\Gamma(\gamma_i)}\right] Q_i \leq 1, \quad for \quad i = 1, 2, \ldots 7. \qquad (4.45)$$

Proof. Assume that there exist another solution of the model (3.1) denoted as ($\hat{S}_v(s)$, $\hat{E}_v(s)$, $\hat{I}_v(s)$, $\hat{S}_b(s)$, $\hat{E}_b(s)$,$\hat{I}_b(s)$ and $\hat{R}_b(s)$), which satisfies the integral system defined by equations (4.10) to (4.16). Then, we obtain:

$$\hat{S}_v(s) = \frac{1-\gamma_1}{B(\gamma_1)} Z_1(s, \hat{S}_v(s)) + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1-1} Z_1(q, \hat{S}_v(q))dq.$$

Apply the norm on both sides,

$$\begin{aligned}\|S_v(s)-\hat{S}_v(s)\| \leq \frac{1-\gamma_1}{B(\gamma_1)}\|Z_1(s,S_v(s))-Z_1(s,\hat{S}_v(s))\| \\ +\frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1-1}\|Z_1(q,s_v(q))-Z_1(q,\hat{S}_v(q))\|dq \end{aligned} \tag{4.46}$$

Z_1 satisfies the Lipschitz property,

$$\|S_v-\hat{S}_v\| \leq \frac{1-\gamma_1}{B(\gamma_1)}Q_1\|S_v-\hat{S}_v\|+\frac{Q_1}{B(\gamma_1)\Gamma(\gamma_1)}\|S_v-\hat{S}_v\|. \tag{4.47}$$

This gives,

$$\left[\frac{1-\gamma_1}{B(\gamma_1)}Q_1+\frac{Q_1}{B(\gamma_1)\Gamma(\gamma_1)}-1\right]\|S_v-\hat{S}_v\| \geq 0. \tag{4.48}$$

implies

$$S_v(s)=\hat{S}_v(s).$$

Similarly,

$$\begin{aligned} E_v(s) &= \hat{E}_v(s) \\ I_v(s) &= \hat{I}_v(s) \\ S_b(s) &= \hat{S}_b(s) \\ E_b(s) &= \hat{E}_b(s) \\ I_b(s) &= \hat{I}_b(s) \\ R_b(s) &= \hat{R}_b(s). \end{aligned}$$

Therefore, the model (3.1) has unique solution. □

5. Hyers - Ulam Stability

In this section, we analyze the Hyers - Ulam stability of the proposed model (3.1).

Definition 5.1: The model (3.1) is said to possess Hyers - Ulam stability if there exist positive constants $\psi_i > 0, i \in N_1^{\gamma}$, ensuring that for every $\varepsilon > 0, i \in N_1^{\gamma}$, the given condition is satisfied

$$\|S_v(s)-\frac{1-\gamma_1}{B(\gamma_1)}Z_1(s,S_v(s))+\frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1-1}Z_1(q,S_v(q))dq\| \leq \varepsilon_1, \tag{5.1}$$

$$\|E_v(s)-\frac{1-\gamma_2}{B(\gamma_2)}Z_2(s,E_v(s))+\frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)}\int_0^s (s-q)^{\gamma_2-1}Z_2(q,E_v(q))dq\| \leq \varepsilon_2 \tag{5.2}$$

$$\|I_v(s)-\frac{1-\gamma_3}{B(\gamma_3)}Z_3(s,I_v(s))+\frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)}\int_0^s (s-q)^{\gamma_3-1}Z_3(q,I_v(q))dq\| \leq \varepsilon_3 \tag{5.3}$$

$$\|S_b(s) - \frac{1-\gamma_4}{B(\gamma_4)} Z_4(s, S_b(s)) + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)} \int_0^s (s-q)^{\gamma_4 - 1} Z_4(q, S_b(q)) dq\| \leq \varepsilon_4 \tag{5.4}$$

$$\|E_b(s) - \frac{1-\gamma_5}{B(\gamma_5)} Z_5(s, E_b(s)) + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)} \int_0^s (s-q)^{\gamma_5 - 1} Z_5(q, E_b(q)) dq\| \leq \varepsilon_5 \tag{5.5}$$

$$\|I_b(s) - \frac{1-\gamma_6}{B(\gamma_6)} Z_6(s, I_b(s)) + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)} \int_0^s (s-q)^{\gamma_6 - 1} Z_6(q, I_b(q)) dq\| \leq \varepsilon_6 \tag{5.6}$$

$$\|R_b(s) - \frac{1-\gamma_7}{B(\gamma_7)} Z_7(s, R_b(s)) + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)} \int_0^s (s-q)^{\gamma_7 - 1} Z_7(q, R_b(q)) dq\| \leq \varepsilon_7 \tag{5.7}$$

and there exist a solution $\hat{S}_v(s)$, $\hat{E}_v(s)$, $\hat{I}_v(s)$, $\hat{S}_b(s)$, $\hat{E}_b(s)$, $\hat{I}_b(s)$ and $\hat{R}_b(s)$ satisfying

$$\hat{S}_v(s) = \frac{1-\gamma_1}{B(\gamma_1)} Z_1(s, \hat{S}_v(s)) + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)} \int_0^s (s-q)^{\gamma_1 - 1} Z_1(q, \hat{S}_v(q)) dq. \tag{5.8}$$

$$\hat{E}_v(s) = \frac{1-\gamma_2}{B(\gamma_2)} Z_2(s, \hat{E}_v(s)) + \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2)} \int_0^s (s-q)^{\gamma_2 - 1} Z_2(q, \hat{E}_v(q)) dq. \tag{5.9}$$

$$\hat{I}_v(s) = \frac{1-\gamma_3}{B(\gamma_3)} Z_3(s, \hat{I}_v(s)) + \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3)} \int_0^s (s-q)^{\gamma_3 - 1} Z_3(q, \hat{I}_v(q)) dq. \tag{5.10}$$

$$\hat{S}_b(s) = \frac{1-\gamma_4}{B(\gamma_4)} Z_4(s, \hat{S}_b(s)) + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4)} \int_0^s (s-q)^{\gamma_4 - 1} Z_4(q, \hat{S}_b(q)) dq. \tag{5.11}$$

$$\hat{E}_b(s) = \frac{1-\gamma_5}{B(\gamma_5)} Z_5(s, \hat{E}_b(s)) + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5)} \int_0^s (s-q)^{\gamma_5 - 1} Z_5(q, \hat{E}_b(q)) dq. \tag{5.12}$$

$$\hat{I}_b(s) = \frac{1-\gamma_6}{B(\gamma_6)} Z_6(s, \hat{I}_b(s)) + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6)} \int_0^s (s-q)^{\gamma_6 - 1} Z_6(q, \hat{I}_b(q)) dq. \tag{5.13}$$

$$\hat{R}_b(s) = \frac{1-\gamma_7}{B(\gamma_7)} Z_7(s, \hat{R}_b(s)) + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7)} \int_0^s (s-q)^{\gamma_7 - 1} Z_7(q, \hat{R}_b(q)) dq. \tag{5.14}$$

such that,

$$\begin{aligned}
|S_v(s) - \hat{S}_v(s)| &\leq \Lambda_1 \varepsilon_1 \\
|E_v(s) - \hat{E}_v(s)| &\leq \Lambda_2 \varepsilon_2 \\
|I_v(s) - \hat{I}_v(s)| &\leq \Lambda_3 \varepsilon_3 \\
|S_b(s) - \hat{S}_b(s)| &\leq \Lambda_4 \varepsilon_4 \\
|E_b(s) - \hat{E}_b(s)| &\leq \Lambda_5 \varepsilon_5 \\
|I_b(s) - \hat{I}_b(s)| &\leq \Lambda_6 \varepsilon_6
\end{aligned}$$

$$|R_b(s) - \hat{R}_b(s)| \leq \Lambda_7 \varepsilon_7$$

Theorem 5.2: The model (3.1) is Hyers - Ulam stable under assumption B.

Proof. Since model (3.1) has a unique solution. Let us consider $\hat{S}_v(s)$,$\hat{E}_v(s)$, $\hat{I}_v(s)$, $\hat{S}_b(s)$,$\hat{E}_b(s)$, $\hat{I}_b(s)$ and $\hat{R}_b(s)$ as approximate solutions satisfying equations (4.10) to (4.16). Thus, we obtain,

$$\begin{aligned}\|S_v(s) - \hat{S}_v(s)\| \leq & \frac{1-\gamma_1}{B(\gamma_1)}\|Z_1(s, S_v(s)) - Z_1(s, \hat{S}_v(s))\| \\ & + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\int_0^s (s-q)^{\gamma_1 - 1}\|Z_1(q, S_v(q)) - Z_1(q, \hat{S}_v(q))\|dq \\ & \leq \left[\frac{1-\gamma_1}{B(\gamma_1)} + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}\right] Q_1\|S_v - \hat{S}_v\|.\end{aligned}$$

Let $\varepsilon_1 = Q_1$, $\psi_1 = \frac{1-\gamma_1}{B(\gamma_1)} + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1)}$. Implies,

$$\|S_v(s) - \hat{S}_v(s)\| \leq \varepsilon_1 \psi_1 \tag{5.15}$$

Similarly,

$$\begin{aligned}\|E_v(s) - \hat{E}_v(s)\| &\leq \varepsilon_2 \psi_2 \\ \|I_v(s) - \hat{I}_v(s)\| &\leq \varepsilon_3 \psi_3 \\ \|S_b(s) - \hat{S}_b(s)\| &\leq \varepsilon_4 \psi_4 \\ \|E_b(s) - \hat{E}_b(s)\| &\leq \varepsilon_5 \psi_5 \\ \|I_b(s) - \hat{I}_b(s)\| &\leq \varepsilon_6 \psi_6 \\ \|R_b(s) - \hat{R}_b(s)\| &\leq \varepsilon_7 \psi_7 \end{aligned} \tag{5.16}$$

Hence, model (3.1) is Hyers - Ulam stable. □

6. Numerical Scheme and Simulations

In this section, we present the numerical simulation conducted for the west nile virus disease (3.1). The fractional Atangana - Baleanu integral is implemented using Euler's method, as described in [4]:

$$\begin{aligned}h_{n+1} = h_0 + & \frac{1-\lambda}{B(\lambda)} Z(S_{n+1}, h_{n+1}(x)) \\ & + \frac{\lambda}{B(\lambda)\Gamma(\lambda+1)}\sum_{j=0}^{n} b^{\lambda}_{n+1,j} f(s_j, h_j(x))\end{aligned} \tag{6.1}$$

For $n = 0, 1, \ldots, N-1$ the coefficients $b_{n+1,j}$ for $j = 0, 1, \ldots, n$ are are determined using the relation: $b^{\lambda}_{n+1,j} = -(n-j)^{\lambda} + (n-j+1)^{\lambda}$. Using the above numerical scheme, we have

$$S_{v_{n+1}} = S_{v_0} + \frac{1-\gamma_1}{B(\gamma_1)} Z_1(x_{n+1}, S_{v_{n+1}}(x)) + \frac{\gamma_1}{B(\gamma_1)\Gamma(\gamma_1+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_1} Z(x_j, S_{v_j}(x)) \tag{6.2}$$

$$E_{v_{n+1}} = E_{v_0} + \frac{1-\gamma_2}{B(\gamma_2)} Z_2(x_{n+1}, E_{v_{n+1}}(x)) + \frac{\gamma_2}{B(\gamma_2)\Gamma(\gamma_2+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_2} Z(x_j, E_{v_j}(x)) \tag{6.3}$$

$$I_{v_{n+1}} = I_{v_0} + \frac{1-\gamma_3}{B(\gamma_3)} Z_3(x_{n+1}, I_{v_{n+1}}(x)) + \frac{\gamma_3}{B(\gamma_3)\Gamma(\gamma_3+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_3} Z(x_j, I_{v_j}(x)) \tag{6.4}$$

$$S_{b_{n+1}} = S_{b_0} + \frac{1-\gamma_4}{B(\gamma_4)} Z_4(x_{n+1}, S_{b_{n+1}}(x)) + \frac{\gamma_4}{B(\gamma_4)\Gamma(\gamma_4+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_4} Z(x_j, S_{b_j}(x)) \tag{6.5}$$

$$E_{b_{n+1}} = E_{b_0} + \frac{1-\gamma_5}{B(\gamma_5)} Z_5(x_{n+1}, E_{b_{n+1}}(x)) + \frac{\gamma_5}{B(\gamma_5)\Gamma(\gamma_5+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_5} Z(x_j, E_{b_j}(x)) \tag{6.6}$$

$$I_{b_{n+1}} = I_{b_0} + \frac{1-\gamma_6}{B(\gamma_6)} Z_6(x_{n+1}, I_{b_{n+1}}(x)) + \frac{\gamma_6}{B(\gamma_6)\Gamma(\gamma_6+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_6} Z(x_j, I_{b_j}(x)) \tag{6.7}$$

$$R_{b_{n+1}} = R_{b_0} + \frac{1-\gamma_7}{B(\gamma_7)} Z_7(x_{n+1}, R_{b_{n+1}}(x)) + \frac{\gamma_7}{B(\gamma_7)\Gamma(\gamma_7+1)} \sum_{j=0}^{n} b_{n+1,j}^{\gamma_7} Z(x_j, R_{b_j}(x)). \tag{6.8}$$

The equations (6.2) to (6.8) present numerical simulations of model (3.1), where the parameter values are considered as

$$\chi_v = 0.05 - 0.08$$
$$\mu_v = 0.05 - 0.33$$

$$\nu_v = 0.07 - 0.14$$
$$\sigma_v = 0.125 - 0.33$$
$$\psi_b = 0 - 0.022$$
$$\mu_b = 0 - 0.01$$
$$\nu_b = 0.33 - 1$$
$$\gamma_b = 0 - 0.2$$
$$\rho_b = 0$$
$$\delta_b = 0.125 - 0.33$$
$$\sigma_b = 0 - \infty$$
$$\beta_{v_b} = 0.65 - 1$$

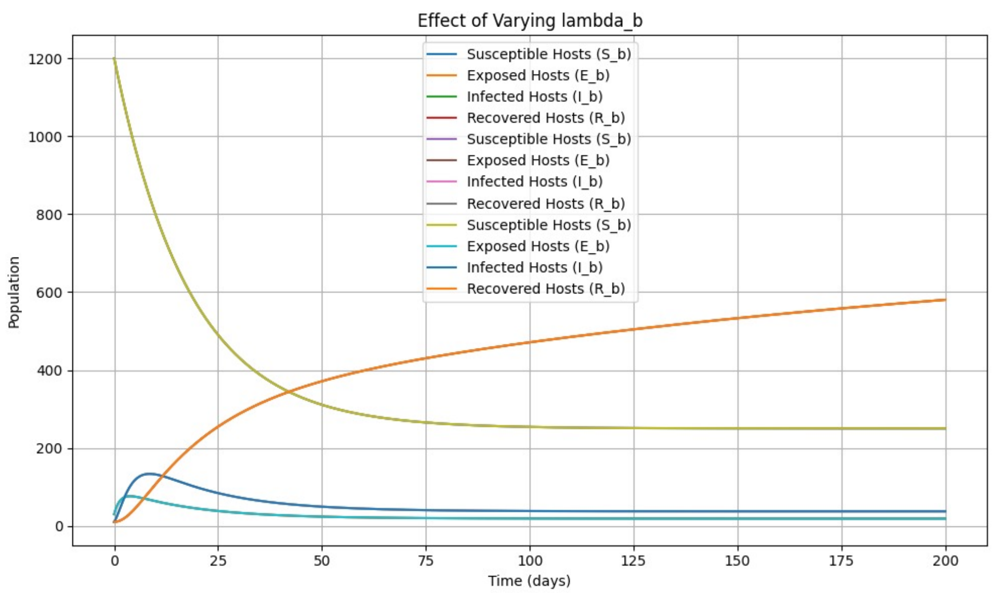

Figure (1) depicts the impact of different values of λ_b on the bird host compartment within the WNV model. As λ_b increases, the susceptible population (S_b) declines more rapidly due to higher infection rates. The exposed (E_b) and infected (I_b) population initially rise but later decline as individuals recover or die. The recovered population (R_b) exhibits a steady increase over time, indicating long-term immunity.

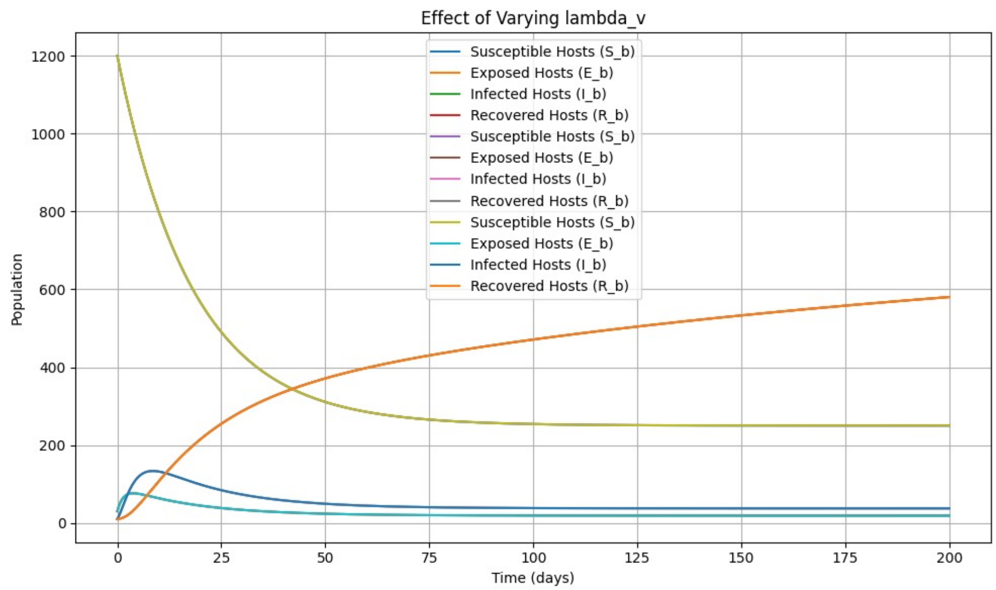

Figure (2) suggests that higher vector-to-host transmission accelerates the epidemic, depleting the susceptible population more quickly and leading to a higher number of recovered individuals.

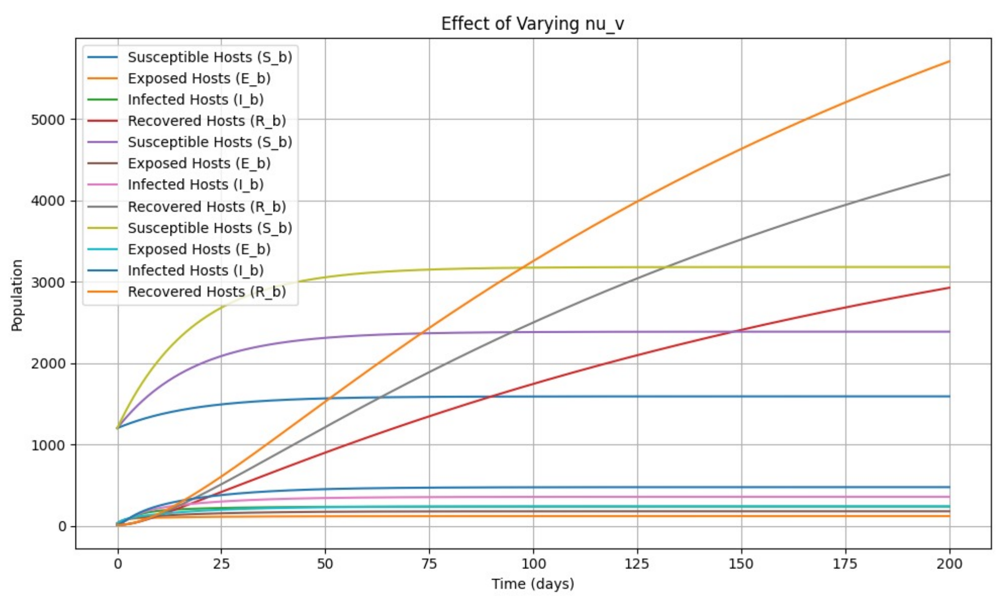

Figure (3) presents the impact of varying the mosquito transmission rate (ν_v) on the population dynamics of West Nile Virus among bird hosts over time. The results suggest that interventions targeting mosquito populations could be effective in mitigating disease spread.

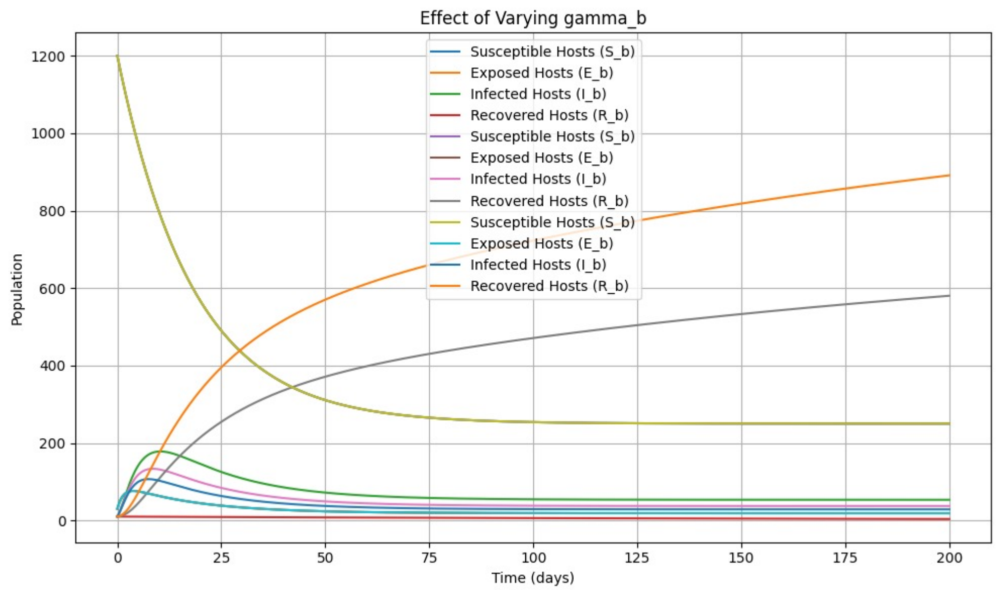

Figure (4) represents that as the recovery rate increases, the infected bird population (I_b) declines more rapidly. The susceptible population decreases initially due to infection but stabilizes over time.

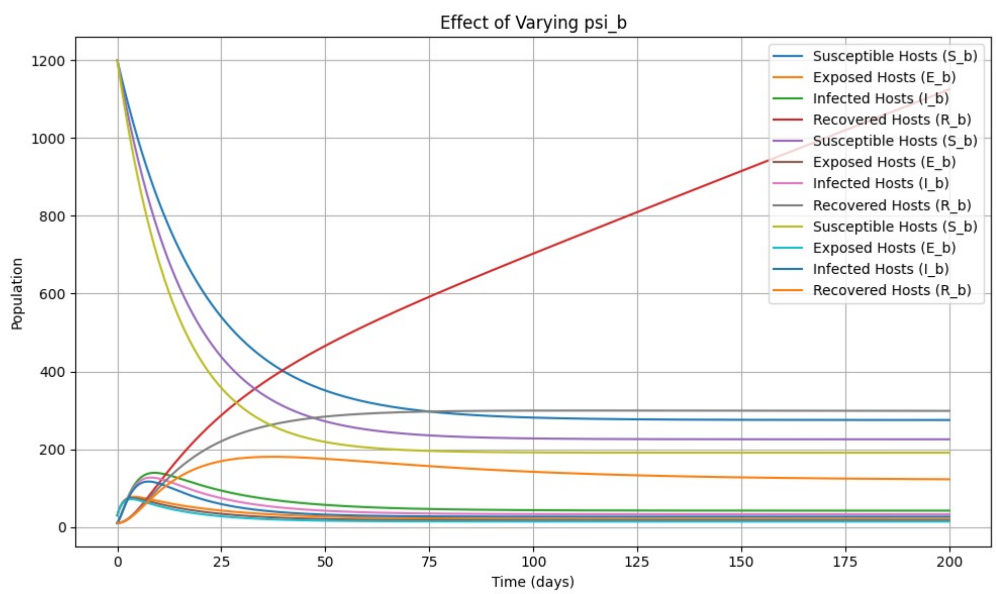

Figure (5) represents that as ϕ_b increases, there is a rapid decline in the susceptible population (S_b) within the first 25 days due to increased exposure and infection. The exposed and infected populations rise sharply during this period, peaking early before gradually declining as more individuals recover.

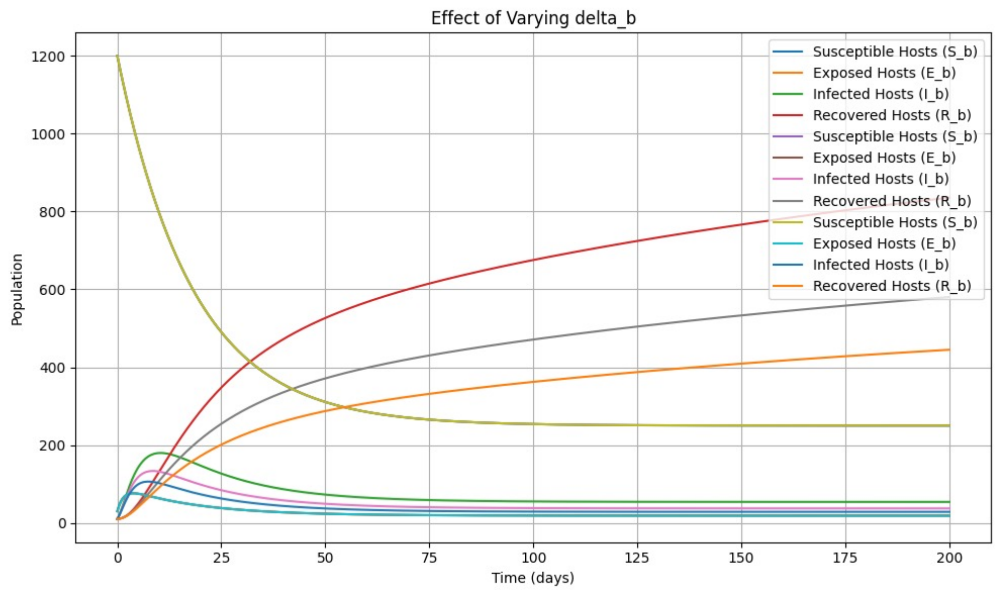

Figure (6) suggests that increasing S_b may reduce the overall disease burden by limiting the duration of infectiousness, but at the cost of higher host mortality.

7. Conclusion:

This study investigated the impact of varying key epidemiological parameters on the population dynamics of WNV. The simulation results reveal significant variations in disease transmission, infection duration, and population stability based on these parameter changes. Higher birth rates (ν_v) lead to increased transmission rates, contributing to a larger infected host population over time. Increased recovery rates (ν_b) reduce infection duration and promote a higher recovered host population. These results highlight the complex interplay between host and vector dynamics in shaping disease outbreaks.

Managing transmission parameters through interventions such as vector control, vaccination, or environmental modifications could significantly alter the epidemic trajectory. Future work should incorporate additional ecological and environmental factors to improve the predictive power of these models.

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References

[1] S. Abbas, M. Nazar, S.F.F. Gillani, M. Naveed, M. Ahmad, Z.U. Nisa, A CPC Fractional Model of the Heat and Mass Transport Mechanism in Carbon Nanotubes with Slip Effects on Velocity, Mod. Phys. Lett. B 38 (2023), 2450100. https://doi.org/10.1142/s0217984924501008.

[2] S. Abbas, M. Ahmad, M. Nazar, M. Amjad, H. Ali, A.Z. Jan, Heat and Mass Transfer Through a Vertical Channel for the Brinkman Fluid Using Prabhakar Fractional Derivative, Appl. Therm. Eng. 232 (2023), 121065. https://doi.org/10.1016/j.applthermaleng.2023.121065. .

[3] P. Agarwal, R. Singh, Modelling of Transmission Dynamics of Nipah Virus (NIV): A Fractional Order Approach, Physica: Stat. Mech. Appl. 547 (2020), 124243. https://doi.org/10.1016/j.physa.2020.124243.

[4] B.S.T. Alkahtani, Chua's Circuit Model with Atangana–baleanu Derivative with Fractional Order, Chaos Solitons Fractals 89 (2016), 547–551. https://doi.org/10.1016/j.chaos.2016.03.020.

[5] D. Baleanu, A. Jajarmi, M. Hajipour, On the Nonlinear Dynamical Systems Within the Generalized Fractional Derivatives with Mittag–leffler Kernel, Nonlinear Dyn. 94 (2018), 397–414. https://doi.org/10.1007/s11071-018-4367-y.

[6] C. Bowman, A. Gumel, P. Vandendriessche, J. Wu, H. Zhu, A Mathematical Model for Assessing Control Strategies Against West Nile Virus, Bull. Math. Biol. 67 (2005), 1107–1133. https://doi.org/10.1016/j.bulm.2005.01.002.

[7] G. Cai, J. Huang, S.X. Yang, An Optimal Control Method for Linear Systems with Time Delay, Comput. Struct. 81 (2003), 1539–1546. https://doi.org/10.1016/s0045-7949(03)00146-9.

[8] B. Chen, S. Wang, H. Lu, Stabilization of Time-Delay Systems Containing Saturating Actuators, Int. J. Control. 47 (1988), 867–881. https://doi.org/10.1080/00207178808906058.

[9] Chyung, D.H., Lee, E.B.: Linear optimal systems with time-delay. SIAM J. Control 4(3), 548–575 (1966)

[10] S. Dadebo, R. Luus, Optimal Control of Time-delay Systems by Dynamic Programming, Optim. Control. Appl. Methods 13 (1992), 29–41. https://doi.org/10.1002/oca.4660130103.

[11] F. Evirgen, Transmission of Nipah Virus Dynamics Under Caputo Fractional Derivative, J. Comput. Appl. Math. 418 (2023), 114654. https://doi.org/10.1016/j.cam.2022.114654.

[12] E. Fesce, G. Marini, R. Rosà, D. Lelli, M.P. Cerioli, M. Chiari, M. Farioli, N. Ferrari, Understanding West Nile Virus Transmission: Mathematical Modelling to Quantify the Most Critical Parameters to Predict Infection Dynamics, PLOS Neglected Trop. Dis. 17 (2023), e0010252. https://doi.org/10.1371/journal.pntd.0010252.

[13] G. Cruzpacheco, L. Esteva, J. Montanohirose, C. Vargas, Modelling the Dynamics of West Nile Virus, Bull. Math. Biol. 67 (2005), 1157–1172. https://doi.org/10.1016/j.bulm.2004.11.008.

[14] C.G. HAYES, West Nile Virus: Uganda, 1937, to New York City, 1999, Ann. N. Y. Acad. Sci. 951 (2001), 25–37. https://doi.org/10.1111/j.1749-6632.2001.tb02682.x.

[15] J. Singh, D. Kumar, D. Baleanu, On the Analysis of Chemical Kinetics System Pertaining to a Fractional Derivative with Mittag-Leffler Type Kernel, Chaos 27 (2017), 103113. https://doi.org/10.1063/1.4995032.

[16] I. Koca, Modelling the Spread of Ebola Virus with Atangana-Baleanu Fractional Operators, Eur. Phys. J. Plus 133 (2018), 100. https://doi.org/10.1140/epjp/i2018-11949-4.

[17] R.S. Lanciotti, J.T. Roehrig, V. Deubel, J. Smith, M. Parker, et al. Origin of the West Nile Virus Responsible for an Outbreak of Encephalitis in the Northeastern United States, Science 286 (1999), 2333–2337. https://doi.org/10.1126/science.286.5448.2333.

[18] T.N. LEE, U.L. RADOVIC, General Decentralized Stabilization of Large-Scale Linear Continuous and Discrete Time-Delay Systems, Int. J. Control. 46 (1987), 2127–2140. https://doi.org/10.1080/00207178708934039.

[19] R. Luus, X. Zhang, F. Hartig, F.J. Keil, Use of Piecewise Linear Continuous Optimal Control for Time-Delay Systems, Ind. Eng. Chem. Res. 34 (1995), 4136–4139. https://doi.org/10.1021/ie00038a060.

[20] M.K. Mondal, M. Hanif, M.H.A. Biswas, A Mathematical Analysis for Controlling the Spread of Nipah Virus Infection, Int. J. Model. Simul. 37 (2017), 185–197. https://doi.org/10.1080/02286203.2017.1320820.

[21] E. Raeisi, M. Yavuz, M. Khosravifarsani, Y. Fadaei, Mathematical Modeling of Interactions Between Colon Cancer and Immune System with a Deep Learning Algorithm, Eur. Phys. J. Plus 139 (2024), 345. https://doi.org/10.1140/epjp/s13360-024-05111-4.

[22] J. Singh, D. Kumar, D. Baleanu, On the Analysis of Chemical Kinetics System Pertaining to a Fractional Derivative with Mittag-Leffler Type Kernel, Chaos 27 (2017), 103113. https://doi.org/10.1063/1.4995032.

[23] M. Syam, M. Al-Refai, Fractional Differential Equations with Atangana–baleanu Fractional Derivative: Analysis and Applications, Chaos Solitons Fractals: X 2 (2019), 100013. https://doi.org/10.1016/j.csfx.2019.100013.

[24] N.H. Sweilam, O.M. Saad, D.G. Mohamed, Fractional Optimal Control in Transmission Dynamics of West Nile Virus Model with State and Control Time Delay: A Numerical Approach, Adv. Differ. Equ. 2019 (2019), 210. https://doi.org/10.1186/s13662-019-2147-8.

[25] K.L. Teo, K.H. Wong, D.J. Clements, Optimal Control Computation for Linear Time-Lag Systems with Linear Terminal Constraints, J. Optim. Theory Appl. 44 (1984), 509–526. https://doi.org/10.1007/bf00935465.

[26] M.J. Turell, M. O'Guinn, J. Oliver, Potential for New York Mosquitoes to Transmit West Nile Virus, Am. J. Trop. Med. Hyg. 62 (2000), 413–414. https://doi.org/10.4269/ajtmh.2000.62.413.

[27] P. Veeresha, E. Ilhan, D. Prakasha, H.M. Baskonus, W. Gao, A New Numerical Investigation of Fractional Order Susceptible-Infected-Recovered Epidemic Model of Childhood Disease, Alex. Eng. J. 61 (2022), 1747–1756. https://doi.org/10.1016/j.aej.2021.07.015.

[28] K.H. Wong, D.J. Clements, K.L. Teo, Optimal Control Computation for Nonlinear Time-Lag Systems, J. Optim. Theory Appl. 47 (1985), 91–107. https://doi.org/10.1007/bf00941318.

[29] J. Yang, T. Soong, Recent Advances in Active Control of Civil Engineering Structures, Probab. Eng. Mech. 3 (1988), 179–188. https://doi.org/10.1016/0266-8920(88)90010-0.

[30] H.G. Zeller, I. Schuffenecker, West Nile Virus: an Overview of Its Spread in Europe and the Mediterranean Basin in Contrast to Its Spread in the Americas, Eur. J. Clin. Microbiol. Infect. Dis. 23 (2004), 147–156. https://doi.org/10.1007/s10096-003-1085-1.