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New Modified Univariate Lindley Distribution: Statistical Properties, Estimation, and Applications

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Abstract. This article centers on the exploration of a new univariate probability distribution. A novel distribution has been formulated using the power transformation technique, termed the new modified univariate Lindley distribution. This model exhibits diverse hazard functions, including J-shaped, reverse-J-shaped, and monotonically increasing patterns. The study examines the fundamental statistical characteristics of this recently introduced distribution, including moments, incomplete moments, hazard rate, mean residual life function, quantile function, skewness, and kurtosis. Estimation of its parameters is conducted through the maximum likelihood estimation method. The precision of this parameter estimation process is verified through Monte Carlo simulation experiments. To illustrate the practical utility of the proposed distribution, two sets of real-world data are employed. The performance of the suggested distribution model is assessed using various model selection criteria and goodness-of-fit test statistics. Empirical findings from these evaluations provide substantial evidence that the proposed model surpasses other existing models.

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1. Introduction

Numerous statistical researchers have made various modifications to existing probability distributions. Among these approaches, augmenting parameters within existing distributions is a commonly employed method to create more adaptable distributions, primarily utilized in survival and reliability analysis, see [1,2]. The Lindley distribution was originally introduced by Lindley (1958) [3] within the realms of Fiducial and Bayesian Statistics, whose PDF is

$$f(x;\delta) = \frac{\delta^2}{\delta+1}(1+x)e^{-\delta x}; x > 0, \delta > 0.$$

In the domain of reliability studies, Ghitany et al. (2008) [4] conducted an exhaustive examination of the Lindley distribution and its practical applications and reported that there are instances where the Lindley distribution might not be appropriate from both a theoretical and applied perspective. To address this, Ghitany et al. (2013) [5] introduced a novel modification of the Lindley distribution, denoted as the power Lindley (PL) distribution, achieved by incorporating the power transformation $X = T^{1/\delta}$. The PDF of the PL distribution is represented as

$$f(x;\alpha,\delta) = \frac{\alpha\delta^2}{\delta+1}x^{\alpha-1}(1+x^{\alpha})e^{-\delta x^{\alpha}}; x > 0, \alpha > 0, \delta > 0.$$

This distribution was subsequently subjected to further investigation by Kumar et al. (2022) [6] in the context of parametric inferences. Furthermore, the Lindley distribution has been employed by several authors to construct new models. For example, Ghitany et al. (2011) [7] developed a two-parameter weighted Lindley distribution and explored its illustrations in survival data analysis. The generalized Lindley distribution was defined by Nadarajah et al. (2011) [8] and its various properties were examined. Bakouch et al. (2012) [9] extended the Lindley distribution and discussed its diverse characteristics. Onyekwere et al. [10] presented an updated version of the Lindley distribution. Merovci and Elbatal (2014) [11] introduced the transmuted Lindley-geometric distribution, utilizing the quadratic rank transmutation map to create a versatile family of probability distributions based on the Lindley-geometric distribution by introducing an additional parameter for enhanced distributional flexibility. Asgharzadeh et al. (2014) [12] presented a comprehensive family of continuous lifetime distributions by compounding continuous distributions with the Poisson–Lindley distribution. Similarly, Ashour and Eltehiwy (2015) [13] introduced a novel model by adding an extra parameter to the PL distribution.

Recent contributions to this field include the weighted power quasi-Lindley distribution by (GANAIE and Rajagopalan, 2023) [14] and a new generalization of the PL distribution by (Guptha and Maruthan, 2023) [15]. Sapkota and Kumar (2023) [16] introduced the Chen exponential distribution, Husain et al. [17] proposed new expansion of Chen distribution, while Sapkota and Kumar (2023) [18] and Sapkota (2022) [19] defined the inverse power Cauchy and exponentiated inverse power Cauchy distributions, which offer greater flexibility compared to the power Cauchy distribution.

In this paper, we propose a new three-parameter Lindley-type univariate distribution using the methodology employed by Ghitany et al. (2013) [5]. We refer to it as the new modified univariate Lindley distribution (NMUL). We anticipate this distribution will find applications across various disciplines, including reliability engineering, life sciences, machine learning, artificial intelligence, and more. One of the primary motivations for introducing this new distribution is its ability to encompass seven existing distributions as sub-models which are presented in Section 2. Further, it has a flexible hazard function having various shapes, as shown in Figure 2. Also, it has simple expressions for both its cumulative distribution function (CDF) and probability density function (PDF). Furthermore, it fits data better than the other seven models found in the literature (see application section 6.)

This study is structured as follows: In Section 2, we outline the methodology for model development and discuss key functions. Section 3 offers an in-depth analysis of its statistical and mathematical properties. Sections 4 and 5 are dedicated to parameter estimation and simulation experiments. Furthermore, we delve into the illustrations and conclusion of our study in Sections 6 and 7.

2. Formulation of New Modified Univariate Lindley Distribution

To develop a new probability model, we have used the Shukla distribution (ShD) as a base model defined by (Shukla and Rama, 2019) [20], and its CDF is defined as follows:

$$G(x) = \frac{\beta^{\alpha+1}\left(1-e^{-\beta x}\right)+\Gamma(\alpha+1)-\Gamma(\alpha+1,\beta x)}{\beta^{\alpha+1}+\Gamma(\alpha+1)};\ x>0, \alpha>0, \beta>0. \tag{2.1}$$

where $\Gamma(a,z)=\int_0^z t^{a-1}e^{-t}dt$. The ShD's PDF is

$$g(x) = \frac{\beta^{\alpha+1}e^{-\beta x}\left(\beta+x^{\alpha}\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}. \tag{2.2}$$

By using the power transformation technique $X = T^{\frac{1}{\delta}}$ to Equation (2.1), where T follows ShD, we can obtain the CDF of the suggested distribution NMULD (for $x>0$ and $\delta>0$) as follows

$$F(x) = \frac{\beta^{\alpha+1}\left(1-e^{-\beta x^{\delta}}\right)+\Gamma(\alpha+1)-\Gamma\left(\alpha+1,\beta x^{\delta}\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}, \tag{2.3}$$

and the survival function (SF) is also can be obtained easily as

$$S(x) = 1-\frac{\beta^{\alpha+1}\left(1-e^{-\beta x^{\delta}}\right)+\Gamma(\alpha+1)-\Gamma\left(\alpha+1,\beta x^{\delta}\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}, \tag{2.4}$$

The PDF of NMULD can be expressed as

$$f(x) = \frac{\delta\beta^{\alpha+1}x^{\delta-1}e^{-\beta x^{\delta}}\left(\beta+x^{\alpha\delta}\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}. \tag{2.5}$$

The PDF plots serve as graphical depictions of the shapes for continuous random variables, offering insight into potential shapes determined by various parameter values. As illustrated in

Figure (1), it becomes apparent that the PDF for NMULD can adopt multiple forms, including left-skewed, right-skewed, reverse-J, and symmetrical distributions. The PDF of the NMULD defined in Equation 2.5 has several sub-models as special cases, which are as follows:

- when $\delta = 1$ NMULD gives ShD (Shukla and Rama, 2019) [20].
- when $\delta = 1$ and $\alpha = 0$ NMULD gives the exponential distribution.
- when $\delta = 1$ and $\alpha = 1$ NMULD gives the distribution defined by (Shankar, 2015) [21].
- when $\delta = 1$ and $\alpha = 2$ NMULD gives the Ishita distribution defined by (Shankar and Shukla, 2017) [22].
- when $\delta = 1$ and $\alpha = 3$ NMULD gives the Pranav distribution defined by (Shukla, 2018) [23].
- when $\delta = 1$ and $\alpha = 4$ NMULD gives the Rani distribution defined by (Shankar, 2017) [24].
- when $\delta = 1$ and $\alpha = 5$ NMULD gives the Ram Awadh distribution defined by (Shukla, 2018) [25].

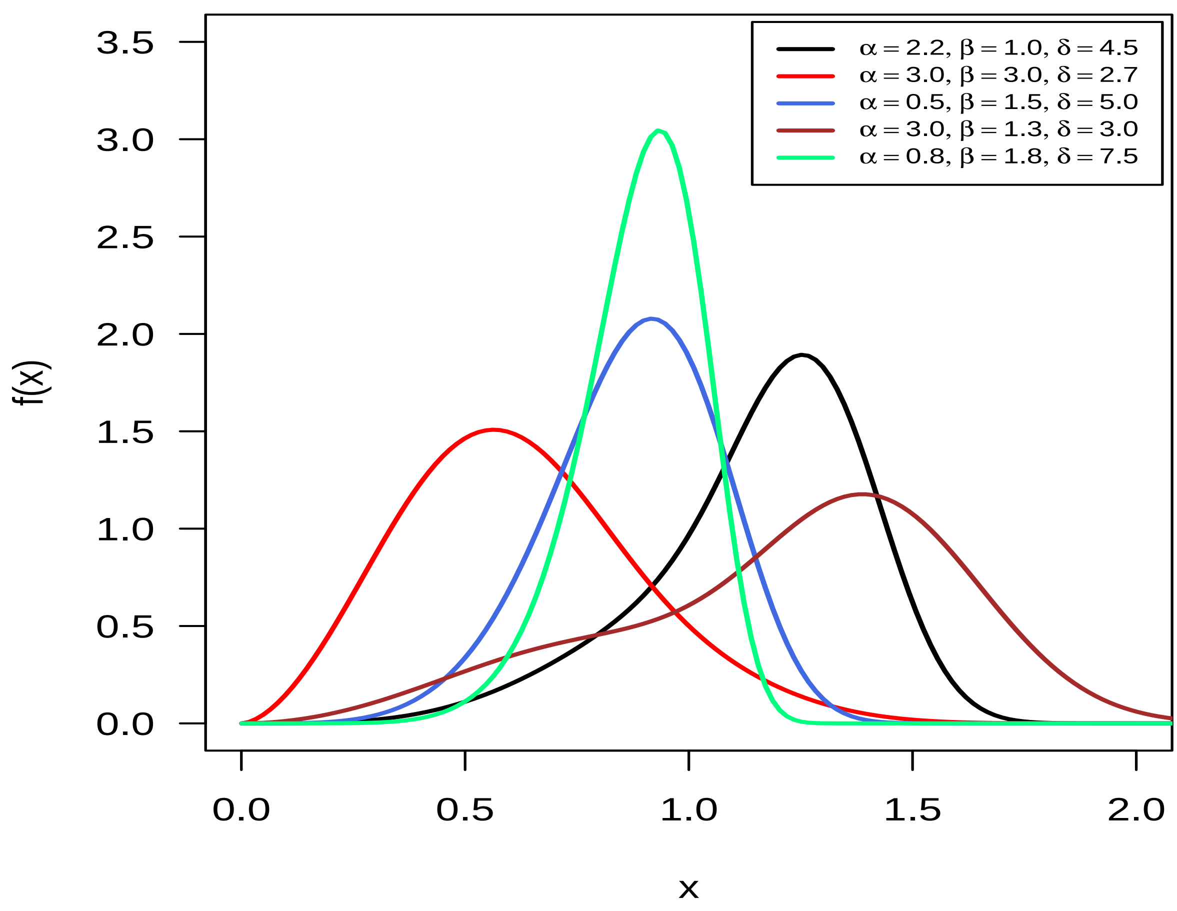

Figure 1. The PDF plot of NMULD for different values of α, β, and δ

3. Statistical properties

3.1. **Moments.** Moments help to describe various aspects of the distribution, including its center, spread, and shape. In practice, raw moments are essential statistical analysis and modeling tools, helping researchers understand the behavior of random variables and their associated probabilities. The r^{th} order raw moment about origin, denoted by $\mu_r{}'$, of the NMULD is obtained as

$$\begin{aligned}\mu_r{}' &= \int_0^\infty x^r f(x)dx\\ &= \frac{\beta^{\alpha+1}}{\beta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty x^r \delta x^{\delta-1}e^{-\beta x^\delta}(\beta+x^{\alpha\delta})dx\\ &= \frac{\beta^{\alpha+1}\Gamma(1+\frac{r}{\delta})+\Gamma(\alpha+1+\frac{r}{\delta})}{\beta^{\frac{r}{\delta}}(\beta^{\alpha+1}+\Gamma(\alpha+1))}. \end{aligned} \tag{3.1}$$

Thus, the first four raw moments (about the origin) of the NMULD are obtained by substituting $r = 1, 2, 3, 4$ respectively in Equation (3.1). By computing these moments, we can effectively characterize the NMUL distribution and demonstrate its flexibility and applicability in various fields. These are presented below.

$$\mu_1{}' = \frac{\beta^{\alpha+1}\,\Gamma(1+\frac{1}{\delta})+\Gamma(\alpha+1+\frac{1}{\delta})}{\beta^{\frac{1}{\delta}}(\beta^{\alpha+1}+\Gamma(\alpha+1))} \tag{3.2}$$

$$\mu_2{}' = \frac{\beta^{\alpha+1}\,\Gamma(1+\frac{2}{\delta})+\Gamma(\alpha+1+\frac{2}{\delta})}{\beta^{\frac{2}{\delta}}(\beta^{\alpha+1}+\Gamma(\alpha+1))} \tag{3.3}$$

$$\mu_3{}' = \frac{\beta^{\alpha+1}\Gamma(1+\frac{3}{\delta})+\Gamma(\alpha+1+\frac{3}{\delta})}{\beta^{\frac{3}{\delta}}(\beta^{\alpha+1}+\Gamma(\alpha+1))} \tag{3.4}$$

$$\mu_4{}' = \frac{\beta^{\alpha+1}\Gamma(1+\frac{4}{\delta})+\Gamma(\alpha+1+\frac{4}{\delta})}{\beta^{\frac{4}{\delta}}(\beta^{\alpha+1}+\Gamma(\alpha+1))}. \tag{3.5}$$

The variance of X that follows the NMULD can be obtained as

$$\mu_2 = \mu_2{}' - (\mu_1{}')^2.$$

The first four central moments can be easily calculated using $r = 1, 2, 3, 4$, followed by other NMULD properties.

- First Moment (μ_1): The mean of the distribution, indicating the central location. It is essential for understanding the average behavior of the random variable.
- Second Moment (μ_2): Related to the variance, which measures the dispersion or spread of the distribution around the mean. The variance is crucial for assessing the variability and consistency of the data.

- Third Moment (μ_3): Provides information about the skewness, describing the asymmetry of the distribution. Skewness helps in understanding whether the data tends to have a longer tail on the left or right side of the mean.
- Fourth Moment (μ_4): Pertains to the kurtosis, indicating the "tailedness" or the peak of the distribution. Kurtosis is important for identifying the presence of outliers and the sharpness of the peak.

Some numerical values of the first four moments and their related measures are determined in Tables 1 and 2. Based on these tables, for $\delta = 0.8$, we have

- Moment values decrease rapidly with increasing β.
- Maximum mean: 43.3651 ($\alpha = 5.0, \beta = 0.3$), minimum mean: 0.1228 ($\alpha = 3.5, \beta = 6.0$), and Maximum variance: 491.9754 ($\alpha = 5.0, \beta = 0.3$).
- Cof. of Variation range: 0.5115 (min) to 1.5997 (max), generally decreases with increasing α, and shows less variation at higher β values.
- Variance decreases monotonically with increasing β for all α values.

and for $\delta = 2.5$, we have

- μ_1' range from 3.2485 ($\alpha = 5.0, \beta = 0.3$) to 0.4347 ($\alpha = 5.0, \beta = 6.0$), μ_4' ranges from 130.1397 to 0.0839, and μ_3' ranges from 37.1186 to 0.1306.
- Maximum variance: 0.3803 ($\alpha = 0.7, \beta = 0.3$), minimum variance: 0.0345 ($\alpha = 0.2, \beta = 6.0$), and Non-monotonic behavior observed at $\alpha \geq 2.5$.
- CV general range from 0.1657 to 0.5164, minimum 0.1657 at $\alpha = 5.0, \beta = 0.3$, maximum 0.5164 at $\alpha = 5.0, \beta = 3.0$, and U-shaped patterns emerge for $\alpha \geq 2.5$.

Table 1. Numerical values for the first four moments and their related measures for $\delta = 0.8$.

Parameters		Measures					
α	β	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	Cof. of Variation
0.2	0.3	6.0551	87.0933	2072.5665	70503.5294	50.4293	1.1728
	0.6	2.4620	14.6672	145.3673	2064.9243	8.6059	1.1916
	0.9	1.4471	5.1343	30.4279	258.9615	3.0401	1.2049
	1.25	0.9402	2.1903	8.5530	48.0430	1.3064	1.2157
	2.5	0.3802	0.3652	0.5917	1.3835	0.2206	1.2354
	3.0	0.3002	0.2286	0.2940	0.5462	0.1385	1.2394
	4.5	0.1782	0.0811	0.0625	0.0696	0.0493	1.2466
	6.0	0.1234	0.0390	0.0209	0.0162	0.0238	1.2503
0.7	0.3	8.9278	160.0124	4441.3086	170076.9160	80.3072	1.0038
	0.6	3.4020	24.7069	282.5463	4490.0994	13.1333	1.0653
	0.9	1.8687	7.8582	52.9009	498.6698	4.3660	1.1181
	1.25	1.1386	3.0465	13.2566	81.4050	1.7502	1.1620
	2.5	0.4100	0.4210	0.7230	1.7798	0.2530	1.2268
	3.0	0.3172	0.2544	0.3426	0.6635	0.1537	1.2361
	4.5	0.1826	0.0854	0.0675	0.0770	0.0520	1.2489
	6.0	0.1249	0.0401	0.0219	0.0172	0.0245	1.2536
1.2	0.3	12.3936	271.0443	8679.5602	371605.1243	117.4439	0.8744
	0.6	4.6645	41.4989	550.2124	9818.0449	19.7409	0.9525
	0.9	2.4351	12.4078	96.6501	1023.7407	6.4781	1.0452
	1.25	1.3821	4.3652	21.7390	149.2945	2.4549	1.1336
	2.5	0.4306	0.4746	0.8777	2.3209	0.2893	1.2492
	3.0	0.3260	0.2745	0.3908	0.8007	0.1682	1.2581
	4.5	0.1830	0.0870	0.0704	0.0825	0.0535	1.2635
	6.0	0.1245	0.0402	0.0221	0.0177	0.0247	1.2630
2.5	0.3	22.4016	729.5440	31630.6584	61.7304×10^6	227.7111	0.6736
	0.6	9.1009	123.8519	2253.1592	51778.6414	41.0251	0.7038
	0.9	4.9353	39.6402	431.4467	5954.7418	15.2831	0.7921
	1.25	2.6186	13.2279	93.7134	850.6719	6.3706	0.9639
	2.5	0.5059	0.7297	1.7975	6.2104	0.4737	1.3604
	3.0	0.3518	0.3527	0.6264	1.6141	0.2290	1.3602
	4.5	0.1828	0.0903	0.0788	0.1026	0.0568	1.3041
	6.0	0.1232	0.0400	0.0225	0.0187	0.0248	1.2787
3.5	0.3	30.4970	1255.2447	65773.7304	4.2189×10^6	325.1784	0.5913
	0.6	12.7351	220.1807	4849.6010	130774.8562	57.9983	0.5980
	0.9	7.3999	76.6812	1016.0269	16496.3842	21.9231	0.6327
	1.25	4.3135	29.0604	253.9790	2729.3543	10.4540	0.7496
	2.5	0.6448	1.2757	4.1232	17.6776	0.8600	1.4383
	3.0	0.3963	0.5009	1.1447	3.6873	0.3438	1.4794
	4.5	0.1842	0.0954	0.0919	0.1375	0.0614	1.3453
	6.0	0.1228	0.0401	0.0231	0.0202	0.0250	1.2876
5.0	0.3	43.3651	2372.5050	157882.3156	1.2444×10^7	491.9754	0.5115
	0.6	18.2266	419.2477	11730.2270	388738.3474	87.0383	0.5119
	0.9	10.9408	151.5452	2554.0580	50986.8016	31.8447	0.5158
	1.25	7.0866	64.9440	725.5541	9604.9598	14.7243	0.5415
	2.5	1.2510	4.1256	18.6886	102.8694	2.5606	1.2792
	3.0	0.5911	1.2435	4.1997	18.0133	0.8941	1.5997
	4.5	0.1913	0.1150	0.1455	0.2976	0.0784	1.4638
	6.0	0.1230	0.0410	0.0253	0.0254	0.0259	1.3079

Table 2. Numerical values for the first four moments and their related measures for $\delta = 2.5$.

Parameters		Measures					
α	β	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	Cof. of Variation
0.2	0.3	1.5468	2.7691	5.5198	11.9780	0.3766	0.3967
	0.6	1.1547	1.5508	2.3254	3.8016	0.2176	0.4040
	0.9	0.9712	1.1009	1.3957	1.9315	0.1577	0.4089
	1.25	0.8439	0.8336	0.9222	1.1148	0.1214	0.4128
	2.5	0.6289	0.4652	0.3865	0.3514	0.0697	0.4196
	3.0	0.5827	0.3997	0.3081	0.2600	0.0602	0.4210
	4.5	0.4923	0.2858	0.1867	0.1335	0.0434	0.4234
	6.0	0.4374	0.2258	0.1312	0.0835	0.0345	0.4246
0.7	0.3	1.8109	3.6597	8.0489	18.9708	0.3803	0.3405
	0.6	1.3103	1.9488	3.1821	5.5975	0.2319	0.3675
	0.9	1.0691	1.3145	1.7877	2.6314	0.1715	0.3874
	1.25	0.9042	0.9496	1.1097	1.4091	0.1320	0.4018
	2.5	0.6444	0.4885	0.4156	0.3865	0.0732	0.4198
	3.0	0.5927	0.4139	0.3248	0.2788	0.0626	0.4221
	4.5	0.4957	0.2901	0.1911	0.1379	0.0444	0.4252
	6.0	0.4387	0.2274	0.1327	0.0849	0.0350	0.4263
1.2	0.3	2.0602	4.6162	11.0644	28.0733	0.3718	0.2960
	0.6	1.4770	2.4286	4.3202	8.1872	0.2469	0.3364
	0.9	1.1743	1.5724	2.3086	3.6404	0.1935	0.3746
	1.25	0.9626	1.0769	1.3372	1.7982	0.1502	0.4026
	2.5	0.6514	0.5018	0.4353	0.4136	0.0775	0.4274
	3.0	0.5954	0.4196	0.3332	0.2902	0.0651	0.4285
	4.5	0.4949	0.2899	0.1915	0.1387	0.0450	0.4287
	6.0	0.4375	0.2265	0.1322	0.0847	0.0351	0.4285
2.5	0.3	2.5760	6.9611	19.6252	57.4815	0.3255	0.2215
	0.6	1.9145	3.8847	8.2580	18.2747	0.2194	0.2447
	0.9	1.5361	2.5741	4.5769	8.5259	0.2144	0.3014
	1.25	1.2019	1.6541	2.4781	3.9458	0.2095	0.3808
	2.5	0.6731	0.5462	0.5068	0.5209	0.0931	0.4533
	3.0	0.6020	0.4345	0.3577	0.3263	0.0721	0.4461
	4.5	0.4927	0.2884	0.1911	0.1395	0.0457	0.4337
	6.0	0.4355	0.2247	0.1309	0.0838	0.0351	0.4301
3.5	0.3	2.8755	8.5746	26.4349	84.0396	0.3062	0.1925
	0.6	2.1703	4.8960	11.4290	27.5225	0.1859	0.1987
	0.9	1.8062	3.4321	6.7798	13.8453	0.1699	0.2282
	1.25	1.4705	2.3660	4.0240	7.1311	0.2038	0.3070
	2.5	0.7127	0.6258	0.6379	0.7257	0.1179	0.4817
	3.0	0.6156	0.4612	0.4000	0.3893	0.0822	0.4657
	4.5	0.4926	0.2889	0.1924	0.1417	0.0463	0.4366
	6.0	0.4349	0.2242	0.1305	0.0836	0.0350	0.4304
5.0	0.3	3.2485	10.8426	37.1186	130.1397	0.2896	0.1657
	0.6	2.4614	6.2256	16.1514	42.9149	0.1671	0.1661
	0.9	2.0882	4.4869	9.8940	22.3483	0.1264	0.1703
	1.25	1.8040	3.3792	6.5161	12.8884	0.1247	0.1957
	2.5	0.8708	0.9552	1.2066	1.6634	0.1970	0.5097
	3.0	0.6737	0.5750	0.5842	0.6737	0.1211	0.5164
	4.5	0.4949	0.2933	0.1992	0.1514	0.0484	0.4446
	6.0	0.4347	0.2241	0.1306	0.0839	0.0351	0.4311

3.2. **Incomplete Moments.** We can calculate the incomplete moments of a distribution over a certain interval. They are the moments of the probability distribution that only consider the values of the random variable within that interval. If the distribution has a PDF of $f(x)$, then the incomplete moments $M_k(a,b)$ are given by

$$M_k(t) = \int_0^t x^k f(x)dx.$$

For NMULD, incomplete moments can be obtained as,

$$\begin{aligned} M_k(t;\alpha,\beta,\delta) &= \int_0^t x^k \frac{\delta\beta^{\alpha+1}x^{\delta-1}e^{-\beta x^\delta}\left(\beta + x^{\alpha\delta}\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}dx \\ &= \frac{\beta^{-\frac{k}{\delta}}\left(\beta^{\alpha+1}\Gamma\left(\frac{k+\delta}{\delta}, t^\delta\beta\right)+\Gamma\left(\frac{k}{\delta}+\alpha+1, t^\delta\beta\right)\right)}{\beta^{\alpha+1}+\Gamma(\alpha+1)}, \end{aligned}$$

where $\Gamma(.,.)$ in lower incomplete gamma function.

3.3. **Hazard Rate Function (HRF) and Mean Residual Life Function.** The HRF, also known as the failure rate function, describes the instantaneous likelihood of failure at a given time, assuming survival up to that time. It plays a crucial role in reliability analysis and survival modeling by characterizing how the risk of failure evolves. For a continuous random variable X with probability density function $f(x)$ and survival function $S(x)$, the hazard function is defined as follows [26]:

$$h(x) = \frac{f(x)}{S(x)}. \tag{3.6}$$

Substituting the pdf f(x) and survival function S(x) of the NMULD into this equation, we obtain the HRF of the NMULD as

$$h(x;\beta,\alpha,\delta) = \frac{\delta\beta^{\alpha+1}x^{\delta-1}\left(\beta + x^{\alpha\delta}\right)}{\beta^{\alpha+1}+e^{\beta x^\delta}\Gamma\left(\alpha+1, x^\delta\beta\right)} \tag{3.7}$$

The HRF plots of the NMULD distribution are shown in Figure (2), illustrating failure rate patterns that include monotonically increasing, J-shaped, and reverse J-shaped curves. These shapes are particularly significant because they are commonly observed in fields such as reliability analysis, biomedical studies, and risk assessment, where understanding the evolution of failure rates over time is critical. The NMULD's capability to capture these hazard rate shapes highlights its potential as a versatile tool for modeling complex real-world scenarios.

The mean residual life function $m(x)$ of the NMULD is further obtained as follows

$$\begin{aligned} m(x;\beta,\alpha,\delta) =& \frac{1}{S(x;\beta,\alpha,\delta)}\int_x^\infty t\, f(t;\beta,\alpha,\delta)dt - x \\ =& \frac{\beta^{-1/\delta}\left(\beta^{\alpha+1}\Gamma\left(1+\frac{1}{\delta}, x^\delta\beta\right)+\Gamma\left(\alpha+\frac{1}{\delta}+1, x^\delta\beta\right)\right)}{S(x;\beta,\alpha,\delta)[\beta^{\alpha+1}+\Gamma(\alpha+1)]} - x \end{aligned}$$

3.4. **The Quantile Function.** Once we have the CDF, we can find the values of x corresponding to specific percentiles/quantiles by using the inverse CDF (quantile) function. Here, for the given $F(x)$, quantile function will be derived by finding the value of X for the desired quantile p (a probability between 0 and 1) as

$$\beta^{\alpha+1}e^{\neg\beta x^{\delta}}+\Gamma(\alpha+1,\beta x^{\delta})-z=0;p\in(0,1). \tag{3.8}$$

where $z = \left\{\beta^{\alpha+1}+\Gamma(\alpha+1)\right\}(1-p)$. The obtained Equation (3.8) can further be solved using numerical methods or statistical software to find quantiles for the specified values of (β,α,δ).

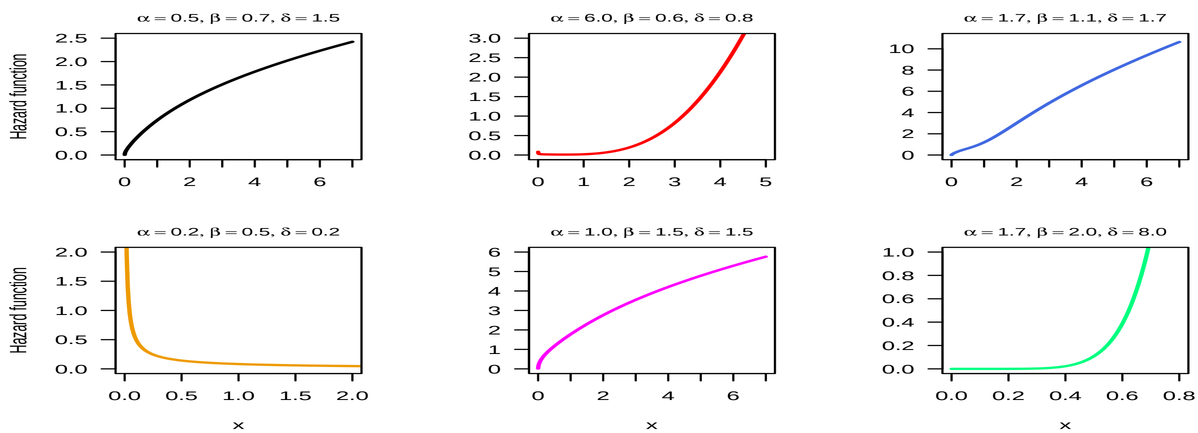

FIGURE 2. Behaviour of hazard function of NMULD for different values of α, β, and δ

3.5. **Skewness and Kurtosis.** Skewness and kurtosis provide information about the shape of a distribution, including its asymmetry (skewness) and the degree of peakedness or flatness of the central part and tails (kurtosis). Interpreting these measures using quantiles allows us to gain insights into how the data values are distributed across the different parts of the distribution. We have computed the skewness and kurtosis using Equation (3.8) on the following relations defined by (Kennedy and Keeping, 1962) [27] developed the quartile-based Bowley's measure of skewness as follows:

$$S_k(\beta,\alpha,\delta)=\frac{Q(1/4;\beta,\alpha,\delta)+Q(3/4;\beta,\alpha,\delta)-2Q(1/2;\beta,\alpha,\delta)}{Q(3/4;\beta,\alpha,\delta)-Q(1/4;\beta,\alpha,\delta)},$$

and the coefficient of kurtosis defined by (Moors, 1988) [28] using octiles is as follows:

$$K_u(\beta,\alpha,\delta)=\frac{Q(0.875;\beta,\alpha,\delta)+Q(0.375;\beta,\alpha,\delta)-Q(0.125;\beta,\alpha,\delta)-Q(0.625;\beta,\alpha,\delta)}{Q(3/4;\beta,\alpha,\delta)-Q(1/4;\beta,\alpha,\delta)}.$$

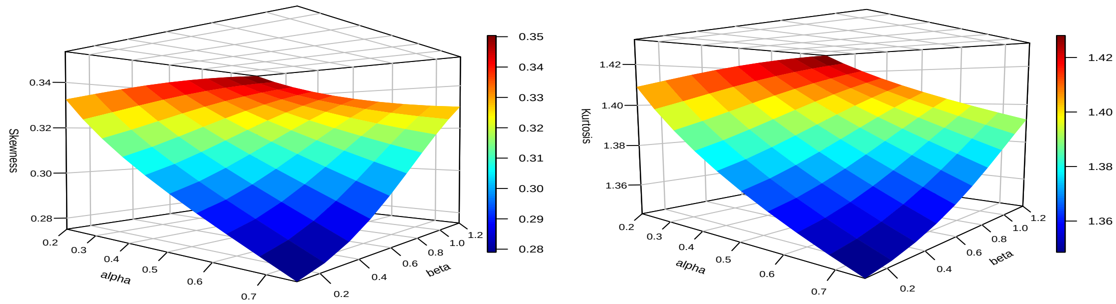

FIGURE 3. Skewness and Kurtosis of NMULD for different values of α and β, keeping $\delta = 0.75$.

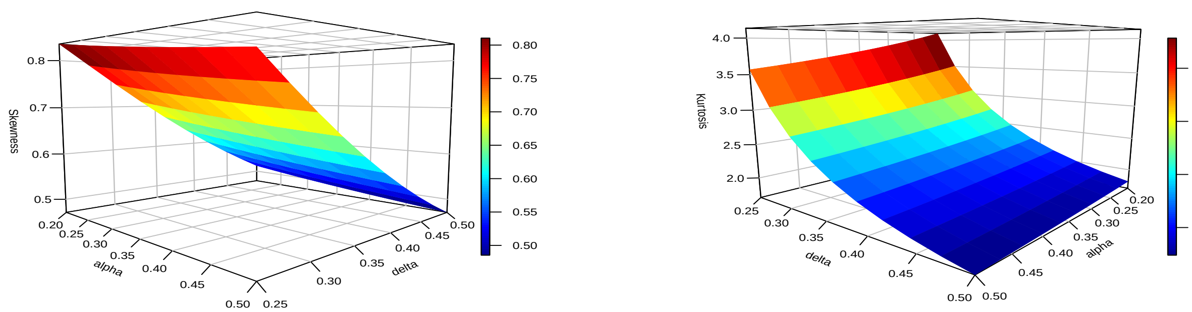

FIGURE 4. Skewness and Kurtosis of NMULD for different values of α and δ, keeping $\beta = 0.5$.

4. The Maximum Likelihood Estimation (MLE)

This section discusses the MLEs for estimating the unknown parameters of the NMULD. Let a random sample $X_1, \ldots, X_n$ of size, n, be taken from NMULD. The likelihood function of NMULD can be obtained as

$$L(\beta, \alpha, \delta) = \left(\frac{\delta \beta^{\alpha+1}}{\beta^{\alpha+1+\Gamma(\alpha+1)}} \right)^n \left(\prod_{i=1}^{n} x_i \right)^{(\delta-1)} \prod_{i=1}^{n} \left(\beta + {x_i}^{\alpha\delta} \right) e^{-\beta \sum_{i=1}^{n} {x_i}^{\delta}}.$$

Then, the log-likelihood function is given as

$$\begin{aligned} \ell(\beta, \alpha, \delta) \quad &= \quad n \log(\delta) + n(\alpha+1)\log(\beta) - n\log\left(\beta^{\alpha+1} + \Gamma(\alpha+1)\right) \\ &\quad + (\delta-1)\sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log(\beta + {x_i}^{\alpha\delta}) - \beta \sum_{i=1}^{n} {x_i}^{\delta} \end{aligned} \tag{4.1}$$

The corresponding likelihood equations are obtained, respectively, by

$$\frac{\partial \ell}{\partial \beta} = \frac{n(\alpha+1)}{\beta} - \frac{n(\alpha+1)\beta^{\alpha}}{\beta^{\alpha+1}+\Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{1}{\beta + x_i^{\alpha\delta}} - \sum_{i=1}^{n} x_i^{\delta} = 0 \tag{4.2}$$

$$\frac{\partial \ell}{\partial \alpha} = n\log(\beta) - \frac{n\left\{\beta^{\alpha+1}\log(\beta) + \phi(\alpha+1)\right\}}{\beta^{\alpha+1}+\Gamma(\alpha+1)} + \delta \sum_{i=1}^{n} \frac{x_i^{\alpha\delta}\log(x_i)}{\beta + x_i^{\alpha\delta}} = 0 \tag{4.3}$$

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \alpha \frac{x_i^{\alpha\delta}\log(x_i)}{\beta + x_i^{\alpha\delta}} - \beta \sum_{i=1}^{n} x_i^{\delta}\log(x_i) = 0 \tag{4.4}$$

where, $\phi(\alpha+1) = \frac{d}{d\alpha}\log\Gamma(\alpha+1)$ is the digamma function. The Equations (4.2)-(4.4) cannot be explicitly solved, and iterative methods can be employed to find their solutions. In this study, the MLEs of β, α, and δ, say $\widehat{\beta}, \widehat{\alpha}$, and $\widehat{\delta}$, are achieved with the optim function in R (R core team, 2023) [29].

5. Numerical simulation

In this section, a Monte Carlo simulation with 10,000 replications is conducted to assess the efficiency of the maximum likelihood estimation method for the proposed NMUL distribution. The sample sizes considered in this study are n = 100, 200, 300, 400, 500, 600, 700, 800, 900 and 1000, ensuring a thorough evaluation of the estimator's performance across different scenarios, with ten distinct initial parameter values (referred to as scenarios), detailed in Table 3.

Table 3. Scenarios for the simulation study

Scenario	α	β	δ
1	0.5	1.3	0.3
2	1.5	0.8	0.7
3	2.0	3.0	1.2
4	0.7	0.3	2.0
5	1.0	1.5	0.5
6	1.2	2.0	1.5
7	3.0	2.0	1.0
8	0.3	0.5	0.7
9	2.5	3.0	2.0
10	0.9	0.9	0.9

In the simulation, samples are generated from the NMULD using an acceptance-rejection sampling method with Algorithm 1.

Algorithm 1:

S1. Generating data from random variable Z follows a Weibull distribution with the following

PDF g

$$g(z) = \frac{a}{b}\left(\frac{z}{b}\right)^{a-1} \exp\left(-\left(\frac{z}{b}\right)^{a}\right).$$

S2. A sample is generated from a standard uniform distribution, denoted as U (independent from Z).

S3. If

$$U < \frac{f(Z)}{a \times g(Z)}$$

then $X = Z$ (ACCEPT), otherwise goto S1(REJECT), where f is the PDF of the NMULD and

$$a = \max_{z \in \mathbb{R}^+} \frac{f(z)}{g(z)}.$$

The output of Algorithm 1 provides random data for the variable X from the NMULD. There are several criteria commonly used to evaluate the performance of an estimator. Some of these include the average absolute bias (ABB), mean squared error (MSE), and mean relative error (MRE). For more details, see [30,31]. In this study, the effectiveness of MLE is demonstrated using these three criteria. The ABB, MSE, and MRE are calculated, respectively, by

$$ABB_{\gamma_j} = \frac{1}{10000} \sum_{i=1}^{10000} \left|\widehat{\gamma_j} - \gamma_j\right|,$$

$$MSE_{\gamma_j} = \frac{1}{10000} \sum_{i=1}^{N} \left(\widehat{\gamma_j} - \gamma_j\right)^2$$

and

$$MRE_{\gamma_j} = \frac{1}{10000} \sum_{i=1}^{10000} \frac{\left|\widehat{\gamma_j} - \gamma_j\right|}{\gamma_j},$$

where $\gamma_j = (\alpha, \beta, \delta)$. All computations, including numerical simulations and real data applications, are performed using the R programming language. The simulation results are reported in Tables 4-6. From Tables 4-6 it is concluded that as the sample size increases, the ABB, MSE, and MRE of the maximum likelihood estimator decrease, and tend to go to zero as expected for all parameters.

Table 4. The average ABB, MSE, and MRE of the maximum likelihood estimators for Scenarios 1-4

		ABB			MSE			MRE		
n	Scenario	α	β	δ	α	β	δ	α	β	δ
100	1	0.7593	0.2390	0.0264	1.2343	0.1019	0.0012	1.5185	0.1838	0.0879
200		0.6397	0.2000	0.0189	0.8056	0.0670	0.0006	1.2794	0.1538	0.0629
300		0.5826	0.1821	0.0159	0.6413	0.0538	0.0004	1.1653	0.1401	0.0529
400		0.5278	0.1667	0.0137	0.5161	0.0441	0.0003	1.0555	0.1283	0.0458
500		0.5081	0.1612	0.0125	0.4786	0.0413	0.0003	1.0162	0.1240	0.0418
600		0.4781	0.1527	0.0114	0.4089	0.0360	0.0002	0.9561	0.1175	0.0381
700		0.4585	0.1477	0.0106	0.3751	0.0335	0.0002	0.9170	0.1136	0.0354
800		0.4411	0.1423	0.0101	0.3456	0.0312	0.0002	0.8821	0.1094	0.0335
900		0.4208	0.1371	0.0097	0.3129	0.0288	0.0002	0.8416	0.1054	0.0324
1000		0.4141	0.1347	0.0092	0.2996	0.0277	0.0001	0.8282	0.1036	0.0307
100	2	0.6635	0.2160	0.0525	0.6781	0.0729	0.0056	0.4423	0.2700	0.0751
200		0.4723	0.1525	0.0350	0.3648	0.0389	0.0026	0.3149	0.1906	0.0501
300		0.3867	0.1239	0.0272	0.2478	0.0262	0.0016	0.2578	0.1548	0.0389
400		0.3288	0.1055	0.0224	0.1804	0.0190	0.0011	0.2192	0.1318	0.0321
500		0.2991	0.0959	0.0199	0.1475	0.0154	0.0008	0.1994	0.1199	0.0284
600		0.2721	0.0869	0.0177	0.1217	0.0126	0.0006	0.1814	0.1087	0.0253
700		0.2495	0.0794	0.0160	0.1031	0.0106	0.0005	0.1663	0.0992	0.0229
800		0.2331	0.0742	0.0150	0.0897	0.0092	0.0004	0.1554	0.0928	0.0214
900		0.2192	0.0697	0.0138	0.0775	0.0080	0.0003	0.1461	0.0871	0.0197
1000		0.2081	0.0660	0.0129	0.0703	0.0071	0.0003	0.1387	0.0825	0.0185
100	3	1.3263	0.3094	0.0932	2.7861	0.1664	0.0142	0.6632	0.1031	0.0777
200		1.0824	0.2174	0.0667	1.6749	0.0773	0.0072	0.5412	0.0725	0.0556
300		0.9827	0.1862	0.0571	1.3481	0.0554	0.0051	0.4913	0.0621	0.0476
400		0.8992	0.1635	0.0500	1.1226	0.0420	0.0039	0.4496	0.0545	0.0417
500		0.8570	0.1520	0.0459	1.0188	0.0360	0.0033	0.4285	0.0507	0.0383
600		0.8117	0.1421	0.0426	0.9227	0.0313	0.0028	0.4059	0.0474	0.0355
700		0.7854	0.1328	0.0408	0.8712	0.0276	0.0025	0.3927	0.0443	0.0340
800		0.7551	0.1277	0.0387	0.8133	0.0255	0.0023	0.3776	0.0426	0.0322
900		0.7261	0.1218	0.0365	0.7625	0.0232	0.0021	0.3630	0.0406	0.0304
1000		0.6984	0.1172	0.0354	0.7137	0.0214	0.0019	0.3492	0.0391	0.0295
100	4	0.5622	0.1673	0.2369	0.5645	0.0502	0.0845	0.8031	0.5577	0.1184
200		0.4595	0.1357	0.2065	0.3381	0.0296	0.0643	0.6564	0.4522	0.1033
300		0.4104	0.1208	0.1936	0.2545	0.0221	0.0572	0.5863	0.4028	0.0968
400		0.3778	0.1112	0.1828	0.2106	0.0183	0.0520	0.5397	0.3706	0.0914
500		0.3603	0.1059	0.1783	0.1885	0.0163	0.0502	0.5148	0.3529	0.0891
600		0.3447	0.1008	0.1696	0.1706	0.0146	0.0457	0.4924	0.3358	0.0848
700		0.3286	0.0958	0.1647	0.1553	0.0132	0.0443	0.4694	0.3195	0.0823
800		0.3146	0.0921	0.1614	0.1414	0.0121	0.0430	0.4494	0.3070	0.0807
900		0.3102	0.0907	0.1599	0.1390	0.0119	0.0429	0.4431	0.3025	0.0800
1000		0.2966	0.0865	0.1528	0.1271	0.0108	0.0396	0.4237	0.2883	0.0764

Table 5. The average ABB, MSE, and MRE of the maximum likelihood estimators for Scenarios 5-7

		ABB			MSE			MRE		
n	Scenario	α	β	δ	α	β	δ	α	β	δ
100	5	0.9625	0.2738	0.0481	1.6081	0.1248	0.0040	0.9625	0.1826	0.0963
200		0.7842	0.2200	0.0339	0.9857	0.0752	0.0020	0.7842	0.1466	0.0678
300		0.7169	0.2012	0.0282	0.7878	0.0605	0.0014	0.7169	0.1341	0.0565
400		0.6531	0.1824	0.0247	0.6505	0.0501	0.0010	0.6531	0.1216	0.0493
500		0.6069	0.1700	0.0218	0.5553	0.0433	0.0008	0.6069	0.1133	0.0437
600		0.5811	0.1626	0.0203	0.5076	0.0398	0.0007	0.5811	0.1084	0.0406
700		0.5522	0.1543	0.0190	0.4576	0.0362	0.0006	0.5522	0.1028	0.0380
800		0.5267	0.1475	0.0177	0.4206	0.0335	0.0005	0.5267	0.0983	0.0355
900		0.5115	0.1429	0.0167	0.3924	0.0314	0.0005	0.5115	0.0953	0.0334
1000		0.4935	0.1380	0.0161	0.3668	0.0296	0.0004	0.4935	0.0920	0.0322
100	6	1.0250	0.2688	0.1418	2.0123	0.1295	0.0353	0.8541	0.1344	0.0945
200		0.8431	0.2090	0.1041	1.2354	0.0737	0.0187	0.7026	0.1045	0.0694
300		0.7569	0.1809	0.0859	0.9396	0.0534	0.0126	0.6307	0.0904	0.0573
400		0.7060	0.1688	0.0762	0.8040	0.0457	0.0100	0.5884	0.0844	0.0508
500		0.6651	0.1562	0.0690	0.6910	0.0387	0.0081	0.5543	0.0781	0.0460
600		0.6412	0.1497	0.0652	0.6382	0.0356	0.0071	0.5343	0.0748	0.0434
700		0.6128	0.1427	0.0615	0.5781	0.0320	0.0064	0.5107	0.0713	0.0410
800		0.5928	0.1377	0.0582	0.5412	0.0299	0.0058	0.4940	0.0689	0.0388
900		0.5748	0.1333	0.0550	0.5037	0.0278	0.0051	0.4790	0.0667	0.0366
1000		0.5603	0.1300	0.0525	0.4699	0.0262	0.0046	0.4669	0.0650	0.0350
100	7	1.0260	0.2921	0.0773	1.6824	0.1390	0.0094	0.3420	0.1460	0.0773
200		0.7233	0.2044	0.0548	0.8906	0.0704	0.0048	0.2411	0.1022	0.0548
300		0.5883	0.1655	0.0458	0.6066	0.0473	0.0033	0.1961	0.0828	0.0458
400		0.5034	0.1417	0.0393	0.4510	0.0351	0.0025	0.1678	0.0709	0.0393
500		0.4423	0.1248	0.0350	0.3319	0.0261	0.0019	0.1474	0.0624	0.0350
600		0.4078	0.1143	0.0320	0.2825	0.0219	0.0016	0.1359	0.0571	0.0320
700		0.3679	0.1040	0.0298	0.2288	0.0179	0.0014	0.1226	0.0520	0.0298
800		0.3421	0.0961	0.0278	0.1939	0.0152	0.0012	0.1140	0.0481	0.0278
900		0.3221	0.0913	0.0256	0.1701	0.0135	0.0010	0.1074	0.0457	0.0256
1000		0.3078	0.0872	0.0246	0.1535	0.0122	0.0010	0.1026	0.0436	0.0246

6. Applications to Real Data

In our study, we selected well-established flexible probability distributions as competing models to evaluate the performance of the proposed NMUL distribution. Specifically, the following models are considered as a competitor: ShD [20], exponentiated exponential distribution (EED) [32], Lindley distribution (LD) [3], Fréchet distribution (FD), xLindley distribution (xLD) [33], Shanker distribution (SD) [21], and xGamma distribution (xGD) [34]. These distributions are chosen based on their relevance in reliability analysis, survival studies, and their ability to capture various hazard function shapes. The PDFs of all distributions are give in Table 7.

To ensure a robust goodness-of-fit evaluation, we employed the MLE procedure to estimate the parameters of all competing distributions, along with their standard errors (SE), log-likelihood values ($\widehat{\ell}$), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). Additionally, we evaluated model fit using the Kolmogorov-Smirnov (KS) test, Anderson-Darling (AD) test, and Cramér-von Mises (CvM) test, along with their corresponding p-values (KS $p$val, AD $p$val, CvM $p$val).

The results, summarized in Tables 8 and 9, demonstrate that the proposed NMUL model provides a superior fit compared to its competitors. Furthermore, Figures 5-12 illustrate the fitted PDF, CDF, and survival function plots, further validating the flexibility and applicability of the proposed model. To enhance clarity and transparency, we have explicitly stated the rationale behind the selection of competing models and provided a comprehensive discussion of their comparative performance.

The first dataset (carbon fiber tensile strength), a well-known dataset in reliability engineering where accurate lifetime modeling is essential for assessing material durability and structural integrity, represents the symmetric behavior of the tensile strength of about 100 observations of carbon fibers and is taken from [35]. The modeling results for the first data are reported in Table 8.

TABLE 6. The average ABB, MSE, and MRE of the maximum likelihood estimators for Scenarios 8-10

		ABB			MSE			MRE		
n	Scenario	α	β	δ	α	β	δ	α	β	δ
100	8	0.4080	0.1515	0.0499	0.4311	0.0483	0.0039	1.3601	0.3030	0.0713
200		0.3347	0.1261	0.0411	0.2619	0.0316	0.0026	1.1158	0.2522	0.0587
300		0.3023	0.1144	0.0371	0.1979	0.0247	0.0021	1.0076	0.2289	0.0530
400		0.2838	0.1084	0.0357	0.1672	0.0214	0.0019	0.9461	0.2167	0.0510
500		0.2679	0.1024	0.0343	0.1435	0.0187	0.0017	0.8932	0.2048	0.0490
600		0.2533	0.0973	0.0329	0.1255	0.0167	0.0016	0.8443	0.1947	0.0470
700		0.2473	0.0955	0.0322	0.1174	0.0158	0.0015	0.8245	0.1910	0.0461
800		0.2360	0.0911	0.0311	0.1056	0.0143	0.0014	0.7866	0.1822	0.0444
900		0.2312	0.0901	0.0310	0.0982	0.0136	0.0014	0.7707	0.1802	0.0443
1000		0.2246	0.0876	0.0303	0.0924	0.0129	0.0013	0.7485	0.1751	0.0433
100	9	1.4126	0.3158	0.1556	2.8822	0.1688	0.0393	0.5650	0.1053	0.0778
200		1.1525	0.2286	0.1123	1.8513	0.0839	0.0199	0.4610	0.0762	0.0562
300		1.0033	0.1920	0.0949	1.4525	0.0587	0.0141	0.4013	0.0640	0.0474
400		0.9263	0.1715	0.0849	1.2547	0.0458	0.0111	0.3705	0.0572	0.0424
500		0.8507	0.1551	0.0775	1.0933	0.0380	0.0092	0.3403	0.0517	0.0387
600		0.8030	0.1457	0.0725	0.9928	0.0335	0.0081	0.3212	0.0486	0.0362
700		0.7518	0.1380	0.0682	0.9010	0.0305	0.0072	0.3007	0.0460	0.0341
800		0.7180	0.1303	0.0636	0.8350	0.0274	0.0064	0.2872	0.0434	0.0318
900		0.6820	0.1224	0.0613	0.7610	0.0247	0.0059	0.2728	0.0408	0.0307
1000		0.6524	0.1176	0.0585	0.7163	0.0230	0.0054	0.2610	0.0392	0.0293
100	10	0.7189	0.2370	0.0663	0.7813	0.0819	0.0074	0.7988	0.2634	0.0737
200		0.5528	0.1840	0.0488	0.4492	0.0492	0.0040	0.6142	0.2045	0.0542
300		0.4762	0.1596	0.0415	0.3335	0.0377	0.0030	0.5291	0.1773	0.0461
400		0.4223	0.1419	0.0359	0.2647	0.0307	0.0024	0.4692	0.1577	0.0399
500		0.3894	0.1311	0.0327	0.2299	0.0269	0.0020	0.4327	0.1456	0.0363
600		0.3619	0.1217	0.0300	0.2005	0.0237	0.0018	0.4021	0.1352	0.0334
700		0.3361	0.1131	0.0280	0.1785	0.0213	0.0016	0.3734	0.1256	0.0311
800		0.3184	0.1065	0.0258	0.1604	0.0189	0.0014	0.3537	0.1184	0.0287
900		0.2949	0.0989	0.0240	0.1418	0.0170	0.0012	0.3276	0.1099	0.0267
1000		0.2803	0.0941	0.0230	0.1294	0.0156	0.0011	0.3114	0.1046	0.0255

TABLE 7. The PDFs of the competitor distributions

Competitor distribution	PDF	Domains of parameter(s)
ShD	$\frac{\beta^{\alpha+1}\exp(-\beta x)(\beta+x^{\alpha})}{\beta^{\alpha+1}+\Gamma(\alpha+1)}, x>0$	$\alpha,\beta>0$
EED	$\alpha\beta\exp\left(-\beta x\right)\left(1-\exp\left(-\beta x\right)\right)^{\alpha-1}, x>0$	$\alpha,\beta>0$
LD	$\frac{\alpha^2}{1+\alpha}\left(x+1\right)\exp\left(-\alpha x\right), x>0$	$\alpha>0$
FD	$\alpha\beta^{\alpha}x^{-\alpha-1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right), x>0$	$\alpha,\beta>0$
xLD	$\frac{\alpha^2}{(1+\alpha)^2}\left(2+\alpha+x\right)\exp\left(-\alpha x\right), x>0$	$\alpha>0$
SD	$\frac{\alpha^2}{\alpha^2+1}\left(\alpha+x\right)\exp\left(-\alpha x\right)$	$\alpha>0$
xGD	$\frac{\alpha^2}{\alpha+1}\left(1+\frac{\alpha x^2}{2}\right)\exp\left(-\alpha x\right)$	$\alpha>0$

Table 8. The goodness of fit results for the carbon fiber dataset

	NMULD	ShD	EED	LD	FD	xLD	SD	xGD
$\widehat{\ell}$	-121.5416	-143.8080	-127.4041	-158.2813	-152.6672	-164.5822	-155.1901	-160.6653
AIC	249.0832	291.6160	258.8082	318.5626	309.3344	331.1643	312.3802	323.3307
BIC	256.4809	296.5478	263.7401	321.0285	314.2662	333.6302	314.8461	325.7966
KS	0.0683	0.1679	0.1191	0.2771	0.1840	0.3003	0.2477	0.3047
AD	0.3908	4.8117	1.3109	11.0191	5.2332	12.7647	9.4160	12.7010
CvM	0.0640	0.7221	0.2363	2.0614	0.8704	2.4400	1.6798	2.4691
KS $p$val	0.8115	0.0148	0.1694	0.0000	0.0055	0.0000	0.0000	0.0000
AD $p$val	0.8575	0.0036	0.2288	0.0000	0.0022	0.0000	0.0000	0.0000
CvM $p$val	0.7903	0.0111	0.2071	0.0000	0.0048	0.0000	0.0001	0.0000
α	0.0059	6.1358	7.7135	0.6156	1.7282	0.5361	0.6274	0.8493
β	0.0447	2.1688	1.0045	–	1.8940	–	–	–
δ	2.8812	–	–	–	–	–	–	–
se of α	0.5184	1.3218	1.5724	0.0485	0.1168	0.0432	0.0444	0.0600
se of β	0.0700	0.4312	0.0923	–	0.1251	–	–	–
se of δ	0.8843	–	–	–	–	–	–	–

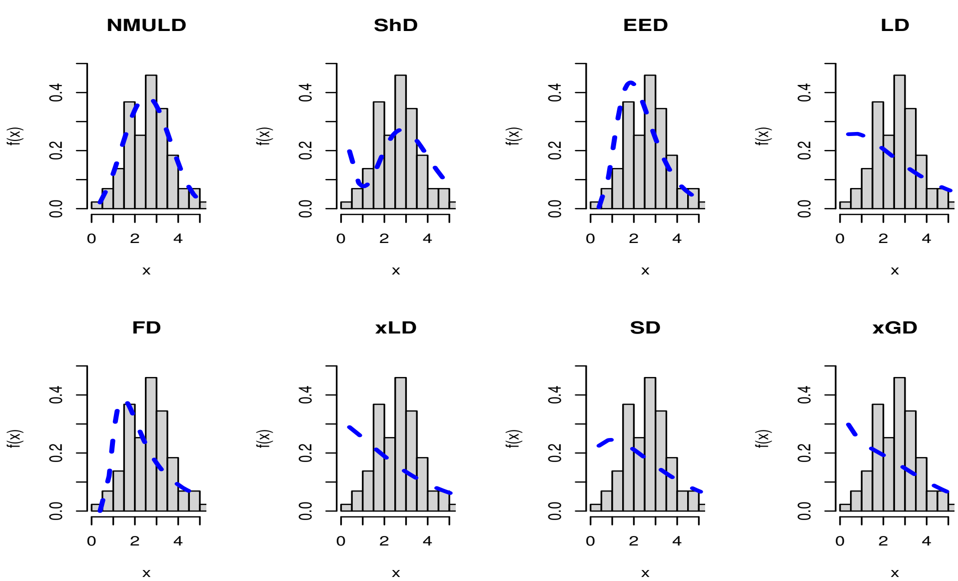

Figure 5. The fitted PDF plots for all models in carbon fiber dataset

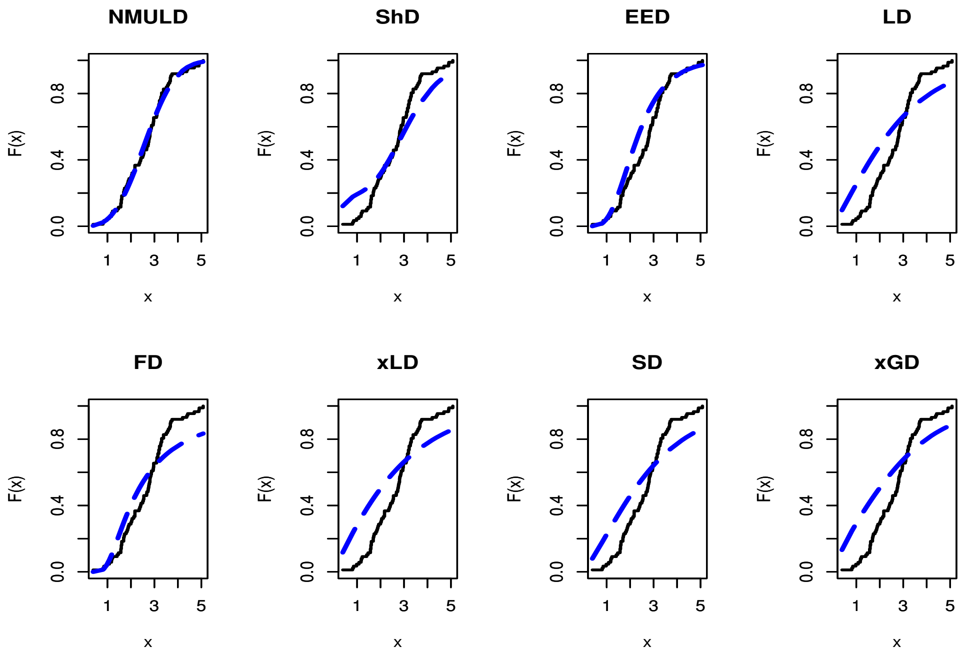

FIGURE 6. The fitted CDF plots for all models in carbon fiber dataset

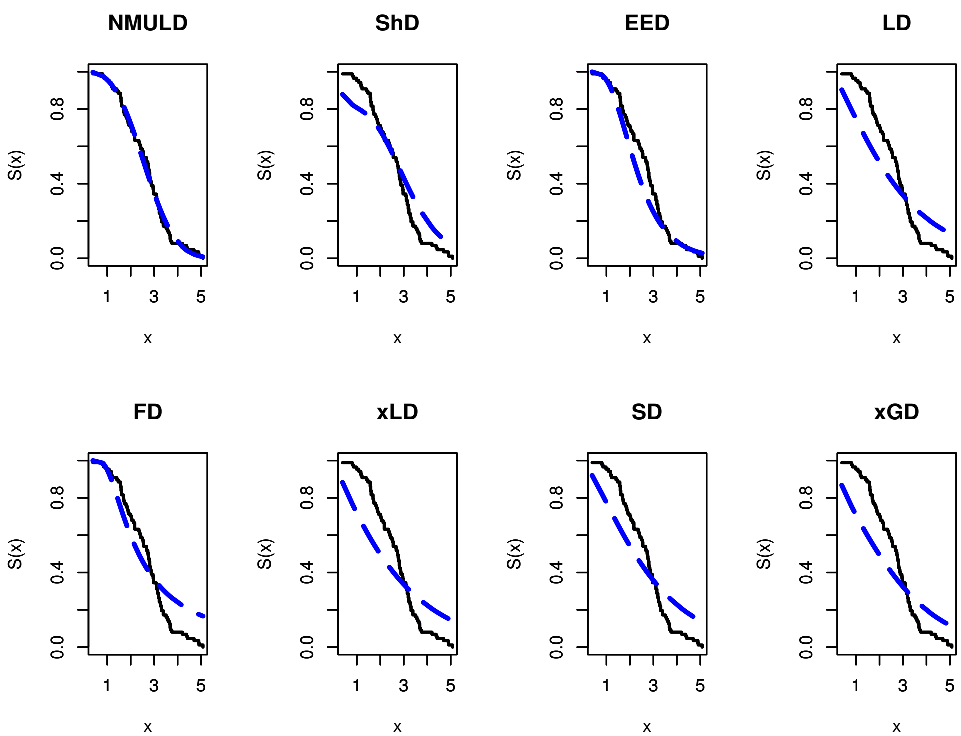

FIGURE 7. The fitted SF plots for all models in carbon fiber dataset

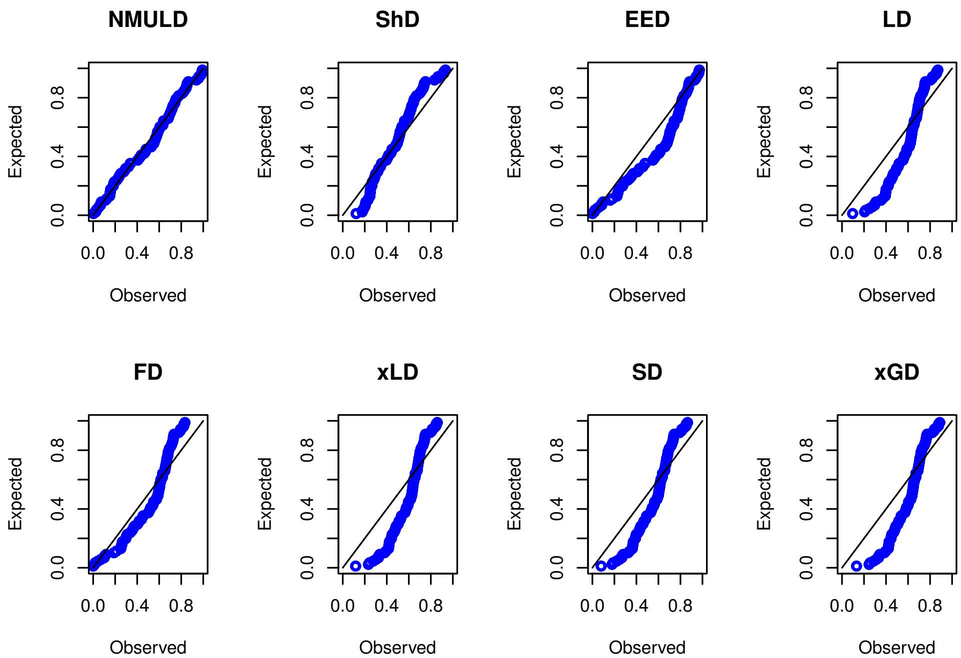

Figure 8. The fitted P–P plots for all models in carbon fiber dataset

The second dataset (bladder cancer remission times), a crucial application in survival analysis demonstrating the proposed model's ability to describe time-to-event data in medical research, consists of remission times (in months) for 128 bladder cancer patients and is taken from [36]. This dataset has previously been analyzed by [37] and [38]. The results of the second data are given in Table 9.

The model selection criteria and goodness-of-fit test statistics indicate that NMULD is the best model according to all criteria. Specifically, in Table 8, the values are AIC = 249.0832, BIC = 256.4809, KS = 0.0683, AD = 0.3908, and CvM = 0.0640. In Table 9, the values are AIC = 808.8236, BIC = 817.3797, KS = 0.0476, AD = 0.3909, and CvM = 0.0596. These minimum statistics highlight the NMULD's suitability for modeling. The flexibility of the NMULD in capturing the data patterns is further demonstrated in Figures 5-12 across both data sets, reinforcing its frequent use in the literature for modeling purposes.

Table 9. The goodness of fit results for the bladder cancer dataset

	NMULD	ShD	EED	LD	FD	XLD	SD	XGD
$\widehat{\ell}$	-401.4118	-402.7599	-402.5108	-417.8172	-428.6759	-412.2702	-427.5825	-425.1403
AIC	808.8236	809.5198	809.0216	837.6345	861.3518	826.5404	857.1650	852.2805
BIC	817.3797	815.2238	814.7257	840.4865	867.0559	829.3925	860.0170	855.1326
KS	0.0476	0.0598	0.0549	0.1329	0.1542	0.1144	0.1405	0.1843
AD	0.3909	0.6389	0.5136	5.9191	5.7172	3.7243	8.8293	8.8633
CvM	0.0596	0.0932	0.0782	0.8198	1.0055	0.5537	1.0150	1.4865
KS *pval*	0.9334	0.7492	0.8355	0.0218	0.0045	0.0701	0.0128	0.0003
AD *pval*	0.8575	0.6116	0.7327	0.0011	0.0013	0.0119	0.0000	0.0000
CvM *pval*	0.8173	0.6198	0.7034	0.0065	0.0023	0.0290	0.0022	0.0002
α	0.4881	0.0000	0.9223	0.2132	0.6904	0.1987	0.2325	0.2864
β	0.2939	0.1169	0.1108	–	2.2874	–	–	–
δ	0.7744	–	–	–	–	–	–	–
se of α	0.5931	0.1220	0.1065	0.0134	0.0420	0.0126	0.0141	0.0160
se of β	0.1897	0.0165	0.0133	–	0.3117	–	–	–
se of δ	0.1262	–	–	–	–	–	–	–

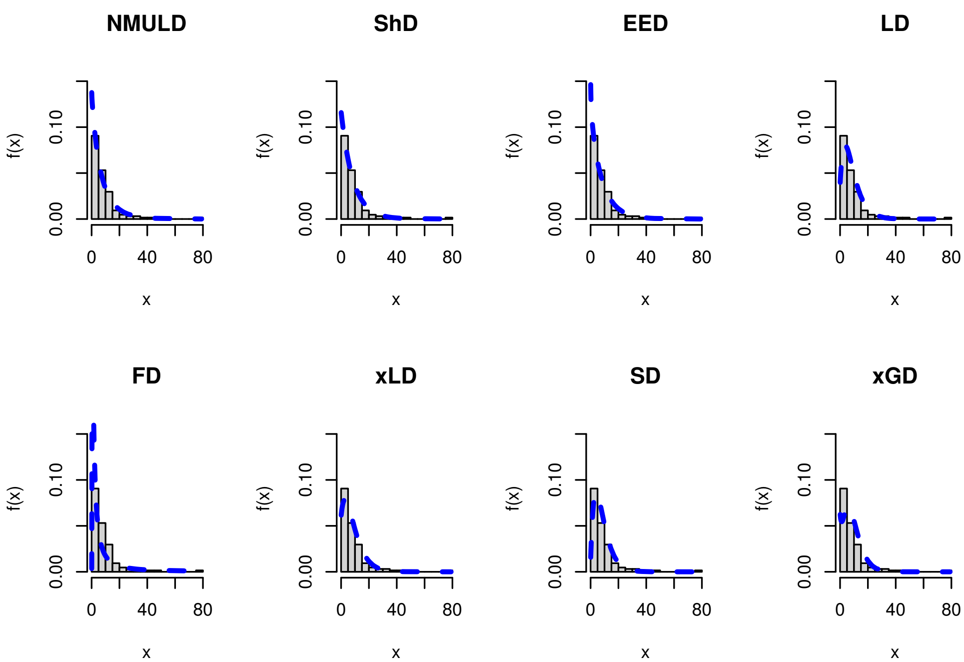

Figure 9. The fitted PDF plots for all models in bladder cancer dataset

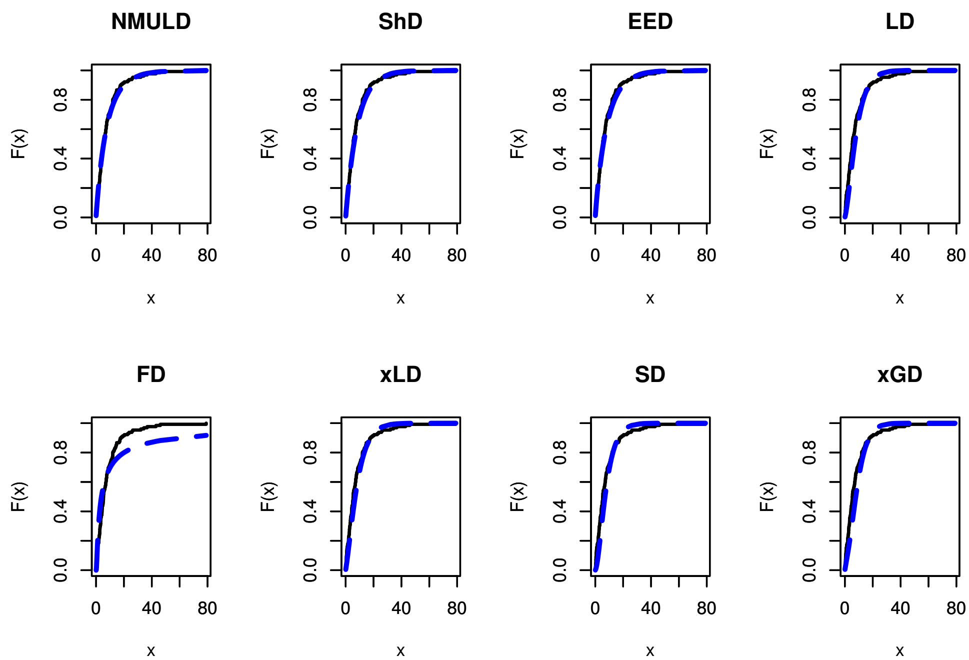

FIGURE 10. The fitted CDF plots for all models in ladder cancer dataset

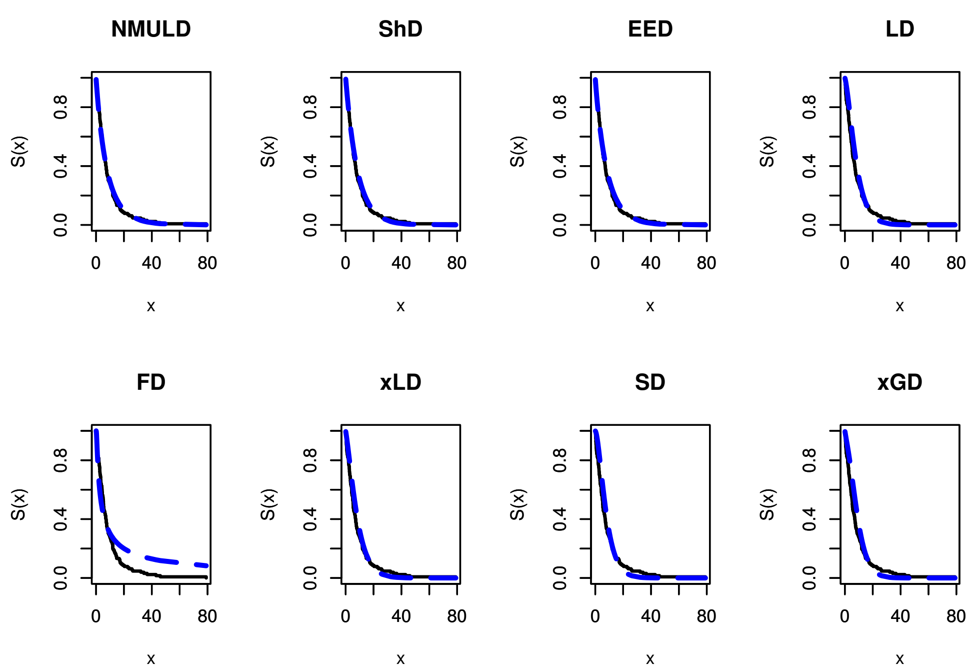

FIGURE 11. The fitted SF plots for all models in ladder cancer dataset

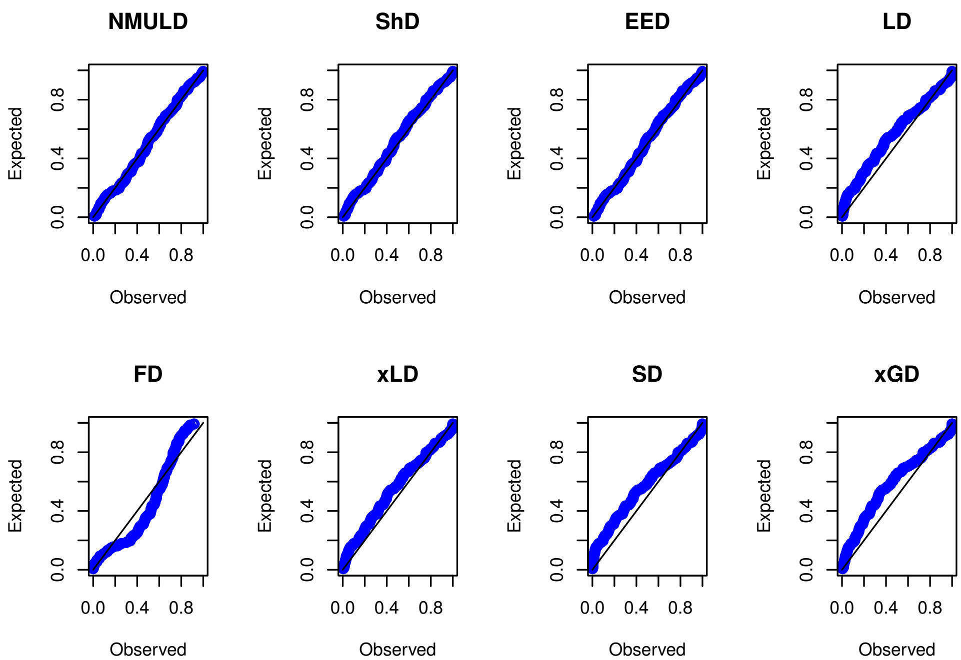

Figure 12. The fitted P–P plots for all models in ladder cancer dataset

7. Conclusion

In this paper, we introduced the NMULD, a flexible extension that generalizes several existing distributions. We thoroughly examined its mathematical properties, including moments, incomplete moments, skewness, kurtosis, and quantile function, and demonstrated its adaptability by accommodating various hazard function shapes, such as J-shaped, reverse J-shaped, and monotonically increasing functions.

To estimate the three unknown parameters, we employed the MLE and rigorously evaluated its performance through extensive Monte Carlo simulations, utilizing an acceptance-rejection algorithm for data generation. The simulation results confirm that as the sample size increases, biases and mean square errors consistently decrease, highlighting the robustness of the estimation approach.

For empirical validation, we applied the NMULD to two real-world datasets, comparing it against multiple established models using model selection criteria and goodness-of-fit tests. The results conclusively demonstrate the superior performance of the proposed distribution, emphasizing its applicability in reliability engineering, survival analysis, and medical research.

Looking ahead, the NMULD has the potential to be widely used across various scientific disciplines, inspiring further advancements in statistical modeling. As a future research direction, alternative estimation techniques—such as Bayesian estimation and the method of moments—can be explored to assess their comparative efficiency against MLE. Such investigations would provide deeper insights into the robustness and adaptability of the NMULD in diverse applications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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