International Journal of Analysis and Applications
ISSN 2291-8639
Volume 6, Number 2 (2014), 205-219
http://www.etamaths.com

TOPOLOGICAL VECTOR-SPACE VALUED CONE BANACH SPACES

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Abstract. In this paper we introduce the notion of tvs-cone normed spaces, discuss related topological concepts and characterize the tvs-cone norm in various directions. We construct generalize locally convex tvs generated by a family of tvs-cone seminorms. The class of weak contractions properly includes large classes of highly applicable contractions like Banach, Kannan, Chatterjea and quasi etc. We prove fixed point results in tvs-cone Banach spaces for nonexpansive self mappings and self/non-self weak contractive mappings. We discuss the necessary conditions for T -stability of Picard iteration. To ensure the novelty of our work we establish an application in homotopy theory without the assumption of normality on cone and many non-trivial examples.

1. Introduction

Recently Beg *et al.* [1] introduced and studied topological vector space-valued cone metric spaces (tvs-cone metric spaces), which generalized the cone metric spaces [2]. Many generalizations and extensions have been made by many researchers, (see [3-6]). For more details about topological vector spaces we refer to [7, 8]. Actually the idea of cone metric space was properly introduced by Huang and Zhang in [2]. In their setting the set of real numbers was replaced by an ordered Banach space and a vector valued metric was defined on a nonempty set. Many authors [9-14] studied the properties of cone metric spaces and generalized important fixed point results of complete metric spaces. The concept of cone metric space in the sense of Huang-Zhang was characterized by Al-Rawashdeh *et al.* in [15].

In [16], the author introduced the notion of cone Banach spaces with normal cones and proved some results regarding fixed points by using nonexpansive mappings. Later on many authors investigated some useful results in fixed and coupled fixed points, (see [17-19]).

Weak contractions were considered in [20], to study the fixed point results for self mappings. It has been shown that the Banach, Kannan, Chatterjea, Zamfirescue, quasi and many other contractions are weak contractions. The importance of non-self mappings is obvious. In fact fixed point theorems for non-self mappings generalized all the corresponding results presented for self-mappings. A variety of results on nonself mappings and weak contractions can be found in [21-27].

In this article, we introduce tvs-cone Banach space and investigate some properties without assumption of normality on cones. We generalize the results of [16] and

2010 *Mathematics Subject Classification.* 47H10; 54H25.
Key words and phrases. tvs-cone Banach space; non-normal cones; weak contraction; nonself mappings; iteration processes; T-stability; fixed points; homotopy; generalized locally convex tvs.

explore some characteristics of norms in cone normed space. We prove fixed point results for Picard, Mann, Ishikawa and Krasnoseskij iterations, we also present results for weak contractive non-self mappings. Many examples have been given and a homotopy result is established for nonexpansive mappings. We discuss the necessary conditions for T-stability of Picard iteration.

2. Preliminaries

Let $\mathbb{E}$ be a topological vector space with its zero vector θ. A nonempty subset K of $\mathbb{E}$ is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be pointed (or proper) if $K \cap (-K) = \{\theta\}$, and K is normal (or saturated) if $\mathbb{E}$ has a base of neighborhoods of zero consisting of order-convex subsets. For a given cone $K \subseteq \mathbb{E}$ we define a partial ordering $\preccurlyeq$ with respect to K by $x \preccurlyeq y$ if and only if $y - x \in K$, $x \prec y$ stands for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of K. The cone K is said to be solid if it has a nonempty interior.

Definition 1. *Let V be a vector space over $\mathbb{R}$. A vector-valued function $\|\cdot\|_K : V \to \mathbb{E}; X \to V$ is called a tvs-cone norm on X if the following conditions are satisfied:*

- $(N1)$ $\|x\|_K \succcurlyeq \theta$ *for all* $x \in V$,
- $(N2)$ $\|x\|_K = \theta$ *if and only if* $x = \theta$,
- $(N3)$ $\|x + y\|_K \preccurlyeq \|x\|_K + \|y\|_K$ *for all* $x, y \in V$,
- $(N4)$ $\|kx\|_K = |k| \, \|x\|_K$ *for all* $k \in \mathbb{R}$.

The pair $(X, \|\cdot\|_K)$ is called a tvs-cone norm space (in brief tvs-CNS).

Definition 2. *Let $(V, \|\cdot\|_K)$ be a tvs-cone norm space and $\{x_n\}$ a sequence in V.*

(i) $\{x_n\}$ tvs-cone converges to $x \in V$ if for every $c \in \mathbb{E}$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x\|_K \ll c$ for all $n \geq n_0$.

(ii) $\{x_n\}$ is a tvs-cone Cauchy sequence if for every $c \in \mathbb{E}$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\|_K \ll c$ for all $n, m \geq n_0$.

(iii) $(V, \|\cdot\|_K)$ is a tvs-cone complete or a tvs-cone Banach space if every tvs-cone Cauchy sequence in V is tvs-cone convergent.

Using the consequences of Lemma 2.4 from [28], we have the following properties.

Lemma 3. *Let $(\mathbb{E}, K)$ be a locally convex tvs. The following properties hold.*

- (a) *For a sequence $\{v_n\}$ in $\mathbb{E}$ with $\theta \preccurlyeq v_n \to \theta$, let $\theta \ll c$ then there exists positive integer n_0 such that $v_n \ll c$ for each $n > n_0$.*
- (b) *There exists a sequence $\{v_n\}$ in $\mathbb{E}$ such that for some positive integer n_0 holds $\theta \preccurlyeq v_n \ll c$ for all $n > n_0$, but $v_n \not\to \theta$.*
- (c) *If there exists v in $\mathbb{E}$ such that $\theta \preccurlyeq v \ll c$ for all $c \in \operatorname{int} K$, then $v = \theta$.*
- (d) *If $a \preccurlyeq \lambda a$, where $a \in K$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Remark 4. *For a Banach space $\mathbb{E}$ with non-normal cone K, with norm $\|\cdot\|$. The following may hold.*

(a) For sequences $\{v_n\}$, $\{u_n\}$ in $\mathbb{E}$ with norm $\|\cdot\|$, it may happen that $v_n \to v$, $u_n \to u$, but $\|v_n - u_n\| \not\to \|v - u\|$ (see Example 5). In particular, $v_n \to v$, $n \to \infty$, may imply that $\|v_n - v\| \not\to \theta$, $n \to \infty$ (this is impossible in CNS defined in [16] if the cone is normal).

(b) If $v_n \to v$ and $v_n \to u$, then $v = u$.

Example 5. *Let $V = \mathbb{R}$ and let $\mathbb{E}$ be the set of all real-valued functions on V which also have continuous derivatives on V. Then $\mathbb{E}$ is a vector space over $\mathbb{R}$ under the following operations:*

$$(x + y)(t) = x(t) + y(t), \quad (\alpha x)(t) = \alpha x(t)$$

for all $x, y \in \mathbb{E}, \alpha \in \mathbb{R}$. Then $\mathbb{E}$ with norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty,$$

has non-normal solid cone, see [5, 8]*:*

$$K = \{x \in \mathbb{E} : \theta \preccurlyeq x\}, \text{ where } \theta(t) = 0 \text{ for all } t \in X.$$

Consider the sequences

$$x_n(t) = \frac{1 + \sin nt}{n+2}, \; y_n(t) = \frac{1 - \sin nt}{n+2}, \quad n \geq 0.$$

in $\mathbb{E}$. We have $x_n \to \theta$, $y_n \to \theta$, $n \to +\infty$, but

$$\begin{aligned} \|x_n - y_n\| &= \left\| \frac{2 \sin nt}{n+2} \right\| = \sup_{t \in V} \left\{ \frac{2 \sin nt}{n+2} \right\} + \sup_{t \in V} \left\{ \frac{2n \cos nt}{n+2} \right\} \\ &= \frac{2 \sin n}{n+2} + 1 \not\to \theta, \quad n \to +\infty. \end{aligned}$$

Also as $x_n \to \theta$, consider

$$\|x_n - \theta\| = \|x_n\| = 1 \not\to \theta.$$

Definition 6 ([1])**.** *Let X be a nonempty set and $(\mathbb{E}, K)$ a tvs. A vector-valued function $d : X \times X \to \mathbb{E}$ is said to be a tvs-cone metric if the following conditions are satisfied:*

- *$(C1)$ $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,*
- *$(C2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$,*
- *$(C3)$ $d(x, z) \preccurlyeq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

The pair (X, d) is called a tvs-cone metric space.

Note that each tvs-CNS is a tvs-cone metric space with induced tvs-cone metric $d : X \times X \to \mathbb{E}$ defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$.

Remark 7 ([1])**.** *The concept of cone metric spaces is more general than that of metric spaces, because each metric space is a cone metric space, and a cone metric space in the sense of Huang and Zhang is a special case of tvs-cone metric spaces when (X, d) is a cone metric space with respect to a normal cone K.*

If K is a normal cone, then a tvs-CNS $(V, \|\cdot\|_K)$ becomes a CNS [16] and with the induced tvs-cone metric [1] this space becomes cone metric space in the sense of [2].

CNS in the case of [16] gives us generalized induced norm known as b-norm $\|\cdot\|_b : V \to \mathbb{R}$ defined by $\|\cdot\|_b = \|\|\cdot\|_K\|$. The triangular property of cone norm

$$\|x + y\|_K \preccurlyeq \|x\|_K + \|y\|_K,$$

gives us the following property of b-norm,

$$\|x + y\|_b \leq k(\|x\|_b + \|y\|_b),$$

where k is a constant of normality.

Obviously every norm is a b-norm, but the contrary is not true, consider the following example

Example 8. *Let $X = \mathbb{R}$ and $\|\cdot\|_b : X \to \mathbb{R}$ defined by $\|x\|_b = |x|^3$. For $x, y \in X$ we have $|x+y|^3 \leq (|x|+|y|)^3 \leq 2^3(|x|^3+|y|^3)$, but $|x+y|^3 \nleq (|x|^3+|y|^3)$. Therefore, $\|x\|_b$ is a b-norm, but it is not a norm on X.*

Let us recall the following definitions.

Definition 9 ([5, 29])**.** *Let X be a nonempty set. A vector-valued function $d : X \times X \to \mathbb{E}$ is said to be cone symmetric if the following conditions are satisfied:*

$(C1)$ *$\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;*
$(C2)$ *$d(x, y) = d(y, x)$ for all $x, y \in X$.*

The pair (X, d) is called a cone symmetric space.

It is clear that the cone symmetric space may not be a cone metric space (see Example 2.2 in [29]). For a given cone symmetric space (X, d) one can deduce (see [29]) a symmetric metric space with $D : X \times X \to \mathbb{R}$ defined by $D(x, y) = \|d(x, y)\|$ for all $x, y \in X$.

For a cone metric space (X, d) with normal cone K with normal constant $k \geq 1$, we have

$$D(x, y) = \|d(x, y)\| \leq k \|d(x, z) + d(z, y)\| \leq k(D(x, z) + D(z, y)).$$

In this case, the metric D becomes b-metric and, hence, the concept of b-metric spaces is more general then that of metric spaces and the topology τ_D generated by D coincides with τ_b generated by b-metric on X.

In the following we explore the concept of tvs-cone seminorm.

Definition 10. *Let V be a vector space over scalars F. If a mapping $\rho_K : X \to (\mathbb{E}, K)$ satisfies:*

$(SN1)$ *$\rho_K(x) \succcurlyeq \theta$ for all $x \in V$,*
$(SN2)$ *$\rho_K(x + y) \preccurlyeq \rho_K(x) + \rho_K(y)$ for all $x, y \in V$,*
$(SN3)$ *$\rho_K(kx) = |k|\rho_K(x)$ for all $x \in V$, $k \in F$.*

Then ρ_K is called a tvs-cone seminorm on X.

Note that a tvs-cone seminorm is a norm if $\rho_K(x) = \theta$ implies $x = \theta$. A tvs-cone seminorm on X induces a pseudo tvs-cone metric defined by $d_p(x, y) = \rho_K(x - y)$ which satisfies:

$(PC1)$ $\theta \preccurlyeq d_p(x, y)$ for all $x, y \in X$,
$(PC2)$ $d_p(x, y) = d_p(y, x)$ for all $x, y \in X$,
$(PC3)$ $d_p(x, z) \preccurlyeq d_p(x, y) + d_p(y, z)$ for all $x, y, z \in X$.

Note that $d_p(x, y) = \theta$ does not imply $x \neq y$.
The class of tvs-cone pseudo metric spaces is larger than the class of tvs-cone metric spaces.

Equivalently, ρ_K is a tvs-cone seminorm on a vector space V if the following conditions are satisfied:

$(SN(i))$ $\rho_K(v + v) \preccurlyeq \rho_K(u) + \rho_K(v)$ for all $u, v \in V$,
$(SN(ii))$ $\rho_K(kv) = |k|\, \rho_K(v)$ for all $v \in V$, $k \in F$.

This cone seminorm gives us generalized seminorm, so called b-seminorm $\|\cdot\|_{bs} : X \to \mathbb{R}$ defined by

$$\|x\|_{bs} = \|\rho_K(x)\|.$$

Using $(SN(i))$, b-seminorm has the following property

$$\|x+y\|_{bs} \leq k(\|x\|_{bs} + \|y\|_{bs}).$$

Note that b-seminorm is a seminorm if $k=1$ and every seminorm is a b-seminorm. The next example shows that the contrary is not true, i.e., b-seminorm does not need to be seminorm.

Example 11. *Let $X=\mathbb{R}$ and $\|\cdot\|_{bs} : X \to \mathbb{R}$ is defined by $\|x\|_{bs} = |x|^3+1$. For $x, y \in X$, we have*

$$\begin{aligned} |x+y|^3+1 &\leq (|x|+|y|)^3+1 \leq 2^3(|x|^3+|y|^3)+1 \leq 2^3(|x|^3+|y|^3)+16 \\ &= 2^3(|x|^3+1+|y|^3+1), \end{aligned}$$

which implies $\|x+y\|_{bs} \leq 2^3(\|x\|_{bs}+\|y\|_{bs})$. This shows that $\|x\|_{bs}$ is a b-seminorm, but not a seminorm on X.

3. Main Results

Let $\{\rho_{Ki} : i \in I\}$ be a family of tvs-cone seminorms on a vector space V. For $\theta \ll \varepsilon$ and $i \in I = \{1, 2, 3, \ldots, n\}$, define

$$\mathcal{U}_{(u_0, \rho_{K_1}, \rho_{K_2}, \rho_{K_3}, \ldots, \rho_{K_n}, \varepsilon)} = \mathcal{U}_{(u_0, \rho_{Kn}, \varepsilon)} = \{u \in V : \rho_{Ki}(u-u_0) \ll \varepsilon,\ i \in I\}.$$

Note that $\mathcal{U}_{(u_0,\rho_{Kn},\varepsilon)} = u_0 + \mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$.

Lemma 12. *The set $\mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$ is balanced and convex in V.*

Proof. For any $w \in \mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$ and $|k| \leq 1$, we have $\rho_{Ki}(kw) \preccurlyeq |k|\rho_{K_i}(w) \ll \varepsilon$, $i \in I$. Thus $\mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$ is absorbing. Now, for $0 \leq t \leq 1$ and $u, v \in \mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$, we obtain

$$\rho_{Ki}(tu+(1-t)v) \preccurlyeq t\rho_{K_i}(u) + (1-t)\rho_{K_i}(v) \ll t\varepsilon + (1-t)\varepsilon = \varepsilon,$$

which implies $t\rho_{K_i}(u)+(1-t)\rho_{K_i}(v) \in \mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$. Therefore, $\mathcal{U}_{(\theta,\rho_{Kn},\varepsilon)}$ is convex. ∎

Lemma 13. *Let $\{\rho_{Ki} : i \in I\}$ be a family of tvs-cone seminorms on a vector space (V, F). For each $v \in V$ denote with $\mathcal{N}_v$ the collection of sets of the form*

$$\mathcal{U}_{(v,\rho_{Kn},\varepsilon)} = \{u \in V : \rho_{Ki}(u-v) \ll \varepsilon, i \in I\}.$$

Let $\mathcal{T}$ be the collection of $\emptyset$ and all subsets G of X such that for each $u \in G$ there exists some $U \in \mathcal{N}_v$ such that $U \subseteq G$. Then $\mathcal{T}$ is topology on V and preserves the structure of vector space. The sets $\mathcal{N}_v$ form an open locally convex neighborhood base at x. The topological space $(V, \mathcal{T})$ is Hausdorff iff the family $\{\rho_{Ki} : i \in I\}$ of tvs-cone seminorms is separating, i.e. for $\theta \neq u \in V$ there exists some $i_0 \in I$ such that $\rho_{Ki_0}(u) \neq \theta$.

Proof. It is clear that V and the union of any number of elements of $\mathcal{T}$ belong to $\mathcal{T}$. We will show that $A, B \in \mathcal{T}$ implies $A \cap B \in \mathcal{T}$. The case $A \cap B = \emptyset$ is obvious. Suppose that $A \cap B \neq \emptyset$ and $v \in A \cap B$. By definition of $\mathcal{T}$ there exist $U_1, U_2 \in \mathcal{T}$ such that $U_1 \subseteq A$ and $U_2 \subseteq B$. Let for comparable $\varepsilon, \delta \in \operatorname{int} K$, we define

$$U_1 := \mathcal{U}_{(v,\rho_{Kn},\varepsilon)} = \{u \in V : \rho_{Ki}(u-v) \ll \varepsilon,\ 1 \leq i \leq n\}$$

and

$$U_2 = \mathcal{U}_{(v,\mu_{Km},\delta)} = \left\{u \in V : \mu_{Kj}(u-v) \ll \delta,\ 1 \leq j \leq m\right\}.$$

If we set

$$U_3 = \mathcal{U}_{(v,\rho_{K_1},\rho_{K_2},\rho_{K_3},\ldots,\rho_{K_n},\mu_{K_1},\mu_{K_2},\mu_{K_3},\ldots,\mu_{K_m},\gamma)},$$

where $\gamma = \varepsilon$ if $\delta - \varepsilon \in \operatorname{int} K$ and $\gamma = \delta$ if $\varepsilon - \delta \in \operatorname{int} K$, then $U_3 \in \mathcal{N}_v$ and $U_3 \subseteq U_1 \cap U_2 \subseteq A \cap B$. Hence $\mathcal{T}$ is topology on V. Let $\mathcal{U}_{(v,\rho_{Kn},\varepsilon)} \in \mathcal{N}_v$ and $w \in \mathcal{U}_{(v,\rho_{Kn},\varepsilon)}$. Then $\rho_{Ki}(w-v) \ll \varepsilon$, $1 \leq i \leq n$. Now choose $\theta \ll \delta$ such that $\delta \ll \varepsilon - \rho_{Ki}(w-v)$ for $1 \leq i \leq n$. For any $1 \leq i \leq n$ and $u \in V$ satisfying $\rho_{Ki}(w-u) \ll \delta$ we have

$$\rho_{Ki}(u-v) \preccurlyeq \rho_{Ki}(u-w) + \rho_{Ki}(w-v) \ll \delta + \rho_{Ki}(w-v) \ll \varepsilon.$$

We see that $\mathcal{U}_{(u,\rho_{Kn},\delta)} \subseteq \mathcal{U}_{(v,\rho_{Kn},\varepsilon)}$, hence $\mathcal{U}_{(v,\rho_{Kn},\varepsilon)}$ is open. Lemma 12 implies that the elements of $\mathcal{N}_v$ are convex. Therefore, $\mathcal{N}_v$ is an open locally convex neighborhood base at v consisting of the open sets $\mathcal{U}_{(v,\rho_{Kn},\varepsilon)}$.

Now we will show that the topology $\mathcal{T}$ is compatible. Let $u, v \in V$ and let $\mathcal{U}_{(u+v,\rho_{Kn},\varepsilon)}$ be a basic neighborhood of $u+v$. Let $(u_n, v_n) \to (u, v)$ in $V \times V$. Then there exists an integer n_0 such that $(u_n, v_n) \in \mathcal{U}_{(u,\rho_{Kn},\frac{\varepsilon}{2})} \times \mathcal{U}_{(v,\rho_{Kn},\frac{\varepsilon}{2})}$ for all $n \geq n_0$. For $1 \leq i \leq n$ and for all $n \geq n_0$, we have

$$\rho_{Ki}(u+v-(u_n+v_n)) \preccurlyeq \rho_{Ki}(u-u_n) + \rho_{Ki}(v-v_n) \ll \varepsilon,$$

which gives $u_n + v_n \in \mathcal{U}_{(u+v,\rho_{Kn},\varepsilon)}$, and, therefore, $u_n + v_n \to u + v$. Now let $(k_n, v_n) \to (k, v)$ in $F \times V$. Let $\mathcal{U}_{(kv,\rho_{Kn},\delta)}$ be a basic neighborhood of kv. Choose $t > 0$ and $\theta \ll \gamma$ such that for $1 \leq i \leq n$ there exists an integer m_0 such that $(k_n, v_n) \in \{\zeta \in F : |\zeta - k| < t\} \times \mathcal{U}_{(v,\rho_{Kn},\gamma)}$ for all $n \geq m_0$, with $t\rho_{Ki}(v) \ll \frac{\delta}{2}$ and $(|k| + t)\gamma \ll \frac{\delta}{2}$. For $n \geq m_0$ we have

$$\begin{aligned}
\rho_{Ki}(kv - k_n v_n) &\preccurlyeq \rho_{Ki}(kv - k_n v) + \rho_{Ki}(k_n v - k_n v_n) \\
&\preccurlyeq |k - k_n|\, \rho_{Ki}(v) + |k_n|\, \rho_{Ki}(v - v_n) \\
&\preccurlyeq |t|\, \rho_{Ki}(v) + |k_n|\, \rho_{Ki}(v - v_n) \\
&\ll |t|\, \rho_{Ki}(v) + (|k| + t)\gamma \\
&\ll \frac{\delta}{2} + \frac{\delta}{2} \\
&= \delta,
\end{aligned}$$

thus $k_n v_n \in \mathcal{U}_{(kv,\rho_{Kn},\delta)}$. Therefore $(V, \mathcal{T})$ is a tvs.

Now, suppose that the family $\mathcal{P} = \{\rho_{Ki} : i \in I\}$ of tvs-cone seminorms is separating. For any $u, v \in V$ with $u \neq v$ there exists some $j_0 \in I$ such that $\theta \ll \delta = \rho_{Kj_0}(v-u)$. Thus, the open sets $\mathcal{U}_{(u,\rho_{Kj_0},\frac{\delta}{2})}$ and $\mathcal{U}_{(v,\rho_{Kj_0},\frac{\delta}{2})}$ are disjoint containing u and v and so the space $(V, \mathcal{T})$ is Hausdorff.

We conclude that the space $(V, \mathcal{T})$ is locally convex tvs. ∎

Definition 14. [20] *Let X be a tvs-cone normed space and $T : X \to X$ an operator.*

(*i*) *T is an almost weak contraction if for all $x, y \in \mathbb{E}$, $L \geq 0$ and $\delta \in (0,1)$, we have*

$$(w1) \qquad \|Tu - Tv\|_K \preccurlyeq \delta \cdot \|u - v\|_K + L \cdot \|u - Tu\|_K \ , \ \forall u, v \in X.$$

(*ii*) *T is a weak contraction if*

$$(w2) \qquad \|Tu - Tv\|_K \preccurlyeq \delta \cdot \|u - v\|_K + L \cdot \|v - Tu\|_K \, .$$

Definition 15. *Let X be a tvs-cone normed space and $T : X \to X$ an operator.*

(*i*) *T is a Zamfirescue contraction if for all $u, v \in X$ and $a \in [0,1)$, $b, c \in [0, \frac{1}{2})$, one of the following conditions is satisfied*

$(z1)$
$$\|Tu - Tv\|_K \preccurlyeq a \cdot \|u - v\|_K \,,$$
$(z2)$
$$\|Tu - Tv\|_K \preccurlyeq b\left(\|u - Tu\|_K + \|v - Tv\|_K\right),$$
$(z3)$
$$\|Tu - Tv\|_K \preccurlyeq c\left(\|u - Tv\|_K + \|v - Tu\|_K\right).$$

(ii) *T is a Quasi contraction if for all $u, v \in X$ and $\alpha \in [0,1)$, holds*
$$\|Tu - Tv\|_K \preccurlyeq c \cdot m,$$
where
$$m \in \left\{\|u - v\|_K \,, \|u - Tu\|_K \,, \|v - Tv\|_K \,, \|u - Tv\|_K \,, \|v - Tu\|_K\right\}.$$

Remark 16. [20](a) *Every Zamfirescue contraction is a weak contraction.*
(b) *Every Quasi contraction is a weak contraction.*

Definition 17. [19] *Let X be a tvs-cone normed space, $T : X \to X$ an operator and $u_0 \in X$. A sequence $\{u_n\}$ is called:*

1) *Picard iteration if*
$$u_{n+1} = Tu_n; \tag{p1}$$
2) *Mann iteration if*
$$u_{n+1} = (1 - \alpha_n)\, u_n + \alpha_n T u_n; \tag{m1}$$
3) *Ishikawa iteration if*
$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)\, u_n + \alpha_n T v_n, \\ v_n &= (1 - \beta_n)\, x_n + \beta_n T x_n, \end{aligned} \tag{i1}$$
where $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq [0,1)$.
4) *Krasnoselskij iteration if*
$$u_{n+1} = (1 - \lambda)\, u_n + \lambda T u_n,$$
where $\lambda \in (0,1)$.

Denote with $F(T)$ the set of all fixed points of T.

Lemma 18. *Let X be a tvs-cone Banach space and $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{E}$ satisfying $a_{n+1} \preccurlyeq \lambda a_n + b_n$, where $\lambda \in (0,1)$ and $b_n \to \theta$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = \theta$.*

Proof. On the contrary, suppose that $\lim_{n\to\infty} a_n \neq \theta$ and $\lim_{n\to\infty} a_n = c$, for some $\theta \ll c$. Then, by lemma 3 (d), we have $a_n = \theta$. ∎

In the following theorem we obtain a fixed point result for nonself weak contractions in a tvs-cone Banach space.

Theorem 19. *Let X be a tvs cone Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is a weak contraction (satisfying $(w2)$), such that $\delta(1 + L) < 1$. If $T(\partial C) \subseteq C$, then T has a fixed point.*

Proof. We construct two sequences $\{u_n\}$ and $\{v_n\}$ in the following way. Let us choose u_0 arbitrary in X and set $v_1 = Tu_0$. If $v_1 \in C$, then set $u_1 = v_1$. If not, then there exists $u_1 \in \partial C$ such that

$$\|u_1 - u_0\|_K + \|u_1 - v_1\|_K = \|u_0 - v_1\|_K .$$

Thus $u_1 \in C$ and let $v_2 = Tu_1$. We have

$$\|v_2 - v_1\|_K = \|Tu_0 - Tu_1\|_K \preccurlyeq \delta \cdot \|u_1 - u_0\|_K + L \cdot \|u_1 - Tu_0\| .$$

If $v_2 \in C$, set $u_2 = v_2$. Otherwise, there exists $u_2 \in \partial C$ such that

$$\|u_2 - u_1\|_K + \|v_2 - u_2\|_K = \|v_2 - u_1\|_K .$$

Thus $u_2 \in C$. Let $v_3 = Tu_2$ and consider

$$\|v_2 - v_3\|_K = \|Tu_1 - Tu_2\|_K \preccurlyeq \delta \cdot \|u_2 - u_1\|_K + L \cdot \|u_2 - Tu_1\|_K .$$

Continuing in the same way, we construct the sequences $\{u_n\}$ and $\{v_n\}$ such that
(i) $v_{n+1} = Tu_n$,
(ii) $\|v_n - v_{n+1}\|_K \preccurlyeq \delta \cdot \|u_{n-1} - u_n\|_K + L \cdot \|u_n - Tu_{n-1}\|_K$,
where
(iii) $v_n \in C$ implies $v_n = u_n$.
(iv) If $v_n \notin C$, then $v_n \neq u_n$, and then $u_n \in \partial C$ is such that

$$\|u_{n-1} - u_n\|_K + \|v_n - u_n\|_K = \|v_n - u_{n-1}\|_K .$$

We will show that $\{u_n\}$ is a Cauchy sequence. Define

$$\begin{aligned} P &= \{u_i \in \{u_n\} : u_i = v_i\} , \\ Q &= \{u_i \in \{u_n\} : u_i \neq v_i\} . \end{aligned}$$

It is obvious that if $u_n \in Q$, then u_{n-1} and u_{n+1} are in P. We have the following three possibilities.
Case 1. If $u_n,\ u_{n+1} \in P$, then

$$\begin{aligned} \|u_n - u_{n+1}\|_K &= \|v_n - v_{n+1}\|_K \preccurlyeq \delta \cdot \|u_{n-1} - u_n\|_K + L \cdot \|u_n - Tu_{n-1}\|_K \\ &\preccurlyeq \delta \cdot \|u_{n-1} - u_n\|_K . \end{aligned}$$

Case 2. If $u_n \in P$, $u_{n+1} \in Q$, then

$$\begin{aligned} \|u_n - u_{n+1}\|_K &\preccurlyeq \|u_n - u_{n+1}\|_K + \|u_{n+1} - v_{n+1}\|_K \\ &= \|u_n - v_{n+1}\|_K \\ &= \|v_n - v_{n+1}\|_K \\ &\preccurlyeq \delta \cdot \|u_{n-1} - u_n\|_K + L \cdot \|u_n - Tu_{n-1}\|_K \\ &\preccurlyeq \delta \cdot \|u_{n-1} - u_n\|_K . \end{aligned}$$

Case 3. If $u_n \in Q$, $u_{n+1} \in P$, then

$$\begin{aligned}
\|u_n - u_{n+1}\|_K &\preccurlyeq \|v_n - u_n\|_K + \|v_n - v_{n+1}\|_K \\
&\preccurlyeq \|v_n - u_n\|_K + \delta \cdot \|u_{n-1} - u_n\|_K + L \cdot \|u_n - Tu_{n-1}\|_K \\
&\preccurlyeq \|v_n - u_n\|_K + \|u_{n-1} - u_n\|_K + L \cdot \|v_n - u_n\|_K \\
&= \|v_n - u_{n-1}\|_K + L \cdot \|v_n - u_n\|_K \\
&= \|v_n - u_{n-1}\|_K + L \cdot \|v_n - u_{n-1}\|_K - L \cdot \|u_{n-1} - u_n\|_K \\
&\preccurlyeq (1+L)\|v_{n-1} - v_n\|_K \\
&\preccurlyeq (1+L)\delta \cdot \|u_{n-2} - u_{n-1}\|_K + (1+L)L \cdot \|u_{n-1} - Tu_{n-2}\|_K \\
&\preccurlyeq (1+L)\delta \cdot \|u_{n-2} - u_{n-1}\|_K \\
&= h\|u_{n-2} - u_{n-1}\|_K,
\end{aligned}$$

where $h = (1+L)\delta < 1$.
Taking $\alpha = \max\{\delta, h\}$, and combining all above three cases we have

$$\|u_n - u_{n+1}\|_K \preccurlyeq \begin{cases} \alpha\|u_{n-1} - u_n\|_K \\ \alpha\|u_{n-2} - u_{n-1}\|_K \end{cases}.$$

By mathematical induction, for all $n > 0$, we have

$$\|u_n - u_{n+1}\|_K \preccurlyeq h^{(n-1)/2} w$$

for $w \in \{\|u_1 - u_0\|_K, \|u_2 - u_1\|_K\}$.
Now for $n > m$, we consider

$$\begin{aligned}
\|u_m - u_n\|_K &\preccurlyeq \|u_n - u_{n-1}\|_K + \|u_{n-1} - u_{n-2}\| + \cdots + \|u_{m-1} - u_m\| \\
&\preccurlyeq (h^{(n-1)/2} + h^{(n-2)/2} + \cdots + h^{(m-1)/2})w \\
&\preccurlyeq \frac{h^{(m-1)/2}}{1 - h^{(n-m)/2}} w.
\end{aligned}$$

As $h < 1$, we have $h^{(m-1)/2} \to 0$ when $n, m \to \infty$, and this gives us $\frac{h^{(m-1)/2}}{1-h^{(n-m)/2}} w \to \theta$, $n \to \infty$, in the locally convex space $\mathbb{E}$. Now, according to Lemma 3-(a), we conclude that for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number k_1 such that $\|u_m - u_n\|_K \ll c$ for all $m, n \geq k_1$, so $\{u_n\}$ is a tvs-cone Cauchy sequence in C. As C is closed, thus there exists some $u \in C$, such that $u_n \to u$ as $n \to \infty$.
By construction of $\{u_n\}$ there exists a subsequence $\{u_{n_q}\}$ such that

$$v_{n_q} = u_{n_q} = Tu_{n_{q-1}}$$

and $u_{n_q} \to u$ as $q \to \infty$. So, for a given $c \in \mathbb{E}$ with $\theta \ll c$, let us choose a natural number k_2 such that $\|u - u_{n_q}\|_K \ll \frac{c}{1+L}$ and $\|u_{n_{q-1}} - u\|_K \ll \frac{c}{\delta}$ for all $q - 1 \geq k_2$. Now, we have

$$\begin{aligned}
\|u - Tu\|_K &\preccurlyeq \|u - u_{n_q}\|_K + \|u_{n_q} - Tu\|_K \\
&\preccurlyeq \|u - u_{n_q}\|_K + \|Tu_{n_{q-1}} - Tu\|_K \\
&\preccurlyeq \|u - u_{n_q}\|_K + \delta\|u_{n_{q-1}} - u\|_K + L\|u - Tu_{n_{q-1}}\|_K \\
&\preccurlyeq (1+L)\|u - u_{n_q}\|_K + \delta\|u_{n_{q-1}} - u\|_K,
\end{aligned}$$

i.e. $\|u - Tu\|_K \ll c(k_2)$ for all $q - 1 \geq k_2$.

This completes the proof. ∎

Theorem 20. *Let X be a tvs cone Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is a weak contraction (satisfying $(w1)$), such that $\delta(1+L) < 1$. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

Corollary 21. *Let X be a cone Banach space with normal cone K and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is a weak contraction/almost weak contraction (satisfying $(w1)/(w2)$), such that $\delta(1+L) < 1$. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

The following corollaries are due to remark 16.

Corollary 22. *Let X be a tvs cone Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is Zamfirecue operator. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

Corollary 23. *Let X be a tvs cone Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is quasi operator. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

Corollary 24. *Let X be a Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is Zamfirecue operator. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

Corollary 25. *Let X be a Banach space and C be a nonempty closed and convex subset of X. Suppose that $T : C \to X$ is a quasi operator. If T satisfies the condition: $u \in \partial C \Rightarrow Tu \in C$, then T has a fixed point.*

Theorem 26. *Let $\mathbb{E}$ be a tvs-normed space, C be a closed and convex subset of $\mathbb{E}$. Let $T : C \to C$ be an almost weak contractive mapping (satisfying $(w1)$) with $F(T) \neq \varphi$. Let $\{u_n\}$ be Ishikawa iteration satisfying*

$$(\alpha) \qquad \sum_{j=0}^{\infty} \alpha_j = \infty,$$

$u_0 \in C$ is arbitrary chosen. Then $\{u_n\}$ converges strongly to a unique fixed point of T.

Proof. It can be shown that $(w1)$ gives us a unique fixed point. Let $p \in F(T)$ be a unique fixed point of T and $\{u_n\}$ be Ishikawa iteration defined in $(i1)$ and $u_0 \in C$. We have

$$\begin{aligned}
\|u_{n+1} - p\|_K &= \|(1-\alpha_n)u_n + \alpha_n T v_n - p\|_K \\
&= \|(1-\alpha_n)(u_n - p) + \alpha_n (T v_n - p)\|_K \\
&\preccurlyeq (1-\alpha_n)\|u_n - p\|_K + \alpha_n \|T v_n - p\|_K \\
&\preccurlyeq (1-\alpha_n)\|u_n - p\|_K + \alpha_n \delta \|v_n - p\|_K, \quad \text{by } (w1),
\end{aligned}$$

and

$$\begin{aligned}
\|v_n - p\|_K &= \|(1-\beta_n)u_n + \beta_n T u_n - p\|_K \\
&= \|(1-\beta_n)(u_n - p) + \beta_n (T u_n - p)\|_K \\
&\preccurlyeq (1-\beta_n)\|u_n - p\|_K + \beta_n \delta \|u_n - p\|_K, \quad \text{by } (w1).
\end{aligned}$$

So, we obtain

$$\begin{aligned}\|u_{n+1}-p\|_K &\preccurlyeq \left(1-(1-\delta)^2\alpha_n\right)\|u_n-p\|_K \\ &\preccurlyeq e^{-(1-\delta)^2\alpha_n}\|u_n-p\|_K \\ &\preccurlyeq \left(e^{-(1-\delta)^2\sum_{j=0}^{n}\alpha_j}\right)\cdot\|u_0-p\|_K\,.\end{aligned}$$

Using (α), this implies $\left(e^{-(1-\delta)^2\sum_{j=0}^{n}\alpha_j}\right)\to 0$, $n\to\infty$, which gives us $\left(e^{-(1-\delta)^2\sum_{j=0}^{n}\alpha_j}\right)\cdot\|u_0-p\|_K\to\theta$, $n\to\infty$, in the locally convex space $\mathbb{E}$. This completes the proof of theorem. ∎

The following corollaries are due to remark 16.

Corollary 27. *Let $\mathbb{E}$ be a tvs-normed space, C be a closed and convex subset of $\mathbb{E}$. Let $T:C\to C$ be Zamfirescue operator, with $F(T)\neq\varphi$. Let $\{u_n\}$ be Ishikawa iteration satisfying*

$$\sum_{j=0}^{\infty}\alpha_j=\infty$$

where $u_0\in C$ is arbitrary chosen. Then $\{u_n\}$ converges strongly to a unique fixed point of T.

Corollary 28. *Let $\mathbb{E}$ be a tvs-normed space, C be a closed and convex subset of $\mathbb{E}$. Let $T:C\to C$ be a quasi operator, with $F(T)\neq\varphi$. Let $\{u_n\}$ be Ishikawa iteration satisfying*

$$\sum_{j=0}^{\infty}\alpha_j=\infty$$

where $u_0\in C$ is arbitrary chosen. Then $\{u_n\}$ converges strongly to a unique fixed point of T.

Corollary 29. [24] *Let $\mathbb{E}$ be a normed space, C be a closed and convex subset of $\mathbb{E}$. Let $T:C\to C$ be a Zamfirescue operator, with $F(T)\neq\varphi$. Let $\{u_n\}$ be Ishikawa iteration satisfying*

$$\sum_{j=0}^{\infty}\alpha_j=\infty$$

and $u_0\in C$ is arbitrary chosen. Then $\{u_n\}$ converges strongly to a unique fixed point of T.

Corollary 30. [24] *Let $\mathbb{E}$ be a normed space, C be a closed and convex subset of $\mathbb{E}$. Let $T:C\to C$ be a quasi operator, with $F(T)\neq\varphi$. Let $\{u_n\}$ be Ishikawa iteration satisfying*

$$\sum_{j=0}^{\infty}\alpha_j=\infty$$

and $u_0\in C$ is arbitrary chosen. Then $\{u_n\}$ converges strongly to a unique fixed point of T.

The following theorem is a result for fixed point of non-expansive mappings in tvs-cone Banach space for Krasnoselskij iteration with $\lambda=\frac{1}{2}$.

Theorem 31. *Let C be a closed and convex subset of a tvs-cone Banach space $(X, \|\cdot\|_K)$. Suppose that the mapping $F : C \to C$ satisfies*

(a) $$\|v - Fv\|_K + \|u - Fu\|_K \preccurlyeq \eta \|v - u\|_K$$

for all $u, v \in C$. Then F has at least one fixed point if $2 \leq \eta \leq 4$.

Proof. Let us choose $v_0 \in C$ arbitrary and define sequence $\{v_n\}$ as follows:

$$v_{n+1} = \frac{v_n + Fv_n}{2}, \ n = 0, 1, 2, 3, \ldots$$

Since

$$v_n - Fv_n = 2\left(v_n - \frac{v_n + Fv_n}{2}\right) = 2(v_n - v_{n+1}),$$

we have

(b) $$\|v_n - Fv_n\|_K = 2\|v_n - v_{n+1}\|_K, \ n = 0, 1, 2, 3, \ldots.$$

Combining (a) and (b), we have

$$2\|v_{n-1} - v_n\|_K + 2\|v_n - v_{n+1}\|_K \preccurlyeq \eta \|v_{n-1} - v_n\|_K,$$

which gives

$$\|v_n - v_{n+1}\|_K \preccurlyeq \lambda \|v_{n-1} - v_n\|_K, \ n = 0, 1, 2, 3\ldots,$$

for $\lambda = \frac{\eta-2}{2} < 1$.

According to the previous inequality, for $m \geq n$, we obtain

$$\|v_n - v_m\|_K \preccurlyeq \frac{\lambda^n}{1-\lambda} \|v_0 - v_1\|_K.$$

Since $\lambda^n \to 0$ as $n \to \infty$, then $\frac{\lambda^n}{1-\lambda} \|v_0 - v_1\|_K \to \theta$, $n \to \infty$, in the locally convex space $\mathbb{E}$. Now, according to Lemma 3 part (a), we conclude that for every $c \in \mathbb{E}$ with $\theta \ll c$ there exists a natural number n_1 such that $\|v_n - v_m\|_K \ll c$ for all $m, n \geq n_1$. Therefore, $\{v_n\}$ is a tvs-cone Cauchy sequence in C. Since C is closed, there exists some $w \in C$, such that $v_n \to w$ as $n \to \infty$. Now, choose a positive integer m_1 such that for every $c \in \mathbb{E}$ with $\theta \ll c$ we have $\|w - v_n\|_K \ll \frac{1}{\eta} c$ for all $n \geq m_1$.

Substituting $v = w$ and $u = v_n$ in (a), for all $n \geq m_1$, we obtain

$$\|w - Fw\|_K + 2\|v_n - v_{n+1}\|_K \preccurlyeq \eta \|w - v_n\|_K,$$

$$\|w - Fw\|_K \preccurlyeq \eta \|w - v_n\|_K - 2\|v_n - v_{n+1}\|_K \ll c.$$

Thus, $w = Fw$ is a fixed point of F. ∎

Corollary 32. [16] *Let C be a closed and convex subset of a cone Banach space $(X, \|\cdot\|_K)$. Suppose that the mapping $F : C \to C$ satisfies*

$$\|v - Fv\|_K + \|u - Fu\|_K \preccurlyeq \eta \|v - u\|_K$$

for all $u, v \in C$. Then F has at least one fixed point if $2 \leq \eta \leq 4$.

The next theorem is an application of above theorem in topological homotopy theory.

Theorem 33. *Let $(X, \|\cdot\|_K)$ be a tvs-cone Banach space, C a closed and convex subset of X and U an open subset of C. Let $\mathcal{K} : [0,1] \times \bar{U} \to C$ be a homotopy mapping with the following conditions:*

(a) $\xi \neq \mathcal{K}(t, \xi)$, for each $\xi \in \partial U$ and each $t \in [0, 1]$,

(*b*) *$\mathcal{K}(t,\cdot):\bar{U}\to C$ is a mapping satisfying the conditions of Theorem 31,*

(*c*) *there exists a continuous increasing function $g:(0,1]\to P$ such that*

$$\left\|\mathcal{K}(s,\xi)-\mathcal{K}(t,\acute{\xi})\right\|_K \preccurlyeq g(s)-g(t),$$

$$g(s)\in g(t)+P,$$

for all $s,t\in[0,1]$, and each $\xi\in\bar{U}$.

Then $\mathcal{K}(0,\cdot)$ has a fixed point if and only if $\mathcal{K}(1,\cdot)$ has a fixed point.

Proof. We first suppose that $\mathcal{K}(0,\cdot)$ has a fixed point z, i.e. $z=\mathcal{K}(0,z)$. From (a), we obtain $z\in U$. Define

$$\Gamma:=\{(t,\xi)\in[0,1]\times C:\xi=\mathcal{K}(\xi,t)\}.$$

Clearly $\Gamma\neq\phi$. We define the partial ordering in Γ as follows:

$$(t,\xi)\precsim(s,\acute{\xi})\Leftrightarrow t\leq s \text{ and } \left\|\acute{\xi}-\xi\right\|_K\preccurlyeq\frac{2}{\eta-2}(g(s)-g(t)).$$

Let $\mathcal{B}$ be a totally ordered subset of Γ and $\mathring{t}=\sup\{t:(t,\xi)\in\mathcal{B}\}$. Consider a sequence $\{(t_n,\xi_n)\}_{n\geq0}$ in $\mathcal{B}$ such that, $(t_n,\xi_n)\precsim(t_{n+1},\xi_{n+1})$ and $t_n\to\mathring{t}$ as $n\to\infty$. For $m>n$, we have

$$\|\xi_m-\xi_n\|_K\preccurlyeq\frac{2}{\eta-2}(g(t_m)-g(t_n))\to\theta,\ \text{as } n,m\to\infty,$$

and conclude that $\{\xi_n\}$ is a tvs-cone Cauchy sequence. There exists $\mathring{\xi}\in C$ such that $\xi_n\to\mathring{\xi}$. Choose $n_0\in N$ such that for $\theta\ll c$ we have $\|\mathring{\xi}-\xi_n\|_K\ll\frac{c}{\eta}$ for all $n\geq n_0$. The mapping $\mathcal{K}(t,\cdot)$ satisfies all the conditions of Theorem 31 and substituting $v=\mathring{\xi}$ and $u=\xi_n$ into (1), for all $n\geq n_0$, we obtain

$$\left\|\mathring{\xi}-\mathcal{K}(\mathring{t},\mathring{\xi})\right\|_K+2\left\|\xi_n-\xi_{n+1}\right\|_K\preccurlyeq\eta\left\|\mathring{\xi}-\xi_n\right\|_K,$$

$$\left\|\mathring{\xi}-\mathcal{K}(\mathring{t},\mathring{\xi})\right\|_K\preccurlyeq\eta\left\|\mathring{\xi}-\xi_n\right\|_K-2\left\|\xi_n-\xi_{n+1}\right\|_K\ll c.$$

We see that $\mathring{\xi}=\mathcal{K}(\mathring{t},\mathring{\xi})$ and, hence, $\mathring{\xi}\in U$, which implies $(\mathring{t},\mathring{\xi})\in\Gamma$. Thus, $(t,\xi)\precsim(\mathring{t},\mathring{\xi})$ for all $(t,\xi)\in\mathcal{B}$ gives us that $(\mathring{t},\mathring{\xi})$ is an upper bound of $\mathcal{B}$. By Zorn's lemma, Γ has maximal element $(\mathring{t},\mathring{\xi})$.

We claim that $\mathring{t}=1$. On the contrary, suppose that $\mathring{t}\leq1$. Let us choose $\theta\ll r$ arbitrary and, for any $t\geq\mathring{t}$, consider

$$B_r(\mathring{\xi})=\left\{\xi:\|\xi-\mathring{\xi}\|_K\preccurlyeq r\right\}\subset U,$$

where $r=\frac{2}{\eta-2}(g(t)-g(\mathring{t}))$.

Using the condition (c), we have

$$\|\mathcal{K}(t,\xi)-\mathcal{K}(\mathring{t},\mathring{\xi})\|_K\preccurlyeq g(t)-g(\mathring{t})=\frac{\eta-2}{2}r\ll r.$$

Hence, for each $t\in[0,1]$, there exists some $\xi\in B_r(\mathring{\xi})\subset U$ such that $\xi=\mathcal{K}(t,\xi)$. Since

$$\|\xi-\mathring{\xi}\|_K\preccurlyeq r=\frac{2}{\eta-2}(g(t)-g(\mathring{t}))$$

implies $(\mathring{t},\mathring{\xi})\precsim(t,\xi)$, we obtain a contradiction. Therefore, $\mathring{t}=1$.

From the above it follows that $\mathcal{K}(1,\cdot)$ has a fixed point $\mathring{\xi}=\mathcal{K}(1,\mathring{\xi})$.

Conversely, if $\mathcal{K}(1,\cdot)$ has a fixed point, then, in the same way, we can prove that $\mathcal{K}(0,\cdot)$ has a fixed point. ∎

Let X be a tvs cone normed space and T be a self operator of X. Let u_0 be any fixed point and $x_{n+1} = \xi(T, x_n)$ is an iteration process involving T, which computes the sequence $\{x_n\}$ in X.

Definition 34. *(see also [30]) The iteration procedure $x_{n+1} = \xi(T, x_n)$ is said to be T-stable with respect to T if $\{x_n\}$ converges to a unique fixed point q of T and whenever $\{y_n\}$ is a sequence in X with*

$$\lim_{n\to\infty} \|y_{n+1} - \xi(T, x_n)\|_K = \theta$$

we have $\lim_{n\to\infty} y_n = q$.

Theorem 35. *Let X be a tvs-cone normed space and T be a weak contraction (satisfying (w1)) with $v = q \in F(T) \neq \varphi$, in addition, whenever $\{y_n\}$ is a sequence with $\lim_{n\to\infty} \|y_{n+1} - Ty_n\|_K = \theta$, then the Picard iteration defined in (p1) is T-stable.*

Proof. We will show that the sequence $\{y_n\}$ with $\lim_{n\to\infty} \|y_{n+1} - \xi(T, x_n)\|_K = \theta$, satisfies $\lim_{n\to\infty} y_n = q$.
We have

$$\begin{aligned} \|y_{n+1} - q\|_K &\preccurlyeq \|y_{n+1} - Ty_n\|_K + \|Ty_n - q\|_K \\ &\preccurlyeq \|y_{n+1} - Ty_n\|_K + \delta \|y_n - q\|_K + L \|y_n - Ty_n\|_K \\ &= \delta \|y_n - q\|_K + (\|y_{n+1} - Ty_n\|_K + L \|y_n - Ty_n\|_K) \\ &= \delta a_n + b_n, \end{aligned}$$

where $a_n = \|y_n - q\|_K$ and $b_n = (\|y_{n+1} - Ty_n\|_K + L \|y_n - Ty_n\|_K)$.
Using Lemma 18, we have $a_n \to \theta$ as $n \to \infty$. Thus, $\lim_{n\to\infty} y_n = q$. ∎

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