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A New Alternative to the Log-Kumaraswamy Distribution: Properties, Estimation, and Fitting Data

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Abstract. Statistical distributions play a crucial role in modelling real-life data in various fields. Recently, various statistical distributions have been proposed and used in real-life data analysis. This paper introduces a novel statistical distribution as an alternative to the log-Kumaraswamy distribution. It is called the power log-Kumaraswamy distribution. We explore several distributional properties of the suggested distribution. We consider nine estimation techniques, namely, maximum likelihood, Cramér-von Mises, maximum product of spacing, least squares, weighted least squares, Anderson–Darling, right-tailed Anderson–Darling, minimum spacing absolute distance, and minimum spacing absolute-log distance methods to estimate the parameters of the introduced distribution. The performances of these estimators are evaluated via an extensive Monte Carlo simulation study. Furthermore, the applicability and superiority of the power log-Kumaraswamy distribution are demonstrated through two practical data examples from engineering and health economics. The goodness-of-fit analysis's results support the proposed distribution's superiority over its main competitors.

1. Introduction

Mudholkar [1] developed the exponentiated family of distributions, one of the first and most basic methods for creating new probability distributions. By adding an extra shape parameter to a

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baseline distribution, this method improves it and makes it more adaptable to modelling a greater range of data patterns. The cumulative distribution function (CDF) of the exponentiated family is given by

$$F(x;\gamma,\eta) = [G(x;\eta)]^{\alpha}, \quad \alpha,\eta > 0, \quad x \in \mathbb{R}. \tag{1.1}$$

Here, $\alpha > 0$ serves as an additional shape parameter, and $G(x;\eta)$ represents the CDF of the baseline random variable with parameter vector η . Several transformation techniques have been introduced in the literature to enhance the flexibility of standard distributions. For instance, the Marshall-Olkin transformation was proposed by Marshall and Olkin [2], with the CDF defined as

$$\bar{F}_{MO}(\alpha;x) = \frac{\alpha\bar{F}(x)}{1-(1-\alpha)\bar{F}(x)}, \quad \alpha > 0.$$

Lone et al. [3] introduced the MIT transformation, providing an alternative modification given by

$$F_{MIT}(x;\alpha) = \frac{\alpha F(x)}{1-(1-\alpha)F(x)}, \quad \alpha > 0.$$

Using the T-X family methodology of Alzaatreh et al. [4] and Shah et al. [5] proposed the NGEP-X family with the CDF

$$F(x;\alpha,\zeta) = 1-\left(e^{\alpha G(x;\zeta)} - e^{\alpha}G(x;\zeta)\right), \quad \alpha > 0.$$

Trigonometric transformations have also gained traction; for example, Odhah et al. [6] defined a trigonometric-based CDF as

$$F(x;\zeta) = \frac{e^{1-\cos\left(\frac{\pi G(x;\zeta)}{1+G(x;\zeta)}\right)}-1}{e-1}.$$

More recently, Mir et al. [7] introduced the ASP family to model skewed and heavy-tailed data, with the CDF expressed as

$$F_{\mathrm{ASP}}(x;\alpha,\zeta) = 2\sin\left(\frac{\pi}{2}[C(x;\zeta)]^{\alpha}\right) - [C(x;\zeta)]^{\alpha}.$$

Modeling and analyzing natural phenomena constitute a fundamental aspect of statistical research across various practical domains, including science, engineering, and environmental studies. Over the past three decades, numerous statistical models have been proposed to capture the complex characteristics of real-world data better. One such contribution is the two-parameter Kumaraswamy distribution introduced by Kumaraswamy for modeling hydrological data [8]. This distribution has applications in diverse contexts involving bounded outcomes, such as hydrology, human body weights, academic scores, species growth rates, wind speed, atmospheric temperature, medicine, physics, and finance.

Several researchers have investigated various aspects of the Kumaraswamy distribution. For instance, [9] discussed different estimation methods, while [10] proposed a generalized modification for modeling real-life data. [11] developed the generalized inverted Kumaraswamy generated family of distributions, exploring its theoretical properties and applications. Despite its relevance, the distribution initially received limited attention in the statistical literature. However, Jones conducted a comprehensive study of its features—including the quantile function, L-moments, and order statistics—highlighting its similarities to the beta distribution [12,13].

Recent advancements have further extended the Kumaraswamy framework. For example, [14] analyzed generalized order statistics from the Kumaraswamy model, while [15] proposed Bayesian and frequentist estimators based on Type II censored data. These works focused on estimating shape parameters, reliability functions, and failure rates. Proposed modified point estimators [16], and classical and Bayesian methods for the Kumaraswamy inverse exponential distribution were studied by [17]. Also, [18] compared six different estimation methods for the LKD. Moreover, several novel families of distributions derived from the Kumaraswamy distribution have been introduced to handle data from domains such as hydrology, medicine, engineering, insurance, and finance; for more details, see [19–26].

More recently, studies have proposed the Harris extended inverted Kumaraswamy distribution [27], the odd generalized exponential Kumaraswamy–Weibull distribution [28], the Kumaraswamy modified Kies-G family [29], and the Kumaraswamy alpha power Lomax distribution [30], further expanding the utility and flexibility of the Kumaraswamy family of distributions in statistical modeling.

This study broadens the applicability and flexibility of the log-Kumaraswamy model, enabling its use for both bounded and unbounded real-world data sets. The motivation for this extension is multifaceted:

(i) To introduce a flexible distribution that serves as an alternative to the bounded log-Kumaraswamy distribution.
(ii) To develop a distribution capable of exhibiting various density shapes and hazard rate behaviors;
(iii) To derive and explore key statistical properties;
(iv) To estimate the model parameters using a range of classical techniques.
(v) To assess the performance and efficacy of the proposed distribution relative to existing models through applications to real-life data sets.

The structure of the paper is organized as follows. Section 2 presents the probability density function (PDF), cumulative distribution function (CDF), survival function, and hazard rate function of the proposed power Log-Kumaraswamy distribution. In Section 3, we investigate key statistical characteristics including moments, the information-generating function, mixture representations, the quantile function, and order statistics. Parameter estimation methods are discussed in Section 4. Section 5 includes a detailed simulation study comparing estimation techniques such as MLE, CVME, MPSE, OLSE, WLSE, ADE, RADE, MSADE, and MSALDE. Section 6 provides empirical analysis based on real data sets to evaluate the flexibility and adequacy of the proposed model. Finally, conclusions and recommendations for future work are presented in Section 7.

2. Power Log-Kumaraswamy Distribution

Lemonte et al. [31], Ishaq et al. [32] developed a novel statistical model called log-Kumaraswamy distribution (LKD). Its CDF is

$$G(x) = 1 - \left[1 - (1 - e^{-x})^{\alpha}\right]^{\beta}; x > 0,\ \alpha,\ \beta > 0. \tag{2.1}$$

By implementing a power transformation on the cumulative distribution function (CDF) (2.1), we get the power LKD (PLKD), with the CDF and PDF given by

$$F(x;\alpha,\beta,\sigma) = 1 - \left[1 - \left(1 - e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta};\ x > 0,\ \alpha,\ \beta,\ \sigma > 0, \tag{2.2}$$

$$f(x;\alpha,\beta,\sigma) = \alpha\beta\sigma e^{-x^{\sigma}} x^{\sigma-1} \left(1 - e^{-x^{\sigma}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta-1}, \tag{2.3}$$

respectively. The PLKD PDF is plotted using values of parameters α, β, and σ in Fig.1.

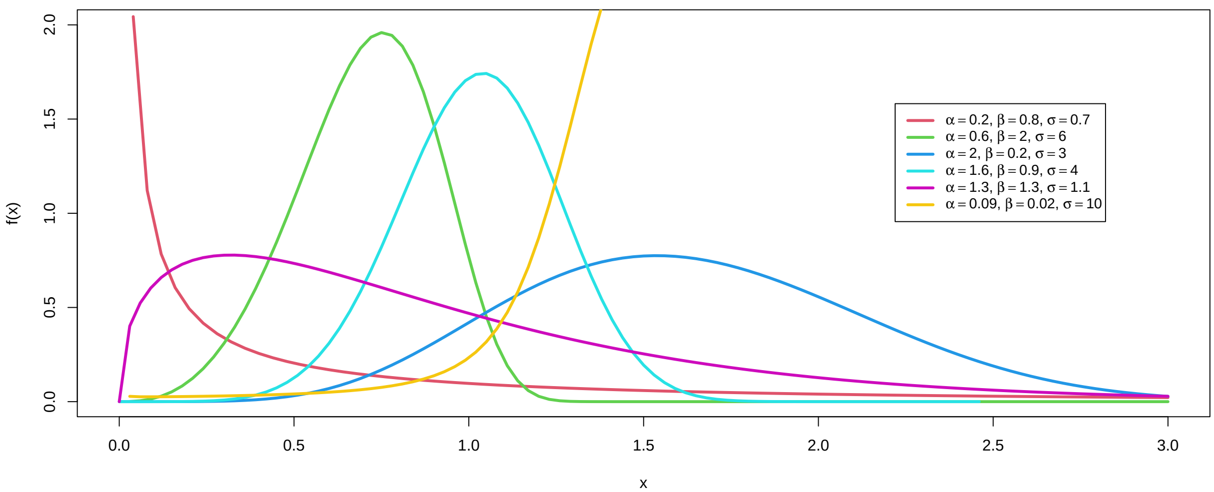

Figure 1. Density plots of PLKD for parameters α, β and σ.

Remark 2.1. *For $\sigma = 1$ in (2.2), the PLKD reduces to the LKD (2.1).*

The survival function of the PKLD is given by

$$S(x;\alpha,\beta,\sigma) = \left[1 - \left(1 - e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta};\ x > 0,\ \alpha,\ \beta,\ \sigma > 0. \tag{2.4}$$

The hazard function of the PKLD is derived as

$$h(x;\alpha,\beta,\sigma) = \frac{\alpha\beta\sigma e^{-x^{\sigma}} x^{\sigma-1} \left(1 - e^{-x^{\sigma}}\right)^{\alpha-1}}{\left[1 - (1 - e^{-x^{\sigma}})^{\alpha}\right]^{\beta}};\ x > 0,\ \alpha,\ \beta,\ \sigma > 0. \tag{2.5}$$

Fig. 2 is characterized by failure rates that are increasing, inverted bathtub-shaped, bathtub-shaped, and bimodal.

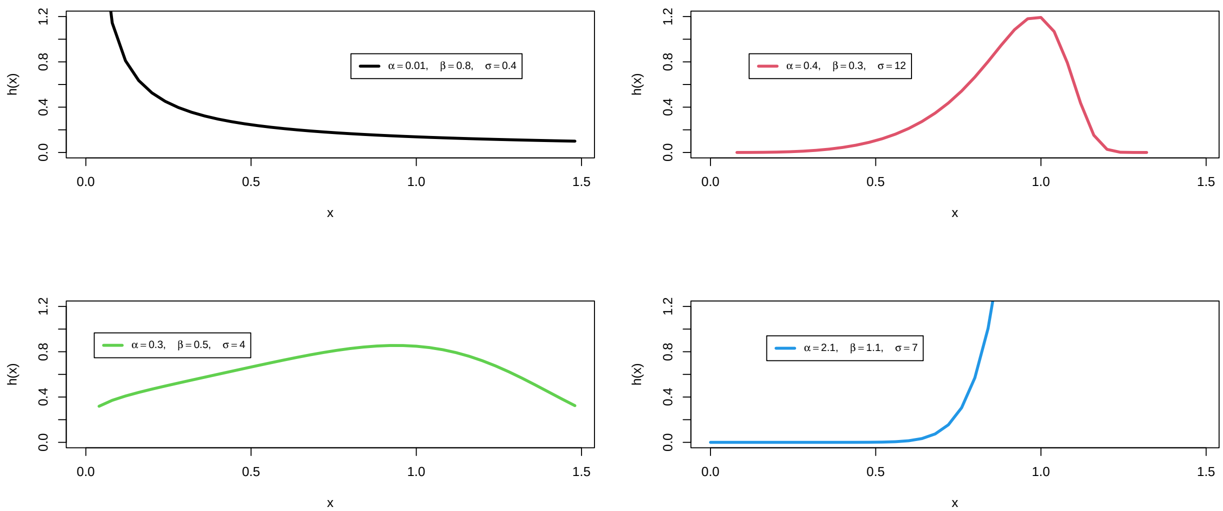

FIGURE 2. Hazard rate plots for different combinations of PLKD for different combinations of parameters α, β, and σ.

3. Statistical Properties of PLKD

3.1. Quantile Function. Let $F(x;\alpha,\beta,\sigma) = u$. The quantile function of the PLKD can be obtained as follows.

$$1-\left[1-\left(1-e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta} = u.$$

Taking the logarithm on both sides and simplifying it further, we obtain the required quantile function as

$$x = \left(-\ln\left(1-\left(1-(1-u)^{1/\beta}\right)^{1/\alpha}\right)\right)^{1/\sigma}. \tag{3.1}$$

The first quartile (Q_1), median (Q_2), and third quartile (Q_3) can be derived by setting $u = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ in Eq. (3.1) respectively.

3.2. Mixture Representation. Using the series expansion:

$$(1-x)^{\tau} = \sum_{j=0}^{\infty} \frac{(-1)^{j}\Gamma(\tau+1)}{j!\Gamma(\tau-j+1)} x^{j}, \quad |x|<1,\ \tau>0. \tag{3.2}$$

Applying (2.3) into (3.2), it will become

$$f(x) = \alpha\beta\sigma e^{-x^{\sigma}} x^{\sigma-1} \sum_{j=0}^{\infty} (-1)^{j} \frac{\Gamma(\beta)}{j!\Gamma(\beta-j)} (1-e^{-x^{\sigma}})^{\alpha(1+j)-1}. \tag{3.3}$$

Further expanding, we express it as:

$$f(x) = \sum_{j,k=0}^{\infty} w_{j,k}\, x^{\sigma-1}\, e^{-x^{\sigma}(1+k)}, \tag{3.4}$$

which is the PDF of the PKLD expressed as a mixture representation, where

$$w_{j,k} = \alpha\beta\sigma\,(-1)^{j+k}\frac{\Gamma(\beta)\Gamma(\alpha(1+j))}{j!k!\Gamma(\beta-j)\Gamma(\alpha(1+j)-k)}. \tag{3.5}$$

3.3. **Moments.** The rth raw moment of the PLKD, corresponding to the pdf given in equation (3.4), is expressed as:

$$E(X^r) = \sum_{j,k=0}^{\infty} w_{j,k}\int_0^{\infty} x^{\sigma+r-1}e^{-x^{\sigma}(1+k)}dx. \tag{3.6}$$

Let

$$z = x^{\sigma}(1+k) \implies dx = \frac{1}{\sigma}\frac{z^{\frac{1}{\sigma}-1}}{(1+k)^{\frac{1}{\sigma}}}dz. \tag{3.7}$$

Inserting (3.7) into (3.6) gives

$$E(X^r) = \frac{1}{\sigma\,(1+k)^{\frac{r+\sigma}{\sigma}}}\sum_{j,k=0}^{\infty} w_{j,k}\int_0^{\infty} z^{\frac{r+\sigma}{\sigma}-1}e^{-z}dz. \tag{3.8}$$

$$= \frac{1}{\sigma}\sum_{j,k=0}^{\infty} w_{j,k}\frac{\Gamma\left(\frac{r+\sigma}{\sigma}\right)}{(k+1)^{\frac{r+\sigma}{\sigma}}}. \tag{3.9}$$

3.4. **Information Generating Function.** Let X follow the PKLD with the PDF given in (2.3). Then, the information-generating function is

$$I_{\phi}(x) = \int_0^{\infty} f^{\phi}(x)dx = \frac{1}{\sigma}\sum_{l,m=0}^{\infty}\frac{\delta_{l,m}}{(m+\phi)^{\frac{(\sigma-1)\phi+1}{\sigma}}}\Gamma(\frac{(\sigma-1)\phi+1}{\sigma}). \tag{3.10}$$

3.5. **Rényi Entropy.** The Rényi entropy of PLKD is

$$R_{\phi}(x) = \frac{1}{1-\phi}\log\left[\int_0^{\infty} f(x)^{\phi}\right]dx; \quad \phi > 0, \quad \phi \neq 1, x \in \mathbb{R}. \tag{3.11}$$

By substituing (3.10) into (3.11), the Rényi entropy of the PLKD as

$$R_{\phi}(x) = \frac{1}{1-\phi}\log\left[\frac{1}{\sigma}\sum_{l,m=0}^{\infty}\frac{\delta_{l,m}}{(m+\phi)^{\frac{(\sigma-1)\phi+1}{\sigma}}}\Gamma(\frac{(\sigma-1)\phi+1}{\sigma})\right]; \quad \phi > 0, \quad \phi \neq 1. \tag{3.12}$$

3.6. **Q-Entropy.** The q-entropy of the PLKD is obtained from (3.10) as

$$Q_{\phi}(x) = \frac{1}{\phi-1}\left[1-\frac{1}{\sigma}\sum_{l,m=0}^{\infty}\frac{\delta_{l,m}}{(m+\phi)^{\frac{(\sigma-1)\phi+1}{\sigma}}}\Gamma(\frac{(\sigma-1)\phi+1}{\sigma})\right]. \tag{3.13}$$

3.7. **Order Statistics.** Suppose $X_1, X_2, \ldots, X_n$ denote the random variables from the sample sizes n with the PDF and CDF defined, respectively, in 2.3 and 2.2. The m^{th} order statistics of these variables $f_{m,k}(x)$ is defined as

$$f_{m,k}(x) = \frac{k!}{(m-1)!(k-m)!}f(x)[F(x)]^{m-1}[1-F(x)]^{k-m}. \tag{3.14}$$

Substituting Eqs. (2.3) and (2.2) into (3.14), one can obtain

$$f_{m,k}(x) = \frac{k!}{(m-1)!(k-m)!}\alpha\beta\sigma e^{-x^{\sigma}}x^{\sigma-1}\left(1-e^{-x^{\sigma}}\right)^{\alpha-1}\left[1-\left(1-e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta-1}$$
$$\times\left[1-\left[1-\left(1-e^{-x^{\sigma}}\right)^{\alpha}\right]^{\beta}\right]^{m-1}\left[1-\left(1-e^{-x^{\sigma}}\right)^{\alpha}\right]^{(k-m)\beta}. \tag{3.15}$$

Using 3.2, we get m^{th} order statistics for PLKD as

$$f_{m,k}(x) = \sum_{l=0}^{k-m}\sum_{p,q=0}^{\infty}\Lambda_{l,p,q}e^{-x^{\sigma}(q+1)}x^{\sigma-1}, \tag{3.16}$$

which is the m^{th} order statistics of the PLKD. Where,

$$\Lambda_{l,p,q} = \frac{\alpha\beta\sigma k!}{(m-1)!(k-m)!}\frac{(-1)^{l+p+q}\Gamma(m)\Gamma((k-m+l+1)\beta)\Gamma(\alpha(1+p))}{l!p!q!\Gamma(m-l)\Gamma((k-m+l+1)\beta-p)\Gamma(\alpha(1+p)-q)}.$$

4. Parameter Estimation

In this section, we explore the estimation of the parameters of the PLKD using various frequent estimation techniques. Nine estimation estimators are used to estimate of the parameters of the PLKD, including the MLE (Δ_1), CvME (Δ_2), MPSE (Δ_3), LSE (Δ_4), WLSE (Δ_5), ADE (Δ_6) and RTADE (Δ_7), MSADE (Δ_8), and MSALDE (Δ_9).

4.1. **Maximum Likelihood Estimation (Δ_1) for Complete Sample.** Let $X_1, X_2, ..., X_n$ be a random sample with observed values $x_1, x_2, ..., x_n$ from the PLKD model with the PDF $f(x;\alpha,\beta,\sigma)$ as defined in Eq. (2.3). The likelihood function is denoted as $\mathcal{L}(\alpha,\beta)$ as follows:

$$\mathcal{L}(x;\alpha,\beta,\sigma) = \prod_{i=1}^{n}\alpha\beta\sigma e^{-x_i^{\sigma}}x_i^{\sigma-1}\left(1-e^{-x_i^{\sigma}}\right)^{\alpha-1}\left[1-\left(1-e^{-x_i^{\sigma}}\right)^{\alpha}\right]^{\beta-1}. \tag{4.1}$$

By evaluating the system of non-linear equations $\frac{\partial}{\partial\alpha}\ell(x;\alpha,\beta,\sigma) = 0$, $\frac{\partial}{\partial\beta}\ell(x;\alpha,\beta,\sigma) = 0$, and $\frac{\partial}{\partial\sigma}\ell(x;\alpha,\beta,\sigma) = 0$, the MLEs of the parameters for the PLKD can be obtained.

4.2. **Cramér-von Mises Estimation (Δ_2).** A study by Macdonald [33] showed that CVME has less bias than other minimum distance estimators. Here, the CVME method is applied to estimate the parameters of the PLKD model. The CVME of unknown parameters can be determined by minimizing the Eq. in (4.2).

$$C(\alpha,\beta,\sigma) = \frac{1}{12n} + \sum_{k=1}^{n+1}\left[1-\left[1-\left(1-e^{-x_{(k)}^{\sigma}}\right)^{\alpha}\right]^{\beta} - \frac{2k-1}{2n}\right]^2. \tag{4.2}$$

4.3. **Maximum Product of Spacing Estimation (Δ_3).** The MPS method was initially introduced by Cheng and Amin [34] as an alternative to the maximum likelihood estimation approach for parameter estimation in continuous univariate distributions. Independently, Ranneby [35] further investigated this method, establishing its consistency and analyzing it as an approximation to the Kullback-Leibler information measure. Cheng and Amin reinforced the reliability of the MPS method by demonstrating its efficiency and consistency under broader conditions compared to the maximum likelihood approach.

The maximum product of spacing estimators is obtained by maximizing the Eq. in (4.3).

$$M(\alpha,\beta) = \frac{1}{n+1}\sum_{k=1}^{n+1}\log\left[1-\left[1-\left(1-e^{-x_k^{\sigma}}\right)^{\alpha}\right]^{\beta}-1-\left[1-\left(1-e^{-x_{(k-1)}^{\sigma}}\right)^{\alpha}\right]^{\beta}\right]. \tag{4.3}$$

By solving the non-linear equations $\frac{\partial}{\partial\alpha}M(\alpha,\beta,\sigma) = 0$,$\frac{\partial}{\partial\beta}M(\alpha,\beta,\sigma) = 0$ and $\frac{\partial}{\partial\sigma}M(\alpha,\beta,\sigma) = 0$, we procure the maximum product of spacing estimates for the parameters of the PLKD model.

4.4. **Least Square Estimation (Δ_4) and Weighted Least Square Estimation (Δ_5).** To estimate the parameters of the proposed distribution, Swain et al. [36] proposed the least-squares and weighted least-squares estimation techniques. The parameters of LKD can be estimated for the least squares estimation by minimizing the following equation.

$$S(\alpha,\beta,\sigma) = \sum_{k=1}^{n+1}\left[1-\left[1-\left(1-e^{-x_k^{\sigma}}\right)^{\alpha}\right]^{\beta}-\frac{k}{n+1}\right]^2. \tag{4.4}$$

Similarly, the WLSEs obtained by minimizing in Eq. (4.5).

$$W(\alpha,\beta,\sigma) = \sum_{k=1}^{n}\frac{(n+1)^2(n+2)}{k(n-k+1)}\left[1-\left[1-\left(1-e^{-x_k^{\sigma}}\right)^{\alpha}\right]^{\beta}-\frac{k}{n+1}\right]^2. \tag{4.5}$$

4.5. **Anderson Darling Estimation (Δ_6).** Anderson and Darling [37] introduced the Anderson-Darling test as an alternative to conventional statistical methods for detecting deviations from normality in sample distributions. In a subsequent study, Boos [38] analyzed the properties of the ADE. Based on his findings, the ADE for the PLKD model can be determined by minimizing the Anderson-Darling statistic, denoted as $A(\alpha,\beta,\sigma)$, which is expressed in Eq. (4.6)

$$A(\alpha,\beta,\sigma) = -n-\frac{1}{n}\sum_{k=1}^{n}(2k-1)\left[\log\left(1-\left[1-\left(1-e^{-x_k^{\sigma}}\right)^{\alpha}\right]^{\beta}\right)+\log\left(\left[1-\left(1-e^{-x_{(n+1-k)}^{\sigma}}\right)^{\alpha}\right]^{\beta}\right)\right]. \tag{4.6}$$

4.6. **Right-Tailed Anderson-Darling Estimation (Δ_7).** The right-tailed Anderson-Darling (AD) estimators are determined by optimizing the following equation.

$$R(\alpha,\beta,\sigma) = \frac{n}{2}-2\sum_{k=1}^{n}1-\left[1-\left(1-e^{-x_k^{\sigma}}\right)^{\alpha}\right]^{\beta}-\frac{1}{n}\sum_{k=1}^{n}(2k-1)\log\left(\left[1-\left(1-e^{-x_{(n+1-k)}^{\sigma}}\right)^{\alpha}\right]^{\beta}\right). \tag{4.7}$$

4.7. **Minimum spacing Absolute Distance Estimation (Δ_8).** In this subsection, we used the minimum spacing absolute distance estimator (MSADE) to estimate the parameters α, β, and σ of PLKD. The MSADE method is a statistical approach used to estimate distribution parameters by minimizing the sum of absolute logarithmic differences between successive datasets. The MSADE statistic, denoted as MS (α,β,σ), is given in Eq. (4.8)

$$MS\,(\alpha,\beta,\sigma) = \sum\left|\left[\left(1-\left(1-e^{-x_{(k-1)}^{\sigma}}\right)^{\alpha}\right)^{\beta}-\left(1-\left(1-e^{-x_{(k)}^{\sigma}}\right)^{\alpha}\right)^{\beta}\right]-\frac{1}{n+1}\right|. \tag{4.8}$$

4.8. **Minimum Spacing Absolute-Log Distance Estimation (Δ_9).** The MSALDE was proposed by Torabi [39]. This method estimates the parameters by minimizing the sum of absolute differences between the logarithmic values of the consecutive gaps between the ordered datasets. Accordingly, MSALDEs of the parameters are obtained by minimizing the Eq. (4.9).

$$MSL\left(\alpha,\beta,\sigma\right)=\sum_{k=1}^{n+1}\left|\left[\left(1-\left(1-e^{-x^{\alpha}{}_{(k-1)}}\right)^{\alpha}\right)^{\beta}-\left(1-\left(1-e^{-x^{\alpha}{}_{(k)}}\right)^{\alpha}\right)^{\beta}\right]-\log\left(\frac{1}{n+1}\right)\right|. \tag{4.9}$$

5. Simulation

In this section, we design a Monte-Carlo (MC) Simulation study to compare and evaluate the performance of all the estimators given in Section 5 for the point estimation of PKLD. We perform the MC simulations with $n = 50, 100, 200, 300, 500, 750, 1000$, the sample sizes, and 5000 repetitions. The initial values of the α, β and σ parameters are given as follows:

$Case_I = (\alpha = 0.7, \beta = 0.5, \sigma = 0.3)$,
$Case_{II} = (\alpha = 1.2, \beta = 0.7, \sigma = 0.5)$,
$Case_{III} = (\alpha = 0.5, \beta = 0.5, \sigma = 0.9)$,
$Case_{IV} = (\alpha = 2.5, \beta = 1.5, \sigma = 2)$.

The bias, mean squared error (MSE), and mean relative error (MRE) are considered to assess the performance of the examined estimator. The formulas of the criteria are given as follows:

$$Bias=\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\Theta}-\Theta\right),$$

$$MSE=\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\Theta}-\Theta\right)^{2},$$

$$MRE=\frac{1}{5000}\sum_{i=1}^{5000}\frac{|\hat{\Theta}-\Theta|}{\Theta},$$

where $\Theta = (\alpha, \beta, \sigma)$.

5.1. **Random sample generation.** This subsection suggests generating random samples from the PLKD (α, β, σ). We generate random samples from the PLKD (α, β, σ) using an acceptance-rejection (AR) sampling algorithm. We consider the Weibull distribution to be the proposal distribution in the algorithm. The AR algorithm is given as follows:

Algorithm 1.

A1. Generate data on random variable $Y \sim Weibull(\theta, \lambda)$ with the PDF g given as follows:

$$g\left(\theta,\lambda\right)=\theta\lambda x^{\lambda-1}e^{-\theta x^{\lambda}}.$$

A2. Generate U from standard uniform distribution(independent of Y).

A3. If

$$U<\frac{f\left(Y;\alpha,\beta,\sigma\right)}{k\times g\left(Y;\theta,\lambda\right)}$$

then set $X = Y$ ("accept"); otherwise go back to A1 ("reject"), where the PDF $f(.)$ is given as in Eq. (2.3) and

$$k = \max_{z \in \mathbb{R}_+} \frac{f(z;\alpha,\beta,\sigma)}{g(z;\theta,\lambda)}.$$

The output of Algorithm 1 provides random data on X from the PLKD (α,β,σ). Algorithm 1 is used in all MC simulations.

Tables 1-4 provide the MC simulation results.

Table 1. The biases, MSEs and MREs for $\alpha = 0.7$, $\beta = 0.5$ and $\sigma = 0.3$

		Bias			MSE			MRE		
Estimator	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
MLE	50	0.1016	0.5258	0.2458	0.6309	2.7757	0.0867	0.5687	1.1543	0.8691
	100	-0.0351	0.2650	0.2455	0.1839	0.7813	0.0728	0.3326	0.5810	0.8300
	250	-0.1066	0.1524	0.2484	0.0332	0.0531	0.0663	0.2142	0.3197	0.8283
	500	-0.1159	0.1424	0.2473	0.0214	0.0286	0.0633	0.1840	0.2875	0.8242
	750	-0.1190	0.1394	0.2470	0.0194	0.0250	0.0624	0.1791	0.2797	0.8233
	1000	-0.1226	0.1354	0.2475	0.0187	0.0221	0.0623	0.1789	0.2711	0.8250
LSE	50	0.2459	0.6148	0.1750	0.8454	1.8338	0.0689	0.8303	1.3716	0.7203
	100	0.0834	0.3736	0.1757	0.3708	0.7121	0.0518	0.5326	0.8249	0.6477
	250	-0.0442	0.2101	0.1796	0.0913	0.1499	0.0406	0.2938	0.4470	0.6091
	500	-0.0783	0.1729	0.1805	0.0305	0.0499	0.0364	0.2045	0.3519	0.6023
	750	-0.0870	0.1651	0.1813	0.0227	0.0394	0.0355	0.1814	0.3327	0.6044
	1000	-0.0954	0.1574	0.1826	0.0192	0.0326	0.0353	0.1706	0.3156	0.6088
WLSE	50	0.2218	0.6526	0.1874	0.8599	2.7184	0.0652	0.7298	1.4123	0.7224
	100	0.0412	0.3598	0.1961	0.2825	2.2196	0.0533	0.4078	0.7694	0.6821
	250	-0.0543	0.3300	0.2116	0.1397	0.0559	0.0663	0.2315	0.6742	0.7083
	500	-0.0731	0.3050	0.2118	0.0872	0.0410	0.0565	0.1716	0.6131	0.7069
	750	-0.0840	0.1720	0.2100	0.0146	0.0371	0.0493	0.1486	0.3446	0.7001
	1000	-0.0859	0.1990	0.2118	0.0443	0.0227	0.0492	0.1477	0.3989	0.7067
ADE	50	0.2567	0.6791	0.1876	0.9122	2.5654	0.0685	0.7860	1.4706	0.7392
	100	0.0749	0.3820	0.1912	0.3546	0.8818	0.0545	0.4756	0.8228	0.6816
	250	-0.0521	0.2048	0.1973	0.0665	0.1302	0.0459	0.2545	0.4279	0.6626
	500	-0.0787	0.1752	0.1979	0.0235	0.0461	0.0424	0.1828	0.3537	0.6599
	750	-0.0860	0.1686	0.1986	0.0184	0.0382	0.0416	0.1648	0.3384	0.6620
	1000	-0.0928	0.1620	0.1997	0.0161	0.0327	0.0415	0.1569	0.3244	0.6657
CvME	50	0.2805	0.6640	0.1811	0.9038	2.0212	0.0801	0.8568	1.4607	0.7417
	100	0.1002	0.3942	0.1779	0.3916	0.7689	0.0532	0.5424	0.8620	0.6562
	250	-0.0390	0.2152	0.1805	0.0928	0.1544	0.0410	0.2940	0.4559	0.6122
	500	-0.0760	0.1750	0.1810	0.0304	0.0509	0.0366	0.2036	0.3559	0.6039
	750	-0.0855	0.1665	0.1816	0.0226	0.0400	0.0356	0.1805	0.3353	0.6054
	1000	-0.0943	0.1584	0.1829	0.0191	0.0330	0.0354	0.1698	0.3175	0.6096
MPSE	50	0.0816	0.4396	0.2156	0.4978	1.5792	0.0710	0.5559	0.9881	0.7712
	100	-0.0357	0.2442	0.2253	0.1456	0.4099	0.0630	0.3313	0.5402	0.7629
	250	-0.1023	0.1552	0.2375	0.0315	0.0464	0.0610	0.2132	0.3250	0.7920
	500	-0.1124	0.1453	0.2407	0.0209	0.0296	0.0601	0.1810	0.2932	0.8022
	750	-0.1160	0.1419	0.2420	0.0188	0.0258	0.0600	0.1760	0.2846	0.8068
	1000	-0.1200	0.1377	0.2434	0.0181	0.0227	0.0603	0.1755	0.2756	0.8115
TADE	50	-0.1644	0.1764	0.3057	0.3192	0.5926	0.1230	0.5787	0.6232	1.0472
	100	-0.2687	0.0452	0.3127	0.1444	0.0975	0.1124	0.4699	0.3120	1.0469
	250	-0.3223	-0.0090	0.3196	0.1155	0.0113	0.1076	0.4638	0.1636	1.0654
	500	-0.3312	-0.0161	0.3200	0.1144	0.0048	0.1049	0.4733	0.1113	1.0666
	750	-0.3335	-0.0181	0.3200	0.1144	0.0034	0.1041	0.4765	0.0943	1.0665
	1000	-0.3375	-0.0218	0.3213	0.1161	0.0026	0.1045	0.4821	0.0824	1.0711
MSADE	50	-0.0703	0.1979	0.2440	0.0782	0.1358	0.0782	0.3019	0.5076	0.8181
	100	-0.0762	0.1879	0.2398	0.0584	0.0982	0.0692	0.2672	0.4451	0.8008
	250	-0.1007	0.1599	0.2419	0.0373	0.0554	0.0647	0.2276	0.3498	0.8065
	500	-0.1125	0.1477	0.2425	0.0254	0.0350	0.0621	0.1957	0.3047	0.8084
	750	-0.1175	0.1426	0.2430	0.0226	0.0293	0.0614	0.1877	0.2892	0.8101
	1000	-0.1244	0.1355	0.2450	0.0218	0.0248	0.0618	0.1878	0.2731	0.8166
MSALDE	50	0.0106	0.2878	0.2314	0.1793	0.3939	0.0774	0.4309	0.6951	0.7852
	100	-0.0572	0.2087	0.2389	0.1074	0.1662	0.0714	0.3413	0.4890	0.8048
	250	-0.1034	0.1590	0.2398	0.0642	0.0750	0.0659	0.2863	0.3544	0.8023
	500	-0.1354	0.1302	0.2427	0.0399	0.0333	0.0634	0.2471	0.2746	0.8090
	750	-0.1454	0.1216	0.2434	0.0357	0.0255	0.0627	0.2381	0.2522	0.8113
	1000	-0.1540	0.1141	0.2444	0.0338	0.0202	0.0623	0.2360	0.2328	0.8146

TABLE 2. The biases, MSEs and MREs for $\alpha = 1.2$, $\beta = 0.7$ and $\sigma = 0.5$

		Bias			MSE			MRE		
Estimator	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
MLE	50	0.2532	0.8839	0.1108	1.4450	3.8766	0.0686	0.6802	1.4236	0.4247
	100	0.0677	0.5586	0.0983	0.7593	1.9483	0.0385	0.4657	0.8914	0.3208
	250	-0.0897	0.2864	0.0937	0.2076	0.3726	0.0198	0.2673	0.4412	0.2335
	500	-0.1449	0.2120	0.0930	0.0734	0.0835	0.0134	0.1876	0.3129	0.1980
	750	-0.1548	0.1990	0.0935	0.0563	0.0623	0.0119	0.1677	0.2889	0.1912
	1000	-0.1553	0.2007	0.0907	0.0481	0.0566	0.0106	0.1566	0.2886	0.1842
LSE	50	0.1929	0.7156	0.1377	1.1757	2.0868	0.1113	0.7187	1.2608	0.5166
	100	0.0731	0.5404	0.1235	0.8198	1.3616	0.0667	0.5714	0.9397	0.4184
	250	-0.0676	0.3330	0.1166	0.3929	0.5493	0.0374	0.3868	0.5615	0.3211
	500	-0.1716	0.2076	0.1227	0.1783	0.1817	0.0267	0.2777	0.3417	0.2782
	750	-0.1960	0.1776	0.1245	0.1329	0.1114	0.0234	0.2467	0.2828	0.2665
	1000	-0.2066	0.1656	0.1212	0.0906	0.0589	0.0199	0.2143	0.2532	0.2496
WLSE	50	0.3282	0.9814	0.1075	1.6948	3.9711	0.0884	0.7873	1.6029	0.4760
	100	0.1107	0.6194	0.1026	0.9366	2.0186	0.0504	0.5492	1.0127	0.3662
	250	-0.0796	0.3118	0.1030	0.3058	0.5005	0.0266	0.3217	0.4973	0.2712
	500	-0.1701	0.1966	0.1087	0.1072	0.1036	0.0188	0.2239	0.3022	0.2352
	750	-0.1832	0.1794	0.1096	0.0826	0.0692	0.0167	0.2024	0.2689	0.2279
	1000	-0.1842	0.1793	0.1058	0.0642	0.0522	0.0145	0.1829	0.2619	0.2156
ADE	50	0.2888	0.8852	0.0987	1.4252	3.2522	0.0703	0.7123	1.4348	0.4315
	100	0.0937	0.5718	0.0975	0.7947	1.6479	0.0434	0.5062	0.9269	0.3427
	250	-0.0822	0.3024	0.1007	0.2703	0.4320	0.0244	0.3060	0.4772	0.2616
	500	-0.1662	0.1985	0.1063	0.0992	0.0975	0.0177	0.2164	0.3019	0.2292
	750	-0.1792	0.1824	0.1077	0.0798	0.0720	0.0160	0.1979	0.2715	0.2234
	1000	-0.1795	0.1826	0.1038	0.0614	0.0528	0.0139	0.1787	0.2657	0.2116
CvME	50	0.2302	0.7677	0.1468	1.2249	2.2732	0.1180	0.7298	1.3230	0.5302
	100	0.0904	0.5608	0.1277	0.8307	1.4139	0.0688	0.5737	0.9623	0.4250
	250	-0.0597	0.3413	0.1180	0.3969	0.5634	0.0380	0.3875	0.5709	0.3239
	500	-0.1679	0.2112	0.1234	0.1791	0.1872	0.0270	0.2772	0.3458	0.2797
	750	-0.1935	0.1800	0.1250	0.1333	0.1150	0.0236	0.2461	0.2856	0.2675
	1000	-0.2050	0.1670	0.1216	0.0901	0.0597	0.0201	0.2135	0.2549	0.2503
MPSE	50	0.2161	0.7783	0.0767	1.2885	2.7985	0.0575	0.6697	1.2854	0.3863
	100	0.0589	0.5113	0.0740	0.6670	1.3650	0.0327	0.4616	0.8270	0.2934
	250	-0.0772	0.2924	0.0783	0.1986	0.3115	0.0170	0.2688	0.4492	0.2137
	500	-0.1321	0.2229	0.0826	0.0739	0.0917	0.0117	0.1867	0.3279	0.1819
	750	-0.1446	0.2075	0.0855	0.0549	0.0670	0.0105	0.1647	0.3007	0.1769
	1000	-0.1466	0.2078	0.0842	0.0467	0.0603	0.0095	0.1531	0.2985	0.1725
TADE	50	0.5439	1.1917	0.0618	1.9345	4.2962	0.0704	0.8786	1.8665	0.4353
	100	0.3410	0.8543	0.0592	1.2198	2.5100	0.0433	0.6554	1.3257	0.3446
	250	0.1438	0.5487	0.0623	0.5924	1.1304	0.0236	0.4220	0.8269	0.2524
	500	0.0047	0.3537	0.0702	0.2118	0.3573	0.0143	0.2637	0.5202	0.1989
	750	-0.0399	0.2944	0.0744	0.1059	0.1682	0.0118	0.2097	0.4282	0.1818
	1000	-0.0360	0.2991	0.0691	0.0878	0.1651	0.0094	0.1800	0.4293	0.1650
MSADE	50	-0.1649	0.1978	0.1093	0.1375	0.1203	0.0408	0.2235	0.3850	0.2679
	100	-0.1531	0.2074	0.0969	0.1154	0.1081	0.0249	0.2104	0.3689	0.2271
	250	-0.1429	0.2124	0.0895	0.0853	0.0901	0.0160	0.1904	0.3395	0.1985
	500	-0.1484	0.2062	0.0880	0.0633	0.0702	0.0126	0.1707	0.3120	0.1860
	750	-0.1536	0.1989	0.0902	0.0560	0.0605	0.0119	0.1630	0.2957	0.1854
	1000	-0.1524	0.2032	0.0876	0.0505	0.0595	0.0106	0.1563	0.2965	0.1794
MSALDE	50	-0.0498	0.3485	0.0994	0.4489	0.6168	0.0508	0.4286	0.6721	0.3328
	100	-0.0470	0.3350	0.0826	0.3116	0.4384	0.0306	0.3586	0.5865	0.2717
	250	-0.0900	0.2697	0.0809	0.1603	0.2036	0.0180	0.2638	0.4282	0.2146
	500	-0.1220	0.2322	0.0813	0.0835	0.1057	0.0126	0.1987	0.3471	0.1862
	750	-0.1394	0.2123	0.0850	0.0608	0.0740	0.0111	0.1727	0.3111	0.1793
	1000	-0.1493	0.2051	0.0857	0.0508	0.0614	0.0101	0.1603	0.2963	0.1763

TABLE 3. The biases, MSEs and MREs for $\alpha = 0.5$, $\beta = 0.5$ and $\sigma = 0.9$

		Bias			MSE			MRE		
Estimator	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
MLE	50	0.2543	0.3167	-0.0632	0.4990	1.0255	0.0541	0.6909	0.7480	0.1993
	100	0.1385	0.1472	-0.0708	0.1044	0.1193	0.0265	0.3912	0.3581	0.1432
	250	0.1001	0.1073	-0.0746	0.0289	0.0240	0.0134	0.2509	0.2370	0.1051
	500	0.0907	0.0991	-0.0772	0.0165	0.0153	0.0097	0.2019	0.2042	0.0927
	750	0.0893	0.0970	-0.0792	0.0131	0.0128	0.0087	0.1874	0.1959	0.0904
	1000	0.0869	0.0955	-0.0783	0.0116	0.0117	0.0080	0.1796	0.1919	0.0883
LSE	50	0.4243	0.4702	-0.0975	0.8871	1.1247	0.1119	1.1085	1.0867	0.2979
	100	0.2785	0.2816	-0.1030	0.4041	0.4240	0.0617	0.7227	0.6477	0.2190
	250	0.1642	0.1575	-0.0939	0.1000	0.0776	0.0284	0.4104	0.3486	0.1506
	500	0.1392	0.1342	-0.0964	0.0458	0.0336	0.0186	0.3140	0.2790	0.1249
	750	0.1314	0.1271	-0.0966	0.0324	0.0245	0.0153	0.2806	0.2581	0.1158
	1000	0.1235	0.1215	-0.0932	0.0260	0.0206	0.0132	0.2594	0.2452	0.1096
WLSE	50	0.3914	0.4778	-0.1044	0.9014	1.5695	0.0883	1.0018	1.0826	0.2601
	100	0.2086	0.2170	-0.0987	0.2468	0.3102	0.0428	0.5495	0.5039	0.1822
	250	0.1292	0.1288	-0.0906	0.0484	0.0363	0.0195	0.3152	0.2818	0.1271
	500	0.1150	0.1166	-0.0917	0.0265	0.0216	0.0137	0.2524	0.2399	0.1106
	750	0.1114	0.1129	-0.0925	0.0204	0.0176	0.0119	0.2322	0.2279	0.1057
	1000	0.1066	0.1097	-0.0901	0.0174	0.0156	0.0107	0.2193	0.2202	0.1021
ADE	50	0.3425	0.3995	-0.1000	0.6737	1.0826	0.0709	0.8730	0.9107	0.2328
	100	0.1988	0.2039	-0.0968	0.2065	0.2364	0.0385	0.5158	0.4709	0.1720
	250	0.1290	0.1288	-0.0905	0.0465	0.0354	0.0190	0.3111	0.2801	0.1247
	500	0.1155	0.1170	-0.0921	0.0262	0.0216	0.0137	0.2522	0.2403	0.1102
	750	0.1119	0.1133	-0.0930	0.0204	0.0176	0.0120	0.2329	0.2285	0.1060
	1000	0.1072	0.1101	-0.0907	0.0175	0.0157	0.0108	0.2201	0.2210	0.1025
CvME	50	0.4563	0.5094	-0.0868	0.9487	1.2491	0.1149	1.1540	1.1544	0.3018
	100	0.2944	0.2978	-0.0982	0.4271	0.4586	0.0622	0.7455	0.6755	0.2196
	250	0.1693	0.1619	-0.0919	0.1034	0.0812	0.0282	0.4176	0.3559	0.1501
	500	0.1416	0.1361	-0.0954	0.0467	0.0344	0.0184	0.3176	0.2824	0.1243
	750	0.1330	0.1284	-0.0959	0.0329	0.0249	0.0152	0.2832	0.2604	0.1153
	1000	0.1246	0.1224	-0.0927	0.0264	0.0209	0.0132	0.2614	0.2469	0.1091
MPSE	50	0.2239	0.2739	-0.1031	0.4327	0.7241	0.0575	0.6688	0.6787	0.2092
	100	0.1284	0.1395	-0.0970	0.1002	0.0976	0.0304	0.3884	0.3492	0.1553
	250	0.0976	0.1065	-0.0890	0.0291	0.0242	0.0157	0.2502	0.2366	0.1155
	500	0.0902	0.0994	-0.0861	0.0166	0.0155	0.0111	0.2020	0.2050	0.1009
	750	0.0894	0.0977	-0.0859	0.0133	0.0130	0.0098	0.1880	0.1972	0.0972
	1000	0.0872	0.0962	-0.0837	0.0117	0.0119	0.0089	0.1805	0.1933	0.0940
TADE	50	0.5578	0.6380	-0.1444	1.2461	1.8834	0.0944	1.3132	1.3921	0.2719
	100	0.3494	0.3498	-0.1370	0.5090	0.6060	0.0556	0.8196	0.7648	0.2077
	250	0.2126	0.1915	-0.1234	0.1222	0.0920	0.0288	0.4747	0.4044	0.1545
	500	0.1759	0.1594	-0.1202	0.0576	0.0411	0.0209	0.3709	0.3247	0.1387
	750	0.1680	0.1520	-0.1204	0.0440	0.0320	0.0186	0.3437	0.3059	0.1352
	1000	0.1599	0.1462	-0.1173	0.0370	0.0276	0.0169	0.3243	0.2929	0.1309
MSADE	50	0.0975	0.1179	-0.0618	0.0997	0.0791	0.0420	0.4170	0.3655	0.1752
	100	0.0878	0.1040	-0.0710	0.0622	0.0484	0.0265	0.3297	0.2912	0.1408
	250	0.0884	0.1009	-0.0778	0.0335	0.0274	0.0167	0.2559	0.2386	0.1143
	500	0.0857	0.0967	-0.0787	0.0198	0.0177	0.0117	0.2102	0.2073	0.0992
	750	0.0878	0.0969	-0.0823	0.0161	0.0150	0.0106	0.1959	0.2000	0.0962
	1000	0.0871	0.0963	-0.0817	0.0142	0.0136	0.0097	0.1890	0.1958	0.0939
MSALDE	50	0.1588	0.1802	-0.0866	0.2006	0.2146	0.0535	0.5577	0.5088	0.2065
	100	0.1248	0.1362	-0.0894	0.0986	0.0853	0.0323	0.4012	0.3587	0.1600
	250	0.1002	0.1099	-0.0864	0.0348	0.0292	0.0175	0.2705	0.2537	0.1202
	500	0.0926	0.1013	-0.0839	0.0198	0.0179	0.0120	0.2161	0.2143	0.1025
	750	0.0907	0.0986	-0.0847	0.0155	0.0147	0.0105	0.1976	0.2024	0.0980
	1000	0.0890	0.0975	-0.0832	0.0136	0.0133	0.0095	0.1887	0.1971	0.0946

Table 4. The biases, MSEs and MREs for $\alpha = 2.5$, $\beta = 1.5$ and $\sigma = 2$

		Bias			MSE			MRE		
Estimator	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
MLE	50	0.0240	0.4228	0.3839	1.8014	2.4229	1.1584	0.4937	0.8245	0.4278
	100	0.1380	0.4912	0.1904	1.5834	2.2273	0.6715	0.4525	0.7838	0.3435
	250	0.2380	0.5223	0.0343	1.2890	2.0448	0.3570	0.3844	0.6933	0.2577
	500	0.2746	0.5100	-0.0330	1.0757	1.8264	0.2421	0.3319	0.6126	0.2087
	750	0.2336	0.4179	-0.0366	0.8717	1.3947	0.1912	0.2919	0.5244	0.1828
	1000	0.2426	0.4040	-0.0580	0.7742	1.2424	0.1616	0.2692	0.4858	0.1660
LSE	50	-0.1961	0.1897	0.4884	1.5848	1.3476	1.6399	0.4608	0.6883	0.4698
	100	-0.0568	0.2653	0.2934	1.3665	1.2532	0.9763	0.4323	0.6678	0.3844
	250	0.1075	0.3324	0.0878	1.0197	1.0067	0.4713	0.3733	0.5956	0.2864
	500	0.2513	0.4493	-0.0298	0.9579	1.1146	0.3100	0.3520	0.5942	0.2424
	750	0.2436	0.4078	-0.0460	0.8276	0.9437	0.2533	0.3229	0.5398	0.2188
	1000	0.3165	0.4726	-0.0957	0.8247	0.9867	0.2220	0.3199	0.5483	0.2075
WLSE	50	0.0596	0.5089	0.3487	1.9228	2.2589	1.3672	0.5244	0.8706	0.4560
	100	0.1548	0.5279	0.1795	1.6346	2.0146	0.7953	0.4782	0.8137	0.3699
	250	0.2795	0.5579	0.0098	1.2918	1.6838	0.4111	0.4094	0.7196	0.2802
	500	0.3291	0.5449	-0.0662	1.0604	1.4148	0.2716	0.3559	0.6338	0.2280
	750	0.3116	0.5003	-0.0728	0.9513	1.2642	0.2307	0.3300	0.5840	0.2079
	1000	0.3295	0.4955	-0.0998	0.8585	1.1539	0.1951	0.3068	0.5471	0.1895
ADE	50	0.0468	0.4397	0.3181	1.6814	1.9000	1.1274	0.4857	0.7985	0.4192
	100	0.1594	0.5076	0.1590	1.5227	1.8712	0.7061	0.4592	0.7823	0.3511
	250	0.2686	0.5299	0.0087	1.2203	1.5547	0.3891	0.3980	0.6941	0.2734
	500	0.3227	0.5285	-0.0666	1.0168	1.3345	0.2630	0.3490	0.6182	0.2245
	750	0.3040	0.4824	-0.0725	0.9080	1.1797	0.2235	0.3237	0.5681	0.2049
	1000	0.3289	0.4900	-0.1009	0.8403	1.1145	0.1923	0.3047	0.5414	0.1886
CvME	50	-0.1661	0.2099	0.5345	1.5689	1.3804	1.7212	0.4590	0.6930	0.4748
	100	-0.0407	0.2745	0.3144	1.3577	1.2572	0.9994	0.4312	0.6686	0.3851
	250	0.1155	0.3377	0.0951	1.0190	1.0100	0.4743	0.3731	0.5964	0.2863
	500	0.2561	0.4528	-0.0267	0.9590	1.1174	0.3102	0.3521	0.5950	0.2423
	750	0.2455	0.4084	-0.0435	0.8256	0.9420	0.2529	0.3224	0.5391	0.2185
	1000	0.3176	0.4724	-0.0938	0.8221	0.9830	0.2213	0.3194	0.5473	0.2072
MPSE	50	0.0443	0.4610	0.2230	1.8065	2.1489	0.9987	0.4986	0.8188	0.4053
	100	0.1598	0.5050	0.0841	1.5201	1.9833	0.5970	0.4449	0.7622	0.3260
	250	0.2357	0.4760	-0.0217	1.1097	1.5049	0.3177	0.3630	0.6326	0.2424
	500	0.2533	0.4312	-0.0639	0.8530	1.1811	0.2106	0.3029	0.5322	0.1939
	750	0.1934	0.3189	-0.0532	0.6290	0.7850	0.1572	0.2558	0.4318	0.1656
	1000	0.2172	0.3299	-0.0742	0.5802	0.7442	0.1366	0.2415	0.4129	0.1533
TADE	50	-0.0166	0.3976	0.3692	1.8080	1.8462	1.3204	0.5000	0.7980	0.4448
	100	0.1000	0.4911	0.2084	1.6885	1.9784	0.8475	0.4851	0.8109	0.3810
	250	0.1001	0.3780	0.1012	1.2288	1.3881	0.4601	0.4024	0.6607	0.2911
	500	0.1482	0.3804	0.0315	1.0541	1.2611	0.3127	0.3597	0.6012	0.2415
	750	0.0960	0.2957	0.0388	0.8931	1.0483	0.2560	0.3263	0.5366	0.2187
	1000	0.1001	0.2700	0.0150	0.7640	0.8873	0.2074	0.2972	0.4864	0.1954
MSADE	50	-0.0973	-0.0256	0.0267	0.1567	0.0820	0.1636	0.0921	0.1236	0.1187
	100	-0.0922	-0.0328	0.0355	0.1509	0.0797	0.1280	0.0842	0.1123	0.1010
	250	-0.0728	-0.0231	0.0286	0.1206	0.0733	0.0718	0.0772	0.1056	0.0778
	500	-0.0487	-0.0083	0.0186	0.1071	0.0714	0.0551	0.0735	0.1028	0.0682
	750	-0.0432	-0.0083	0.0188	0.1081	0.0744	0.0493	0.0763	0.1073	0.0666
	1000	-0.0289	0.0037	0.0093	0.1044	0.0763	0.0429	0.0775	0.1108	0.0644
MSALDE	50	-0.1492	0.0600	0.1987	0.8448	0.6437	0.7162	0.2791	0.3947	0.2746
	100	-0.0904	0.0864	0.1436	0.7789	0.6434	0.4563	0.2818	0.4076	0.2418
	250	0.0147	0.1592	0.0532	0.6298	0.6406	0.2493	0.2589	0.3995	0.1937
	500	0.0743	0.1895	0.0062	0.5288	0.5558	0.1746	0.2350	0.3734	0.1662
	750	0.0821	0.1786	-0.0051	0.4652	0.4932	0.1407	0.2194	0.3472	0.1519
	1000	0.1057	0.1899	-0.0261	0.4198	0.4500	0.1193	0.2070	0.3296	0.1406

The results of the MC simulation study are listed as follows:

- The bias, MSE, and MRE values regarding nine estimators decrease and approach zero as the sample size increases.
- When evaluating the case of all parameters, it is observed that the MSADE outperforms the other estimators for small sample sizes ($n = 50, 100$). As the sample size increases, although the performance of all estimators tends to be similar, it is generally observed that the MSADE method outperforms.
- Although MSADE is generally the best estimator, MLE can also be suggested as an alternative because MLE gives consistent results with low bias, MSE, and MRE values, particularly for large sample sizes ($n = 1000$). Moreover, TADE may be a good alternative for small sample sizes.
- In the small sample sizes, the WLSE method typically demonstrates the highest bias, RMSE, and MRE values.
- The lowest bias, MSE, and MRE values are obtained under the Case I parameter.
- For all parameter cases, the bias, MSE, and MRE values of the σ parameter are the smallest among all estimators across the three parameters. This indicates that all estimators estimate the σ parameter more accurately. Moreover, the bias, MSE, and MRE values of the β parameter are greater than the others.

6. Real Data Analysis

In this section, two real data studies from the engineering and health economic fields are presented to demonstrate the effectiveness of the PLKD in real-life applications. In this regard, the PLKD is compared with several competing distributions such as LKD, log-logistic (LL) [40], Kumaraswamy inverse Rayleigh (KIR) [41], exponential power (EP) [42], and generalized binomial exponential-II (GBE-II) [43]. The PDFs of competing distributions are given as;

Table 5. The PDFs of lifetime distributions for competing distributions

$f_{LKD}(x;\alpha,\beta) = \alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}\left(1-(1-e^{-x})^{\alpha}\right)^{\beta-1}$	, $\alpha,\beta > 0$
$f_{KIR}(x;\alpha,\beta) = 2\alpha\beta x^{-3} e^{-\beta x^{-2}}\left(1-e^{-\beta x^{-2}}\right)^{\alpha-1}$	, $\alpha,\beta > 0$
$f_{LL}(x;\alpha,\beta) = \frac{\beta\left(\frac{x}{\alpha}\right)^{\beta}}{\alpha\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{2}}$	, $\alpha,\beta > 0$
$f_{EP}(x;\alpha,\beta) = \frac{\beta x^{\beta-1} e^{\left(1+\left(\frac{x}{\alpha}\right)^{\beta}-e^{\left(\frac{x}{\alpha}\right)^{\beta}}\right)}}{\alpha^{\beta}}$	, $\alpha,\beta > 0$
$f_{GBE-II}(x;\alpha,\beta,\sigma) = \alpha\beta\left(1+\frac{(\beta x-1)\sigma}{2-\sigma}\right)e^{-\beta x}\left[1-\left(1+\frac{\beta\sigma x}{2-\sigma}\right)e^{-\beta x}\right]^{\alpha-1}$	, $\alpha,\beta,\sigma > 0$

To compare the fit of the distributions to the datasets, selection criteria are utilized, such as -2ℓ, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov (KS), Anderson-Darling (AD), Cramer-Von Mises (CvM), and their corresponding p-values.

6.1. **Carbon Fibers Data.** The first dataset includes strength data, quantified in gigapascals (GPA), obtained from single carbon fibers subjected to tensile testing at a gauge length of 20 mm [44]. The dataset has been previously used in studies by [45,46]. The dataset I consist of values 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585,3.585. The MLEs and corresponding standard errors (SEs) of the distribution parameters are presented in Table 6. The selection criteria are presented in Table 7. Figure 3 illustrates the nonparametric plots of carbon fiber data, while the fitted CDFs and PDFs are given in Figure 4.

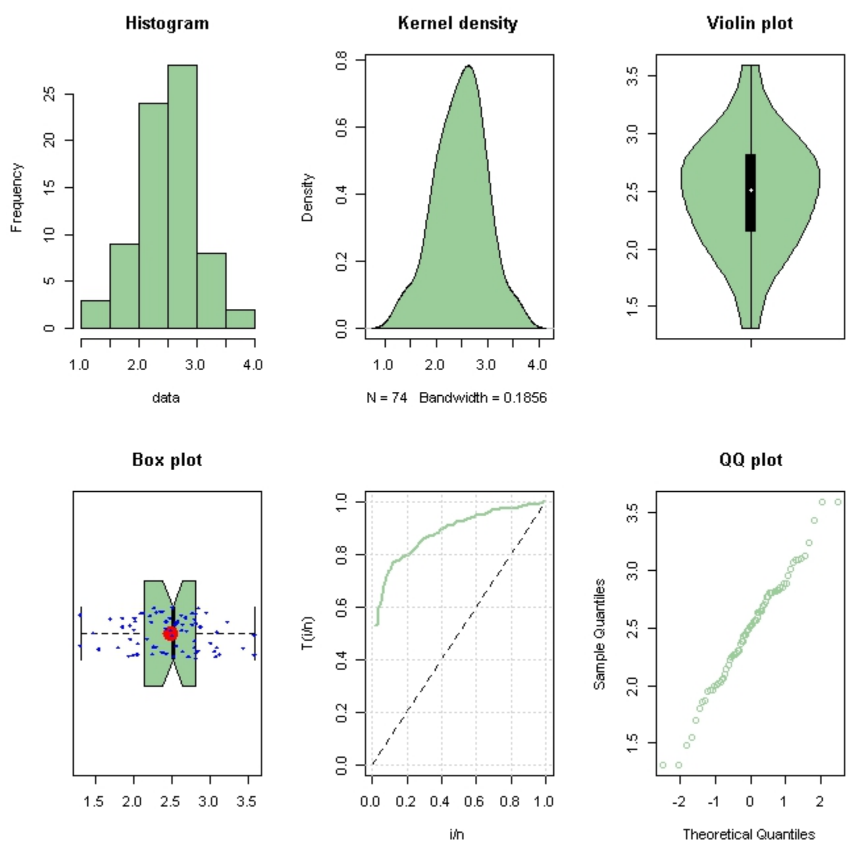

FIGURE 3. Non-parametric plots for carbon fibers dataset

TABLE 6. The MLEs and SEs of the fitting model parameters for the carbon fibers dataset

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$	$SE(\hat{\sigma})$
PLKD	25.85528	97.22275	0.61439	3.56802	203.38838	0.18059
LKD	23.92191	5.64850	-	2.71513	1.28510	-
EP	3.00062	3.86614	-	0.05628	0.34752	-
LL	2.46477	8.63147	-	0.05767	0.83741	-
KIR	11.02911	16.40902	-	3.02972	1.79059	-
GBE-II	34.10474	2.38589	0.96521	33.60585	0.19086	0.60774

TABLE 7. The selection criteria for the carbon fibers data set

Distribution	-2ℓ	AIC	CAIC	HQIC	KS	AD	CVM	p(KS)	p(AD)	p(CVM)
PLKD	**102.45478**	**108.45478**	**108.79764**	111.21214	**0.05472**	**0.23335**	**0.03149**	**0.97967**	**0.97846**	**0.97182**
LKD	105.22846	109.22846	109.39747	**111.0667**	0.06524	0.47991	0.07353	0.91114	0.76684	0.73156
EP	110.81181	114.81181	114.98083	116.65005	0.11126	1.03717	0.14092	0.31885	0.33809	0.41926
LL	108.08447	112.08447	112.25348	113.92271	0.05931	0.56027	0.06756	0.95704	0.68575	0.76812
KIR	111.14937	115.14937	115.31838	116.98761	0.08654	0.92472	0.14027	0.63646	0.39895	0.42141
GBE-II	115.90591	121.90591	122.24877	124.66327	0.09054	1.35718	0.20157	0.57887	0.21455	0.26475

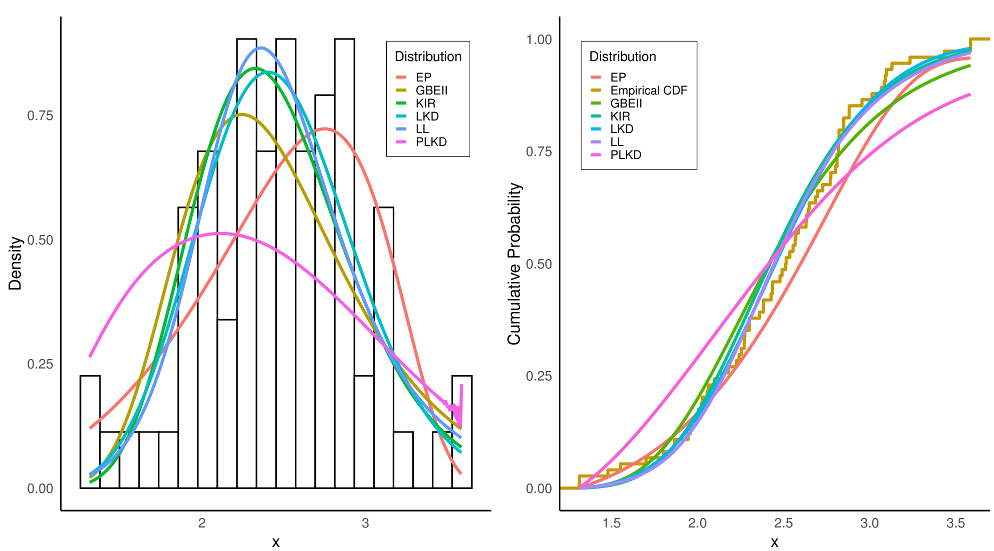

FIGURE 4. The fitted PDFs (on left) and the fitted CDFs (on right) for the carbon fiber data

Table 7 indicates that the PLKD provides the best fit for the carbon fibers data according to all selection criteria except for the HQIC.

6.2. **Private Health Premiums Data.** The second dataset consists of the average annual percent changes in Private Health Insurance Premiums between 1969 and 2007. The data was previously used in [47]. The data set consists of the values: 14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8 and nonparametric plots of the real dataset are presented in Figure 5. The MLEs and their SEs for the distribution parameters are reported in Table 8. The results of the model selection criteria are summarized in Table 9. Figure 6 displays the CDF and PDF plots corresponding to the fitted distributions.

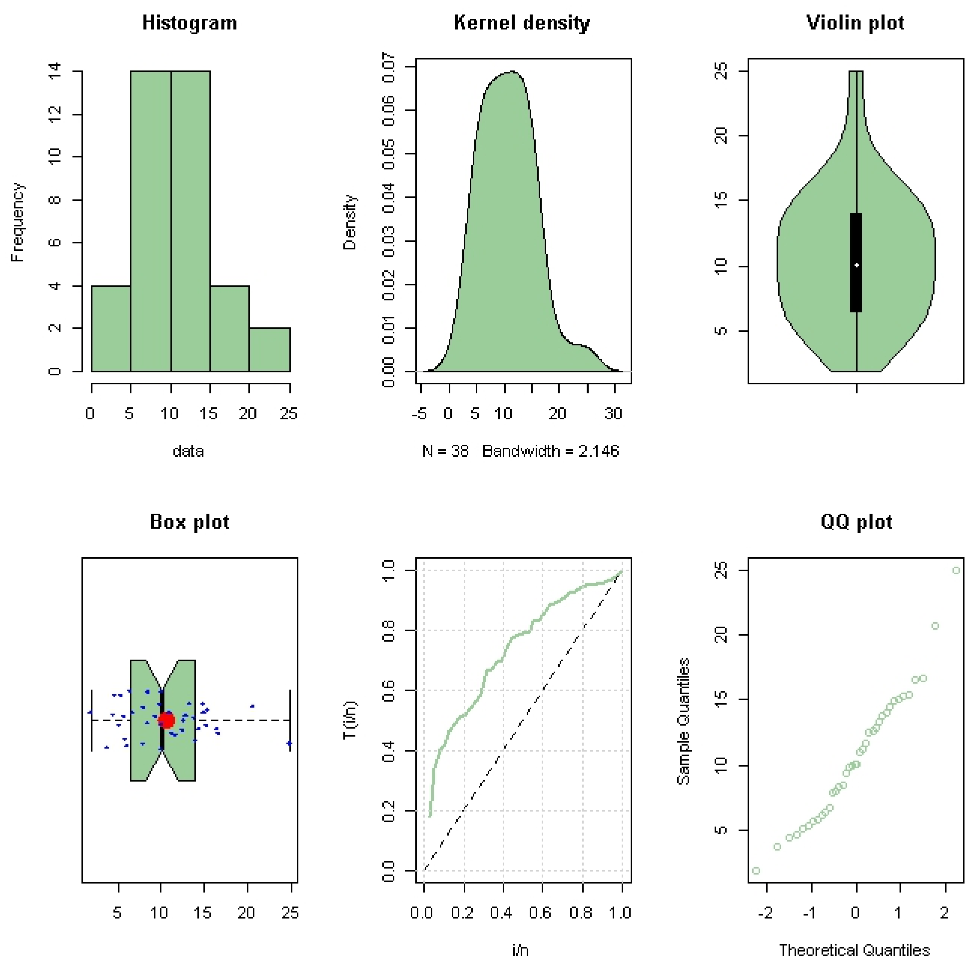

Figure 5. Non-parametric plots for the private health premiums data

Table 8. The MLEs and SEs of the fitting model parameters for the private health premiums data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$SE\,(\hat{\alpha})$	$SE(\hat{\beta})$	$SE\,(\hat{\sigma})$
PLKD	26.135136	90.51995	0.248206	4.508151	228.3323	0.088256
LKD	29.386052	0.136299	-	4.079889	0.022245	-
EP	1.95013	1.57845	-	1.02154	0.19785	-
LL	9.8336612	3.404747	-	0.820664	0.45682	-
KIR	0.883776	40.52798	-	0.195283	9.475106	-
GBE-II	3.28333	0.27737	0.81430	2.68359	0.03881	0.57233

Table 9. The selection criteria for the private health premiums data

Distribution	-2ℓ	AIC	CAIC	HQIC	KS	AD	CVM	p(KS)	p(AD)	p(CVM)
PLKD	**225.25423**	**231.25423**	**231.96011**	**233.00215**	**0.08036**	**0.25355**	**0.04033**	**0.96682**	**0.96836**	**0.93379**
LKD	235.76961	239.76961	240.11246	240.93489	0.17081	1.71987	0.31860	0.21749	0.13194	0.11965
EP	228.72930	232.72930	233.07216	233.89459	0.10161	0.48259	0.05455	0.82758	0.76357	0.85113
LL	228.86607	232.86607	233.20892	234.03135	0.08884	0.47869	0.06903	0.92518	0.76758	0.76047
KIR	247.03894	251.03894	251.38180	252.20422	0.19205	2.13264	0.38955	0.12122	0.07808	0.07654
GBE-II	226.21607	232.21607	232.92195	233.96399	0.08818	0.32824	0.05548	0.92913	0.91502	0.84532

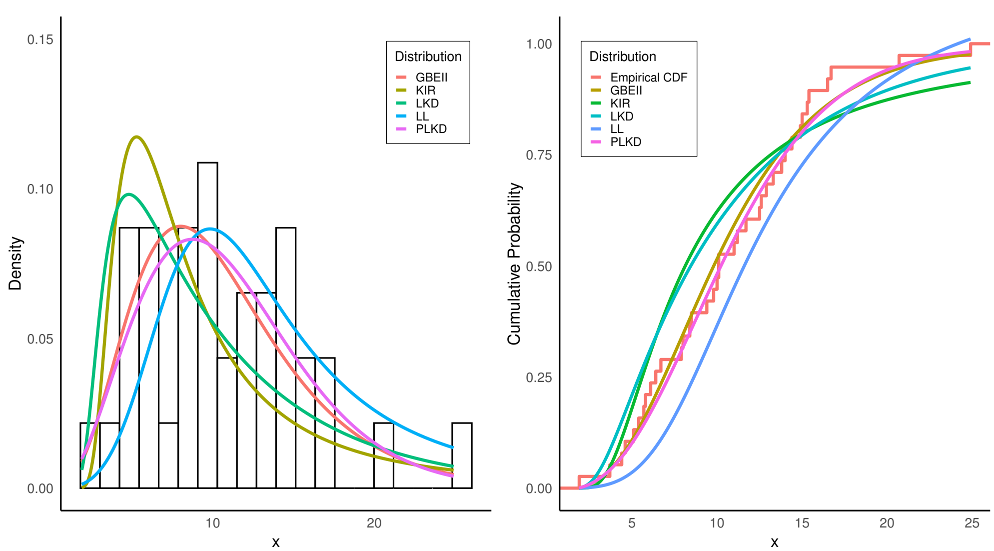

Figure 6. The fitted PDFs (on left) and the fitted CDFs (on right) for the private health premiums data

Table 9 indicates that the PLKD is the best-fitted model for the private health premiums data according to all selection criteria.

7. Conclusion

In this study, we proposed a novel and flexible distribution, the PLKD, as an extension of the LKD. This new model exhibits a wide range of density and hazard function shapes, making it highly adaptable for modeling diverse real-world data types. We established several key statistical properties of the distribution through comprehensive theoretical investigations.

The parameters of the PLKD were estimated using nine classical frequentist methods, and their finite-sample performance was evaluated via extensive MC simulations. Notably, the MSADE method showed superior performance in most settings, while the MLE demonstrated strong consistency, particularly for large samples. On the other hand, the WLSE was less reliable for small sample sizes due to relatively higher bias and error metrics.

The utility of the proposed model was further validated through two real data applications from engineering and health economics. In both cases, the PLKD provided a superior fit compared to several well-established competing models, underscoring its practical relevance.

Given its flexibility, strong inferential properties, and empirical superiority, the PLKD holds great promise as a robust statistical tool for modeling continuous data across various domains, including engineering, medicine, finance, reliability, hydrology, and insurance. Future work may explore its potential in regression frameworks and Bayesian settings to expand its applicability further.

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