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Filters of Sheffer Stroke BL-Algebras Associated With a New Type of Fuzzy Set and Fuzzy Points

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Abstract. (Ultra) filters in Sheffer stroke BL-algebras based on the Jun's J_Y^δ-fuzzy set are studied. The concept of (ultra) $(\in, \in)$-J_Y^δ-fuzzy filters in Sheffer stroke BL-algebras is introduced, and various fundamental properties are investigated. Characterizations of $(\in, \in)$-J_Y^δ-fuzzy filters are discussed, and the relationship between the fuzzy filter and the $(\in, \in)$-J_Y^δ-fuzzy filter is considered. The conditions for the $(\in, \in)$-J_Y^δ-fuzzy filter to become ultra are explored.

1. Introduction

The fuzzy set is introduced by Zadeh [22], and it is an extremely useful mathematical framework for expressing and manipulating uncertainty and ambiguity in data with applications such as decision-making, pattern recognition, image processing, control systems, data mining, expert systems, natural language processing, risk assessment, and decision analysis. Along with the expansion or generalization of fuzzy sets, various types of fuzzy sets have emerged and are being applied in various fields. (see [1, 3, 9–11, 17, 21]). Jun [4] introduced a new type of fuzzy set called J_Y^δ-fuzzy set by introducing the J-operator in relation to the existing fuzzy set, and applied it to BCK-algebras and BCI-algebras (see also [18]). The Sheffer operation (or Sheffer stroke)is a logical operation in Boolean algebra, and it is equivalent to negation of the conjunction operation (AND) in classical logic. It can be observed that the Sheffer stroke is applied in various ways (see [2, 6–8, 12–15, 19]). Jun-Yang-Roh ([5]) applied the J_Y^δ-fuzzy set to Sheffer stroke BL-algebras.

In this paper, we apply Jun's J_Y^δ-fuzzy set to the (ultra) filter of Sheffer stroke BL-algebras. We introduce the concept of an (ultra) $(\in, \in)$-J_Y^δ-fuzzy filter in Sheffer stroke BL-algebras, and

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investigate several properties. We discuss characterizations of $(\in, \in)$-J_Y^δ-fuzzy filters. We consider the relationship between the fuzzy filter and the $(\in, \in)$-J_Y^δ-fuzzy filter. We explore the conditions for the $(\in, \in)$-J_Y^δ-fuzzy filter to become ultra.

2. Preliminaries

Definition 2.1 ([20]). *Let $\mathcal{A} := (D, |)$ be a groupoid. Then the operation $|$ is said to be Sheffer operation or Sheffer stroke if it satisfies:*

(s1) $\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a}$,
(s2) $(\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a}$,
(s3) $\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c}$,
(s4) $(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in D$.

Definition 2.2 ([12]). *An algebra $(X, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Sheffer stroke BL-algebra (briefly, SsBL-algebra) if it satisfies:*

(SBL01) $(X, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
(SBL02) $(X, |)$ *is a groupoid with the Sheffer stroke* $|$,
(SBL03) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $(\mathfrak{a} \wedge \mathfrak{b} = (\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))))$,
(SBL04) $(\forall \mathfrak{a}, \mathfrak{b} \in X)$ $((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \vee (\mathfrak{b}|(\mathfrak{a}|\mathfrak{a})) = 1)$

where $1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of X.

The SsBL-algebra $(X, \vee, \wedge, |, 0, 1)$ is simply denoted by X.

Proposition 2.1 ([12]). *Every SsBL-algebra X satisfies:*

$$\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1, \tag{2.1}$$

$$1|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}, \tag{2.2}$$

$$\mathfrak{a}|(1|1) = 1, \tag{2.3}$$

$$\mathfrak{a} \leq_X \mathfrak{b} \text{ if and only if } \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 1, \tag{2.4}$$

$$\begin{cases} (\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))) \leq_X \mathfrak{a}, \\ (\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))) \leq_X \mathfrak{b}, \end{cases} \tag{2.5}$$

$$\mathfrak{a} \vee \mathfrak{b} = (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}), \tag{2.6}$$

$$\mathfrak{a}|((\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))) = (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|((\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))), \tag{2.7}$$

$$(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = (\mathfrak{b}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a}), \tag{2.8}$$

$$((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \tag{2.9}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Definition 2.3 ([12]). *A filter of X is defined to be a nonempty subset D of X that satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a}, \mathfrak{b} \in D \;\Rightarrow\; (\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}) \in D), \tag{2.10}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in D,\; \mathfrak{a} \leq_X \mathfrak{b} \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.11}$$

Lemma 2.1 ([12]). *A nonempty subset D of X is a filter of X if and only if the following is true.*

$$1 \in D, \tag{2.12}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \in D,\; \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in D \;\Rightarrow\; \mathfrak{b} \in D). \tag{2.13}$$

Definition 2.4 ([12]). *A filter D of X is said to be ultra if it satisfies:*

$$(\forall \mathfrak{a} \in X)(\mathfrak{a} \in D \text{ or } \mathfrak{a}|\mathfrak{a} \in D). \tag{2.14}$$

Lemma 2.2 ([12]). *A filter D of X is ultra if and only if the following is true*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \vee \mathfrak{b} \in D \;\Rightarrow\; \mathfrak{a} \in D \text{ or } \mathfrak{b} \in D). \tag{2.15}$$

Definition 2.5 ([12]). *A fuzzy filter of X is defined to be a fuzzy set ð in X that satisfies:*

$$(\forall \mathfrak{a} \in X)(ð(\mathfrak{a}) \leq ð(1)), \tag{2.16}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(ð(\mathfrak{b}) \geq \min\{ð(\mathfrak{a}), ð(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))\}). \tag{2.17}$$

Given $\mathfrak{a}, \mathfrak{b} \in [0,1]$, we use the notations $\mathfrak{a} \vee \mathfrak{b}$ and $\mathfrak{a} \wedge \mathfrak{b}$ instead of $\max\{\mathfrak{a}, \mathfrak{b}\}$ and $\min\{\mathfrak{a}, \mathfrak{b}\}$, respectively.

The *complement* of a fuzzy set ð in a set X, written by $ð^c$, is defined by

$$ð^c : X \to [0,1],\; \mathfrak{b} \mapsto 1 - ð(\mathfrak{b})).$$

A fuzzy set ð in a set X of the form

$$ð(\mathfrak{b}) := \begin{cases} \tilde{t} \in (0,1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value $\tilde{t}$ and is denoted by $\mathfrak{a}_{\tilde{t}}$.

For a fuzzy set ð in a set X, we say that a fuzzy point $\mathfrak{a}_{\tilde{t}}$ is

(i) *contained* in ð, denoted by $\mathfrak{a}_{\tilde{t}} \in ð$, (see [16]) if $ð(\mathfrak{a}) \geq \tilde{t}$.

(ii) *quasi-coincident* with ð, denoted by $\mathfrak{a}_{\tilde{t}}\, q\, ð$, (see [16]) if $ð(\mathfrak{a}) + \tilde{t} > 1$.

If $\mathfrak{a}_{\tilde{t}}\, \alpha\, ð$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\mathfrak{a}_{\tilde{t}}\, \overline{\alpha}\, ð$.

Given $\tilde{t} \in (0,1]$ and a fuzzy set ð in a set X, consider the following sets

$$(ð, \tilde{t})_{\in} := \{\mathfrak{a} \in X \mid \mathfrak{a}_{\tilde{t}} \in ð\} \text{ and } (ð, \tilde{t})_q := \{\mathfrak{a} \in X \mid \mathfrak{a}_{\tilde{t}}\, q\, ð\}$$

which are called the *level set* and the *q-set* of ð related to $\tilde{t}$, respectively, in X.

In [4], Jun introduced a new type of fuzzy sets generated by the J-operator in the closed interval $[0,1]$. We display the basic notions about the J_Y^{δ}-fuzzy sets.

We use the notation I instead of the closed interval $[0,1]$. Let $\ll$ be the order relation in I^2 defined as follows:

$$(\forall (k,r),(i,j) \in I^2)((k,r) \ll (i,j) \;\Leftrightarrow\; k \leq i,\, r \leq j)$$

Consider a binary operation J_Y in I given as follows:

$$J_Y : I^2 \to I,\; (k,\delta) \mapsto (1-k) \wedge (1-\delta).$$

We will call this binary operation J_Y the *J-operator* in I (see [4]).

Let X be a set. Given a fuzzy set ð in X and $\delta \in I$, let $ð_\delta$ be a mapping defined by

$$ð_\delta : X \to I,\; x \mapsto J_Y(ð(x),\delta).$$

It is clear that $ð_\delta$ is a fuzzy set in X determined by the J-operator and δ. So we can say that $ð_\delta$ is the J_Y^δ*-fuzzy set* of ð in X (see [4]).

Let ð be a fuzzy set in X, $\delta \in I$ and $\tilde{t} \in I \setminus \{0\}$. Given a J_Y^δ-fuzzy set $ð_\delta$, we consider the sets:

$$ð_\delta(\in,\tilde{t}) := \{x \in X \mid J_Y(ð(x),\delta) \leq \tilde{t}\},$$
$$ð_\delta(q,\tilde{t}) := \{x \in X \mid J_Y(ð(x),\delta) < 1-\tilde{t}\},$$

which is called the $Y_\in$*-set* and Y_q*-set* of $ð_\delta$, respectively, related to $\tilde{t}$. We call $\tilde{t}$ the *level degree* of $ð_\delta$.

3. (Ultra) $(\in,\in)$-J_Y^δ-filters

In what follows, let X be an SsBL-algebra and $ð_\delta$ be the J_Y^δ-fuzzy set of a fuzzy set ð in X, where $\delta \in I \setminus \{0,1\}$, unless otherwise specified.

Given a fuzzy point $x_{\tilde{t}}$ and a fuzzy set ð in X, we say that

- $x_{\tilde{t}} \in ð_\delta$ if $J_Y(ð(x),\delta) \leq \tilde{t}$,
- $x_{\tilde{t}}\, q\, ð_\delta$ if $J_Y(ð(x),\delta) + \tilde{t} < 1$,

For every $\alpha \in \{\in, q\}$, if $x_{\tilde{t}}\alpha\, ð_\delta$ is not true, we denote it as $x_{\tilde{t}}\,\overline{\alpha}\, ð_\delta$.

We can observe that

$$ð_\delta(\in,\tilde{t}) := \{x \in X \mid x_{\tilde{t}} \in ð_\delta\} \text{ and } ð_\delta(q,\tilde{t}) := \{x \in X \mid x_{\tilde{t}}\, q\, ð_\delta\}.$$

Definition 3.1. *A fuzzy set ð in X is called a J_Y^δ-fuzzy filter of X if it satisfies:*

$$(\forall x \in X)(J_Y(ð(1),\delta) \leq J_Y(ð(x),\delta)), \tag{3.1}$$

$$(\forall x,y \in X)(J_Y(ð(y),\delta) \leq J_Y(ð(x),\delta) \vee J_Y(ð(x|(y|y)),\delta)). \tag{3.2}$$

Example 3.1. *Consider a set $X := \{\imath_0, \imath_1, \imath_a, \imath_b, \imath_c, \imath_d, \imath_e, \imath_f\}$ with the Hasse diagram, the Sheffer stroke $|$, and binary operations $\vee$ and $\wedge$ given by Figure 1, Table 1, Table 2 and Table 3, respectively.*

Figure 1. Hasse Diagram

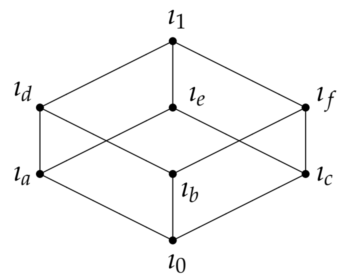

Table 1. Cayley table of the Sheffer stroke |

$\mid$	ι_0	ι_a	ι_b	ι_c	ι_d	ι_e	ι_f	ι_1
ι_0	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1
ι_a	ι_1	ι_f	ι_1	ι_1	ι_f	ι_f	ι_1	ι_f
ι_b	ι_1	ι_1	ι_e	ι_1	ι_e	ι_1	ι_e	ι_e
ι_c	ι_1	ι_1	ι_1	ι_d	ι_1	ι_d	ι_d	ι_d
ι_d	ι_1	ι_f	ι_e	ι_1	ι_c	ι_f	ι_e	ι_c
ι_e	ι_1	ι_f	ι_1	ι_d	ι_f	ι_b	ι_d	ι_b
ι_f	ι_1	ι_1	ι_e	ι_d	ι_e	ι_d	ι_a	ι_a
ι_1	ι_1	ι_f	ι_e	ι_d	ι_c	ι_b	ι_a	ι_1

Table 2. Cayley table of the binary operation $\vee$

$\vee$	ι_0	ι_a	ι_b	ι_c	ι_d	ι_e	ι_f	ι_1
ι_0	ι_0	ι_a	ι_b	ι_c	ι_d	ι_e	ι_f	ι_1
ι_a	ι_a	ι_a	ι_d	ι_e	ι_d	ι_e	ι_1	ι_1
ι_b	ι_b	ι_d	ι_b	ι_f	ι_d	ι_1	ι_f	ι_1
ι_c	ι_c	ι_e	ι_f	ι_c	ι_1	ι_e	ι_f	ι_1
ι_d	ι_d	ι_d	ι_d	ι_1	ι_d	ι_1	ι_1	ι_1
ι_e	ι_e	ι_e	ι_1	ι_e	ι_1	ι_e	ι_1	ι_1
ι_f	ι_f	ι_1	ι_f	ι_f	ι_1	ι_1	ι_f	ι_1
ι_1	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1	ι_1

TABLE 3. Cayley table of the binary operation $\wedge$

$\wedge$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$
$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_0$
$\imath_a$	$\imath_0$	$\imath_a$	$\imath_0$	$\imath_0$	$\imath_a$	$\imath_a$	$\imath_0$	$\imath_a$
$\imath_b$	$\imath_0$	$\imath_0$	$\imath_b$	$\imath_0$	$\imath_b$	$\imath_0$	$\imath_b$	$\imath_b$
$\imath_c$	$\imath_0$	$\imath_0$	$\imath_0$	$\imath_c$	$\imath_0$	$\imath_c$	$\imath_c$	$\imath_c$
$\imath_d$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_0$	$\imath_d$	$\imath_a$	$\imath_b$	$\imath_d$
$\imath_e$	$\imath_0$	$\imath_a$	$\imath_0$	$\imath_c$	$\imath_a$	$\imath_e$	$\imath_c$	$\imath_e$
$\imath_f$	$\imath_0$	$\imath_0$	$\imath_b$	$\imath_c$	$\imath_b$	$\imath_c$	$\imath_f$	$\imath_f$
$\imath_1$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$

Then $(X, \vee, \wedge, |, \imath_0, \imath_1)$ *is an SsBL-algebra (see [12]). Define a fuzzy set* ð *in* X *as follows:*

$$ð : X \to [0,1],\ x \mapsto \begin{cases} 0.74 & \text{if } x = \imath_1, \\ 0.59 & \text{if } x = \imath_d, \\ 0.48 & \text{if } x \in \{\imath_a, \imath_e\}, \\ 0.36 & \text{otherwise.} \end{cases}$$

If we take $\delta := 0.44$*, then the* J_Y^δ*-fuzzy set* $ð_\delta$ *of* ð *is given by Table 4.*

TABLE 4. Tabular representation of $ð_\delta$

$w \in X$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$
$ð_\delta(w)$	0.64	0.52	0.64	0.64	0.41	0.52	0.64	0.26

Then ð *is a* J_Y^δ*-fuzzy filter of* X *for* $\delta = 0.44$.

Theorem 3.1. *A fuzzy set* ð *in* X *is a* J_Y^δ*-fuzzy filter of* X *if and only if the following conditions are valid.*

$$ð_\delta(\in, \tilde{t}) \neq \emptyset \ \Rightarrow\ 1 \in ð_\delta(\in, \tilde{t}), \tag{3.3}$$

$$(\forall x, y \in X)(\forall \tilde{t}, \tilde{s} \in I \setminus \{0\})(x_{\tilde{t}} \in ð_\delta,\, x|(y|y)_{\tilde{s}} \in ð_\delta \vdash y_{\tilde{t}\vee\tilde{s}} \in ð_\delta). \tag{3.4}$$

Proof. Assume that ð is a J_Y^δ-fuzzy filter of X. Let $x, y \in X$ and $\tilde{t}, \tilde{s} \in I \setminus \{0\}$. If $ð_\delta(\in, \tilde{t}) \neq \emptyset$, then there exists $\mathfrak{a} \in ð_\delta(\in, \tilde{t})$, and so $J_Y(ð(\mathfrak{a}), \delta) \leq \tilde{t}$. It follows from (3.1) that $J_Y(ð(1), \delta) \leq J_Y(ð(\mathfrak{a}), \delta) \leq \tilde{t}$. Hence $1 \in ð_\delta(\in, \tilde{t})$. Let $x_{\tilde{t}} \in ð_\delta$ and $x|(y|y)_{\tilde{s}} \in ð_\delta$. Then $J_Y(ð(x), \delta) \leq \tilde{t}$ and $J_Y(ð(x|(y|y)), \delta) \leq \tilde{s}$. Thus $J_Y(ð(y), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta) \leq \tilde{t} \vee \tilde{s}$, and so $y_{\tilde{t}\vee\tilde{s}} \in ð_\delta$. This proves (3.4).

Conversely, suppose that ð satisfies (3.3) and (3.4) for all $x, y \in X$ and $\tilde{t}, \tilde{s} \in I \setminus \{0\}$. If (3.1) is not valid, then there exists $\mathfrak{b} \in X$ and $\tilde{t} \in I \setminus \{0\}$ such that $J_Y(ð(1), \delta) > \tilde{t} \geq J_Y(ð(\mathfrak{b}), \delta)$. Then $\mathfrak{b} \in ð_\delta(\in, \tilde{t})$, and so $1 \in ð_\delta(\in, \tilde{t})$ by (3.3). This is a contradiction. Hence (3.1) is valid. Suppose that (3.2) is not valid. Then $J_Y(ð(\mathfrak{b}), \delta) > \tilde{s} \geq J_Y(ð(\mathfrak{a}), \delta) \vee J_Y(ð(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})), \delta)$ for some $\mathfrak{a}, \mathfrak{b} \in X$ and $\tilde{s} \in I \setminus \{0\}$. Then $\mathfrak{a} \in ð_\delta(\in, \tilde{s})$ and $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in ð_\delta(\in, \tilde{s})$, that is, $\mathfrak{a}_{\tilde{s}} \in ð_\delta$ and $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})_{\tilde{s}} \in ð_\delta$. But $\mathfrak{b}_{\tilde{s}} \overline{\in} ð_\delta$ which shows that ð does not satisfy (3.4). This is a contradiction, and thus (3.2) is valid. Therefore ð is a J_Y^{δ}-fuzzy filter of X. □

In the sense of Theorem 3.1, the J_Y^{δ}-fuzzy filter may also be called the $(\in,\in)$-J_Y^{δ}-fuzzy filter.

Corollary 3.1. *A fuzzy set ð in X is an $(\in,\in)$-J_Y^{δ}-fuzzy filter of X if and only if the nonempty $Y_{\in}$-set $ð_{\delta}(\in,\tilde{t})$ is a filter of X for all $\tilde{t}\in I\setminus\{0\}$.*

Proposition 3.1. *Every $(\in,\in)$-J_Y^{δ}-fuzzy filter ð of X satisfies:*

$$z_{\tilde{t}}\in ð_{\delta},\ (z|q)_{\tilde{s}}\in ð_{\delta}\vdash(((x|(y|y))|(y|y))|(x|x))_{\tilde{t}\vee\tilde{s}}\in ð_{\delta}, \tag{3.5}$$

where $q=((y|(x|x))|(y|(x|x)))$ for all $x,y,z\in X$ and $\tilde{t},\tilde{s}\in I\setminus\{0\}$.

Proof. Let $x,y,z\in X$ and $\tilde{t},\tilde{s}\in I\setminus\{0\}$ be such that $z_{\tilde{t}}\in ð_{\delta}$ and $(z|q)_{\tilde{s}}\in ð_{\delta}$ where $q=((y|(x|x))|(y|(x|x)))$. Then $(z|((y|(x|x))|(y|(x|x))))_{\tilde{s}}\in ð_{\delta}$, and so

$$\begin{aligned}(((x|(y|y))|(y|y))|(x|x))_{\tilde{t}\vee\tilde{s}}&=(((y|(x|x))|(x|x))|(x|x))_{\tilde{t}\vee\tilde{s}}\\&=(y|(x|x))_{\tilde{t}\vee\tilde{s}}\in ð_{\delta}\end{aligned}$$

by (2.8), (2.9) and (3.4). This completes the proof. □

Proposition 3.2. *Every $(\in,\in)$-J_Y^{δ}-fuzzy filter ð of X satisfies:*

$$z_{\tilde{t}}\in ð_{\delta},\ (z|p)_{\tilde{s}}\in ð_{\delta}\vdash((x|(y|y))|(y|y))_{\tilde{t}\vee\tilde{s}}\in ð_{\delta}, \tag{3.6}$$

where $p=(((y|(x|x))|(x|x))|((y|(x|x))|(x|x)))$ for all $x,y,z\in X$ and $\tilde{t},\tilde{s}\in I\setminus\{0\}$.

Proof. Let $x,y,z\in X$ and $\tilde{t},\tilde{s}\in I\setminus\{0\}$. If $z_{\tilde{t}}\in ð_{\delta}$ and $(z|p)_{\tilde{s}}\in ð_{\delta}$, then

$$\begin{aligned}&(z|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))))_{\tilde{s}}\\&=(z|(((y|(x|x))|(x|x))|((y|(x|x))|(x|x))))_{\tilde{s}}\\&=(z|p)_{\tilde{s}}\in ð_{\delta}\end{aligned}$$

by (2.8). It follows from (3.4) that $((x|(y|y))|(y|y))_{\tilde{t}\vee\tilde{s}}\in ð_{\delta}$. □

Proposition 3.3. *Every $(\in,\in)$-J_Y^{δ}-fuzzy filter ð of X satisfies:*

$$(\forall x,y\in X)\ (J_Y(ð(x\vee(x|x)),\delta)=J_Y(ð(1),\delta)). \tag{3.7}$$

Proof. Let ð be an $(\in,\in)$-J_Y^{δ}-fuzzy filter of X. Then

$$\begin{aligned}J_Y(ð(x\vee(x|x)),\delta)&=J_Y(ð((x|((x|x)|(x|x)))|((x|x)|(x|x))),\delta)\\&=J_Y(ð((x|x)|x),\delta)\\&=J_Y(ð(x|(x|x)),\delta)\\&=J_Y(ð(1),\delta)\end{aligned}$$

for all $x\in X$ by (s1), (s2), (2.1) and (2.6). □

Proposition 3.4. *Every* $(\in, \in)$-J_Y^δ*-fuzzy filter* ð *of* X *satisfies:*

$$(\forall x, y, z \in X)\left(\begin{array}{l} (z|((y|(x|x))|(y|(x|x))))_{\tilde{t}} \in ð_\delta,\, z_{\tilde{s}} \in ð_\delta \\ \vdash (((x|(y|y))|(y|y))|(x|x))_{\tilde{t}\vee\tilde{s}} \in ð_\delta \end{array}\right), \tag{3.8}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} (x|((y|(z|z))|(y|(z|z))))_{\tilde{t}} \in ð_\delta,\, (x|(y|y))_{\tilde{s}} \in ð_\delta \\ \vdash (x|(z|z))_{\tilde{t}\vee\tilde{s}} \in ð_\delta \end{array}\right). \tag{3.9}$$

Proof. Let $x, y, z \in X$ be such that $(z|((y|(x|x))|(y|(x|x))))_{\tilde{t}} \in ð_\delta$ and $z_{\tilde{s}} \in ð_\delta$. Then

$$(((x|(y|y))|(y|y))|(x|x))_{\tilde{t}\vee\tilde{s}} = (((y|(x|x))|(x|x))|(x|x))_{\tilde{t}\vee\tilde{s}} = (y|(x|x))_{\tilde{t}\vee\tilde{s}} \in ð_\delta$$

by (2.8), (2.9) and (3.4). Hence (3.8) is valid. If $(x|((y|(z|z))|(y|(z|z))))_{\tilde{t}} \in ð_\delta$ and $(x|(y|y))_{\tilde{s}} \in ð_\delta$, then

$$((x|(y|y))|((x|(z|z))|(x|(z|z))))_{\tilde{t}} = (x|((y|(z|z))|(y|(z|z))))_{\tilde{t}} \in ð_\delta$$

by (2.7), and so $(x|(z|z))_{\tilde{t}\vee\tilde{s}} \in ð_\delta$ by (3.4). Hence (3.9) is valid. □

Lemma 3.1. *A fuzzy set* ð *in* X *is an* $(\in, \in)$-J_Y^δ*-fuzzy filter of* X *if and only if it satisfies:*

$$(\forall x, y, z \in X)\left(\begin{array}{l} x|((y|(z|z))|(y|(z|z))) = 1 \;\Rightarrow \\ J_Y(ð(z), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \end{array}\right). \tag{3.10}$$

Proof. Assume that ð is an $(\in, \in)$-J_Y^δ-fuzzy filter of X, and let $x, y, z \in X$ be such that $x|((y|(z|z))|(y|(z|z))) = 1$. Then

$$\begin{aligned} J_Y(ð(x), \delta) &= J_Y(ð(x), \delta) \vee J_Y(ð(1), \delta) \\ &= J_Y(ð(x), \delta) \vee J_Y(ð(x|((y|(z|z))|(y|(z|z)))), \delta) \\ &\geq J_Y(ð(y|(z|z)), \delta), \end{aligned}$$

by (3.1) and (3.2), and so

$$\begin{aligned} J_Y(ð(z), \delta) &\leq J_Y(ð(y), \delta) \vee J_Y(ð(y|(z|z)), \delta) \\ &\leq J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \end{aligned}$$

by (3.2).

Conversely, suppose the condition (3.10) is valid. Using (2.3) induces

$$x|((x|(1|1))|(x|(1|1))) = 1$$

for all $x \in X$. It follows from (3.10) that

$$J_Y(ð(1), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(x), \delta) = J_Y(ð(x), \delta).$$

For every $x, y \in X$, we have

$$x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 1$$

by the combination of (s1), (s3) and (2.1). Hence

$$J_Y(ð(y), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta)$$

by (3.10). Therefore ð is an $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

Theorem 3.2. *A fuzzy set ð in X is an $(\in, \in)$-J_Y^δ-fuzzy filter of X if and only if it satisfies:*

$$x_{\tilde{t}} \in ð_\delta,\ y_{\tilde{s}} \in ð_\delta \vdash ((x|y)|(x|y))_{\tilde{t}\vee\tilde{s}} \in ð_\delta, \tag{3.11}$$

$$x \leq_X y,\ x_{\tilde{t}} \in ð_\delta \vdash y_{\tilde{t}} \in ð_\delta \tag{3.12}$$

for all $x, y \in X$ and $\tilde{t}, \tilde{s} \in I \setminus \{0\}$.

Proof. Let $x, y \in X$ and $\tilde{t}, \tilde{s} \in I \setminus \{0\}$. Suppose that ð is an $(\in, \in)$-J_Y^δ-fuzzy filter of X. Suppose that $x \leq_X y$ and $x_{\tilde{t}} \in ð_\delta$. Then $x|(y|y) = 1$ by (2.4), and $J_Y(ð(x), \delta) \leq \tilde{t}$. Hence

$$\begin{aligned} J_Y(ð(y), \delta) &\leq J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta) \\ &= J_Y(ð(x), \delta) \vee J_Y(ð(1), \delta) \\ &= J_Y(ð(x), \delta) \leq \tilde{t} \end{aligned}$$

by (3.1) and (3.2). Thus $y_{\tilde{t}} \in ð_\delta$, which shows that (3.12) is valid. Let $x_{\tilde{t}} \in ð_\delta$ and $y_{\tilde{s}} \in ð_\delta$. Then $J_Y(ð(x), \delta) \leq \tilde{t}$ and $J_Y(ð(y), \delta) \leq \tilde{s}$. If we take $\mathfrak{a} := (x|y)$, then

$$\begin{aligned} &x|((y|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{a})))|(y|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{a})))) \\ &= x|((y|\mathfrak{a})|(y|\mathfrak{a})) = (\mathfrak{a}|\mathfrak{a})|\mathfrak{a} = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1 \end{aligned}$$

by (s1), (s2), (s3) and (2.1). It follows from Lemma 3.1 that

$$J_Y(ð((x|y)|(x|y)), \delta) = J_Y(ð(\mathfrak{a}|\mathfrak{a}), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \leq \tilde{t} \vee \tilde{s},$$

that is, $((x|y)|(x|y))_{\tilde{t}\vee\tilde{s}} \in ð_\delta$ and (3.11) is valid.

Conversely, suppose that ð satisfies (3.11) and (3.12). If (3.1) is false, then $J_Y(ð(1), \delta) > \tilde{t} \geq J_Y(ð(\mathfrak{b}), \delta)$ for some $\mathfrak{b} \in X$. Hence $\mathfrak{b}_{\tilde{t}} \in ð_\delta$. Since $\mathfrak{b} \leq 1$ by (2.3) and (2.4), it follows from (3.12) that $1_{\tilde{t}} \in ð_\delta$. Thus $J_Y(ð(1), \delta) \leq \tilde{t}$, a contradiction. Therefore $J_Y(ð(1), \delta) \leq J_Y(ð(x), \delta)$ for all $x \in X$. Suppose that

$$J_Y(ð(y), \delta) > J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta)$$

for some $x, y \in X$. If we take $\tilde{s} := J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta)$, then $x_{\tilde{s}} \in ð_\delta$ and $x|(y|y)_{\tilde{s}} \in ð_\delta$. Hence $((x|(x|(y|y)))|(x|(x|(y|y))))_{\tilde{s}} \in ð_\delta$ by (3.11), and so $y_{\tilde{s}} \in ð_\delta$ by (2.5) and (3.12). Hence $J_Y(ð(y), \delta) \leq \tilde{s} = J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta)$ Therefore ð is an $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

Theorem 3.3. *If a fuzzy set ð in X satisfies $ð(x) \leq \delta$ for all $x \in X$, then it is an $(\in, \in)$-J_Y^δ-fuzzy filter of X.*

Proof. This is straightforward. □

Let ð be a fuzzy set in X. If there exists $\mathfrak{b} \in X$ such that $ð(\mathfrak{b}) > \delta$, then ð may not be an $(\in, \in)$-J_Y^δ-fuzzy filter of X as seen in the following example.

Example 3.2. *Consider the SsBL-algebra* $(X, \vee, \wedge, |, \imath_0, \imath_1)$ *in Example 3.1. Let* ð *be a fuzzy set in* X *defined by Table 5.*

Table 5. Tabular representation of ð

$w \in X$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$
ð(w)	0.59	0.48	0.41	0.38	0.27	0.34	0.48	0.63

If we take $\delta := 0.47$*, then* ð$(w) > \delta$ *for* $w \in \{\imath_0, \imath_a, \imath_f, \imath_1\}$*. We can observe that*

$$\begin{aligned} J_Y(ð(\imath_c), 0.47) &= J_Y(0.38, 0.47) = 0.53 \nleq 0.52 \\ &= J_Y(0.48, 0.47) \wedge J_Y(ð(0.48, 0.47) \\ &= J_Y(ð(\imath_a), 0.47) \wedge J_Y(ð(\imath_a|(\imath_c|\imath_c)), 0.47). \end{aligned}$$

Hence ð *is not an* $(\in, \in)$*-*J_Y^δ*-fuzzy filter of* X*.*

Theorem 3.4. *Every fuzzy filter is an* $(\in, \in)$*-*J_Y^δ*-fuzzy filter.*

Proof. Let ð be a fuzzy filter of X. Then $ð^c(1) \leq ð^c(x)$ and

$$\begin{aligned} ð^c(y) = 1 - ð(y) &\leq 1 - \min\{ð(x), ð(x|(y|y))\} \\ &= \max\{1 - ð(x), 1 - ð(x|(y|y))\} \\ &= ð^c(x) \vee ð^c(x|(y|y)) \end{aligned}$$

for all $x, y \in X$. Hence $J_Y(ð(1), \delta) = ð^c(1) \wedge (1 - \delta) \leq ð^c(x) \wedge (1 - \delta) = J_Y(ð(x), \delta)$ and

$$\begin{aligned} J_Y(ð(y), \delta) = ð^c(y) \wedge (1 - \delta) &\leq (ð^c(x) \vee ð^c(x|(y|y))) \wedge (1 - \delta) \\ &= (ð^c(x) \wedge (1 - \delta)) \vee (ð^c(x|(y|y)) \wedge (1 - \delta)) \\ &= J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta) \end{aligned}$$

for all $x, y \in X$. Therefore ð is an $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

The converse of Theorem 3.4 may not be true as seen in the following example.

Example 3.3. *Consider the SsBL-algebra* $(X, \vee, \wedge, |, \imath_0, \imath_1)$ *in Example 3.1, and define a fuzzy set* ð *in* X *by Table 6.*

Table 6. Tabular representation of ð

$w \in X$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$
ð(w)	0.23	0.52	0.28	0.34	0.68	0.52	0.23	0.73

Then ð *is an* $(\in, \in)$*-*J_Y^δ*-fuzzy filter of* X *for* $\delta := 0.51$*. But* ð *is not a fuzzy filter of* X *since* ð$(\imath_b) = 0.28 \ngeq 0.34 = \min\{ð(\imath_c), ð(\imath_c|(\imath_b|\imath_b))\}$*.*

In the sense of Theorem 3.4 and Example 3.3, we know that the $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter is a generalization of the fuzzy filter.

Given a fuzzy set ð in X, consider the following condition:

$$(\forall x \in X)\, (J_Y(\eth(x),\delta) = J_Y(\eth(1),\delta) \text{ or } J_Y(\eth(x|x),\delta) = J_Y(\eth(1),\delta)). \tag{3.13}$$

We can see that there is an $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter ð of X that does not meet the condition (3.13). In fact, if we get the $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter ð of X for $\delta = 0.44$ in Example 3.1, then $J_Y(\eth(\imath_a),\delta) \neq J_Y(\eth(\imath_1),\delta) \neq J_Y(\eth(\imath_a|\imath_a),\delta)$.

Definition 3.2. *An $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter ð of X is said to be ultra if it satisfies the condition* (3.13).

Example 3.4. *Consider the SsBL-algebra $(X, \vee, \wedge, |, \imath_0, \imath_1)$ in Example 3.1, and define a fuzzy set ð in X by Table 7.*

Table 7. Tabular representation of ð

$w \in X$	$\imath_0$	$\imath_a$	$\imath_b$	$\imath_c$	$\imath_d$	$\imath_e$	$\imath_f$	$\imath_1$
$\eth(w)$	0.33	0.77	0.33	0.33	0.77	0.77	0.33	0.77

For every $\delta \in I \setminus \{0,1\}$, we know that $J_Y(\eth(\imath_a),\delta) = J_Y(\eth(\imath_d),\delta) = J_Y(\eth(\imath_e),\delta) = 0.33 \wedge (1-\delta) = J_Y(\eth(\imath_1),\delta)$, and $J_Y(\eth(\imath_0|\imath_0),\delta) = 0.33 \wedge (1-\delta) = J_Y(\eth(\imath_1),\delta)$, $J_Y(\eth(\imath_b|\imath_b),\delta) = J_Y(\eth(\imath_e),\delta) = 0.33 \wedge (1-\delta) = J_Y(\eth(\imath_1),\delta)$, $J_Y(\eth(\imath_c|\imath_c),\delta) = J_Y(\eth(\imath_d),\delta) = 0.33 \wedge (1-\delta) = J_Y(\eth(\imath_1),\delta)$ and $J_Y(\eth(\imath_f|\imath_f),\delta) = J_Y(\eth(\imath_a),\delta) = 0.33 \wedge (1-\delta) = J_Y(\eth(\imath_1),\delta)$. Hence ð is an ultra $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter of X.

We explore conditions for an $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter to be ultra.

Theorem 3.5. *An $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter ð of X is ultra if and only if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} J_Y(\eth(x),\delta) \neq J_Y(\eth(1),\delta) \Rightarrow \\ J_Y(\eth(x|(y|y)),\delta) = J_Y(\eth(1),\delta) \end{array}\right). \tag{3.14}$$

Proof. Assume that ð is an ultra $(\in, \in)$-$J_Y^{\eth}$-fuzzy filter of X. Let $x, y \in X$ be such that $J_Y(\eth(x),\delta) \neq J_Y(\eth(1),\delta)$. Then $J_Y(\eth(x|x),\delta) = J_Y(\eth(1),\delta)$ by (3.13). Since $(x|x)|((x|(y|y))|(x|(y|y))) = 1$ by (s1), (s3), (2.1) and (2.3), it follows from (3.2) that

$$\begin{aligned} J_Y(\eth(1),\delta) &= J_Y(\eth(1),\delta) \vee J_Y(\eth(1),\delta) \\ &= J_Y(\eth(x|x),\delta) \vee J_Y(\eth((x|x)|((x|(y|y))|(x|(y|y)))),\delta) \\ &\geq J_Y(\eth(x|(y|y)),\delta). \end{aligned}$$

The combination of this and (3.1) induces $J_Y(\eth(x|(y|y)),\delta) = J_Y(\eth(1),\delta)$.

Conversely, let ð be an $(\in, \in)$-J_Y^δ-fuzzy filter of X satisfying (3.14). Suppose that $J_Y(ð(x), \delta) \neq J_Y(ð(1), \delta)$ for all $x \in X$. Then $J_Y(ð(1|1), \delta) \neq J_Y(ð(1), \delta)$, and so

$$\begin{aligned} J_Y(ð(x|x), \delta) &= J_Y(ð(1|((x|x)|(x|x)), \delta) \\ &= J_Y(ð(((x|x)|(x|x))|1), \delta) \\ &= J_Y(ð(x|1), \delta) \\ &= J_Y(ð(x|((1|1)|(1|1))), \delta) \\ &= J_Y(ð(1), \delta) \end{aligned}$$

by (s1), (s2), (2.2) and (3.14). Therefore ð is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

Theorem 3.6. *An $(\in, \in)$-J_Y^δ-fuzzy filter ð of X is ultra if and only if it satisfies:*

$$(\forall x, y \in X)\ (J_Y(ð(x \vee y), \delta) \geq J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta))\,. \tag{3.15}$$

Proof. Assume that ð is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X. For every $x, y \in X$, if $J_Y(ð(x), \delta) = J_Y(ð(1), \delta)$ or $J_Y(ð(x), \delta) = J_Y(ð(1), \delta)$, then

$$J_Y(ð(x \vee y), \delta) \geq J_Y(ð(1), \delta) = J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta).$$

Suppose that $J_Y(ð(x), \delta) \neq J_Y(ð(1), \delta)$ and $J_Y(ð(x), \delta) \neq J_Y(ð(1), \delta)$. Then $J_Y(ð(x|(y|y)), \delta) = J_Y(ð(1), \delta)$ and $J_Y(ð(y|(x|x)), \delta) = J_Y(ð(1), \delta)$ by Theorem 3.5. Hence

$$\begin{aligned} J_Y(ð(x \vee y), \delta) &= J_Y(ð(1), \delta) \vee J_Y(ð(x \vee y), \delta) \\ &= J_Y(ð(x|(y|y)), \delta) \vee J_Y(ð((x|(y|y))|(y|y)), \delta) \\ &\geq J_Y(ð(y), \delta) \end{aligned}$$

by (2.6), (3.1) and (3.2). Similarly, we get $J_Y(ð(x \vee y), \delta) \geq J_Y(ð(x), \delta)$. Thus $J_Y(ð(x \vee y), \delta) \geq J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta)$.

Conversely, let ð be an $(\in, \in)$-J_Y^δ-fuzzy filter of X satisfying (3.15). Then

$$\begin{aligned} J_Y(ð(1), \delta) &= J_Y(ð(x|(x|x)), \delta) \\ &= J_Y(ð((x|((x|x)|(x|x)))|((x|x)|(x|x))), \delta) \\ &= J_Y(ð(x \vee (x|x)), \delta) \\ &\geq J_Y(ð(x), \delta) \vee J_Y(ð(x|x), \delta) \end{aligned}$$

by (s2), (2.1), (2.6) and (3.15). It follows from (3.1) that $J_Y(ð(x), \delta) = J_Y(ð(1), \delta)$ or $J_Y(ð(x|x), \delta) = J_Y(ð(x), \delta)$. Therefore ð is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

Theorem 3.7. *A fuzzy set ð in X is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X if and only if the nonempty $Y_\in$-set $ð_\delta(\in, \tilde{t})$ is an ultra filter of X for all $\tilde{t} \in I \setminus \{0\}$.*

Proof. Assume that ð is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X. Let $\tilde{t} \in I \setminus \{0\}$ and $x, y \in X$ be such that $x \vee y \in ð_\delta(\in, \tilde{t})$. Then $J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \leq J_Y(ð(x \vee y), \delta) \leq \tilde{t}$, and so $J_Y(ð(x), \delta) \leq \tilde{t}$ or

$J_Y(ð(y), \delta) \leq \tilde{t}$. This shows that $x \in ð_\delta(\in, \tilde{t})$ or $y \in ð_\delta(\in, \tilde{t})$. Hence $ð_\delta(\in, \tilde{t})$ is an ultra filter of X by Lemma 2.2 and Corollary 3.1.

Conversely, suppose that the nonempty $Y_\in$-set $ð_\delta(\in, \tilde{t})$ is an ultra filter of X for all $\tilde{t} \in I \setminus \{0\}$. Then $ð$ is an $(\in, \in)$-J_Y^δ-fuzzy filter of X by Corollary 3.1. Let $\tilde{t} := J_Y(ð(x \vee y), \delta) \leq \tilde{t}$. Then $x \vee y \in ð_\delta(\in, \tilde{t})$, and so $x \in ð_\delta(\in, \tilde{t})$ or $y \in ð_\delta(\in, \tilde{t})$ by Lemma 2.2. Hence $J_Y(ð(x), \delta) \leq \tilde{t} = J_Y(ð(x \vee y), \delta)$ or $J_Y(ð(y), \delta) \leq \tilde{t} = J_Y(ð(x \vee y), \delta)$, and thus

$$J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \leq J_Y(ð(x \vee y), \delta).$$

It follows from Theorem 3.6 that $ð$ is an ultra $(\in, \in)$-J_Y^δ-fuzzy filter of X. □

Theorem 3.8. *If $ð$ is an $(\in, \in)$-J_Y^δ-fuzzy filter of X, then the nonempty Y_q-set $ð_\delta(q, \tilde{t})$ of $ð_\delta$ is a filter of X for all $\tilde{t} \in I \setminus \{0\}$. Moreover, if $ð$ is ultra, then so is $ð_\delta(q, \tilde{t})$.*

Proof. Suppose that $ð$ be an $(\in, \in)$-J_Y^δ-fuzzy filter of X. Let $\tilde{t} \in I \setminus \{0\}$ be such that $ð_\delta(q, \tilde{t}) \neq \emptyset$. Then there exists $\mathfrak{a} \in ð_\delta(q, \tilde{t})$, and so $J_Y(ð(\mathfrak{a}), \delta) < 1 - \tilde{t}$. Using (3.1), we have $J_Y(ð(1), \delta) \leq J_Y(ð(\mathfrak{a}), \delta) < 1 - \tilde{t}$ which implies that $1 \in ð_\delta(q, \tilde{t})$. Let $x, y \in X$ be such that $x \in ð_\delta(q, \tilde{t})$ and $x|(y|y) \in ð_\delta(q, \tilde{t})$. Then $J_Y(ð(x), \delta) < 1 - \tilde{t}$ and $J_Y(ð(x|(y|y)), \delta) < 1 - \tilde{t}$. It follows from (3.2) that

$$J_Y(ð(y), \delta) \leq J_Y(ð(x), \delta) \vee J_Y(ð(x|(y|y)), \delta) < 1 - \tilde{t}.$$

Hence $y \in ð_\delta(q, \tilde{t})$, which shows that $ð_\delta(q, \tilde{t})$ is a filter of X by Lemma 2.1. Suppose $ð$ is ultra. If $x \vee y \in ð_\delta(q, \tilde{t})$, then

$$J_Y(ð(x), \delta) \vee J_Y(ð(y), \delta) \leq J_Y(ð(x \vee y), \delta) < 1 - \tilde{t}$$

by Theorem 3.6. Hence $J_Y(ð(x), \delta) < 1 - \tilde{t}$ or $J_Y(ð(y), \delta) < 1 - \tilde{t}$, that is, $x \in ð_\delta(q, \tilde{t})$ or $y \in ð_\delta(q, \tilde{t})$. Therefore $ð_\delta(q, \tilde{t})$ is an ultra filter of X Lemma 2.2. □

4. Conclusions

To develop a new type of fuzzy set, Jun introduced the concept of the J-operator in unit interval $[0, 1]$, and then formed a J_Y^δ-fuzzy set. He (together with Yang and Roh) applied it to the subalgebras and ideals of BCK/BCI-algebras and to the quasi-subalgebras of Sheffer stroke BL-algebras. In this paper, we have studied filter theory in Sheffer stroke BL-algebras with the J_Y^δ-fuzzy set. We have introduced the concept of an (ultra) $(\in, \in)$-J_Y^δ-fuzzy filter in Sheffer stroke BL-algebras, and have investigated various fundamental properties. We have discussed characterizations of $(\in, \in)$-J_Y^δ-fuzzy filters, and have considered the relationship between the fuzzy filter and the $(\in, \in)$-J_Y^δ-fuzzy filter. We have explored the conditions for the $(\in, \in)$-J_Y^δ-fuzzy filter to become ultra.

Based on the ideas and results of this paper, we will apply the J_Y^δ-fuzzy set to several logical algebras. We will also explore the possibility of fusion with soft sets.

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References

[1] M. Anbarloei, Interval Valued (α, β)-Fuzzy Hyperideals in Krasner (m, N)-Hyperrings, J. Algebr. Hyperstruct. Log. Algebr. 4 (2023), 13–25. https://doi.org/10.61838/kman.jahla.4.1.2.

[2] I. Chajda, R. Halaš, H. Länger, Operations and Structures Derived From Non-Associative MV-Algebras, Soft Comput. 23 (2018), 3935–3944. https://doi.org/10.1007/s00500-018-3309-4.

[3] Y. Jun, Positive Implicative Be-Filters of BE-Algebras Based on Lukasiewicz Fuzzy Sets, J. Algebr. Hyperstruct. Log. Algebr. 4 (2023), 1–11. https://doi.org/10.61838/kman.jahla.4.1.1.

[4] Y. Jun, A New Form of Fuzzy Set and Its Application in BCK-Algebras and BCI-Algebras, Ann. Fuzzy Math. Inf. 26 (2023), 165–175. https://doi.org/10.30948/afmi.2023.26.2.165.

[5] Y. Jun, E. Yang, E. Hwan Roh, Quasi-Subalgebras of Sheffer Stroke BL-Algebras Associated with Fuzzy Points and a New Type of Fuzzy Set, J. Algebr. Hyperstruct. Log. Algebr. 5 (2024), 19–32. https://doi.org/10.61838/kman.jahla.5.2.3.

[6] T. Oner, T. Katican, A. Borumand Saeid, On Sheffer Stroke Be-Algebras, Discuss. Math. - Gen. Algebr. Appl. 42 (2022), 293–314. https://doi.org/10.7151/dmgaa.1391.

[7] T. Katican, T. Oner, A. Borumand Saeid, Stabilizers on Sheffer Stroke BL-Algebras, Honam Math. J. 44 (2022), 78–97. https://doi.org/10.5831/HMJ.2022.44.1.78.

[8] T. Katican, T. Oner, A. Rezaei, F. Smarandache, Neutrosophic -Structures Applied to Sheffer Stroke Bl-Algebras, Comput. Model. Eng. Sci. 129 (2021), 355–372. https://doi.org/10.32604/cmes.2021.016996.

[9] S. Mirvakili, H. Naraghi, M. Shirvani, Connections Between Fuzzy Multi-Groups and Fuzzy Multi-Graphs, J. Algebr. Hyperstruct. Log. Algebr. 2 (2021), 47–60. https://doi.org/10.52547/hatef.jahla.2.2.47.

[10] S. Mirvakili, H. Naraghi, M. Dehghanizadeh, H. Naraghi, On the Equivalence of Sequences Dependent on Fuzzy Ideals in the Bci-Algebra, J. Algebr. Hyperstruct. Log. Algebr. 3 (2022), 65–73. https://doi.org/10.52547/hatef.jahla.3.4.5.

[11] A. Namdar, Fuzzy N-Fold Fantastic Filters on Hoop Algebras, J. Algebr. Hyperstruct. Log. Algebr. 4 (2023), 189–203. https://doi.org/10.61838/kman.jahla.4.2.12.

[12] T. Oner, T. Katican, A. Borumand Saeid, (Fuzzy) Filters of Sheffer Stroke BL-Algebras, Kragujev. J. Math. 47 (2023), 39–55. https://doi.org/10.46793/kgjmat2301.039o.

[13] T. Oner, T. Kalkan, A.B. Saeid, Class of Sheffer Stroke Bck-Algebras, An. Univ. "Ovidius" Constanta - Ser. Mat. 30 (2022), 247–269. https://doi.org/10.2478/auom-2022-0014.

[14] T. Oner, T. Kalkan, A. Borumand Saeid, (Anti) Fuzzy Ideals of Sheffer Stroke BCK-Algebras, J. Algebr. Syst. 11 (2023), 105–135. https://doi.org/10.22044/jas.2022.11512.1582.

[15] T. Oner, I. Senturk, The Sheffer Stroke Operation Reducts of Basic Algebras, Open Math. 15 (2017), 926–935. https://doi.org/10.1515/math-2017-0075.

[16] P. Pao-Ming, L. Ying-Ming, Fuzzy Topology. I. Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence, J. Math. Anal. Appl. 76 (1980), 571–599. https://doi.org/10.1016/0022-247x(80)90048-7.

[17] G. Rezaei, Y. Jun, Commutative Ideals of BCI-Algebras Based on Łukasiewicz Fuzzy Sets, J. Algebr. Hyperstructures Log. Algebr. 3 (2022), 25–36. https://doi.org/10.52547/hatef.jahla.3.4.2.

[18] E.H. Roh, E. Yang, Y.B. Jun, Ideals of Bck-Algebras and Bci-Algebras Based on a New Form of Fuzzy Set, Eur. J. Pure Appl. Math. 16 (2023), 2009–2024. https://doi.org/10.29020/nybg.ejpam.v16i4.4933.

[19] I. Senturk, A View on State Operators in Sheffer Stroke Basic Algebras, Soft Comput. 25 (2021), 11471–11484. https://doi.org/10.1007/s00500-021-06059-8.

[20] H.M. Sheffer, A Set of Five Independent Postulates for Boolean Algebras, Trans. Amer. Math. Soc. 14 (1913), 481–488.

[21] X. Xin, F. Xiao, X. Yang, L-fuzzy Algebraic Substructure, J. Algebr. Hyperstruct. Log. Algebr. 4 (2023), 17–35. https://doi.org/10.61838/kman.jahla.4.2.2.

[22] L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.