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EXISTENCE OF QUASILINEAR NEUTRAL IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACE

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Abstract. In this paper, we devoted to study the existence of mild solutions for quasilinear impulsive integrodifferential equation in Banach spaces. The results are established by using Hausdorff's measure of noncompactness and the fixed point theorems. Application is provided to illustrate the theory.

1. **Introduction**

In various fields of engineering and physics, many problems that are related to linear viscoelasticity, nonlinear elasticity have mathematical models and are described by the problems of differential or integral equations or integrodifferential equations. Our work centers on the problems described by the integrodifferential models. It is important to note that when we describe the systems which are functions of space and time by partial differential equations, in some situations, such a formulation may not accurately model the physical system because, while describing the system as a function at a given time, it may fail to take into account the effect of past history. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention during the last few decades [1, 2, 3]. A good guide to the literature for neutral functional differential equations is the book by Hale and Verduyn Lunel [4] and the references therein. The existence of solution to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [5, 6]. Byszewski and Lakshmikanthan [7] proved an existence and uniqueness of solutions of a nonlocal Cauchy problem in Banach spaces. Ntouyas and Tsamatos [8] studied the existence for semilinear evolution equations with nonlocal conditions. The problem of existence of solutions of evolution equations in Banach space has been studied by several authors [9, 10].

However, one may easily visualize that abrupt changes such as shock, harvesting and disasters may occur in nature. These phenomena are short time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equation [11, 12, 13] is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)),\ i = 1,\ 2, \ldots, m,$$

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is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Liu [14] discussed the iterative methods for the solution of impulsive functional differential systems.

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operators theory, theory of differential and integral equations and others [15]. There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\sigma(X)$ of a bounded set X in a metric space is defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of sets of diameter smaller than r. The Hausdorff measure of noncompactness $\chi(X)$ defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. There exist many formulae on $\chi(X)$ in various spaces [15, 18].

Let $\mathbb{E}$ be a Banach space and $\mathbb{F}$ be a subspace of $\mathbb{E}$. Let $\chi_{\mathbb{E}}(X)$, $\chi_{\mathbb{F}}(X)$, $\sigma_{\mathbb{E}}(X)$, $\sigma_{\mathbb{F}}(X)$ denote Hausdorff and Kuratowski measures in spaces $\mathbb{E},\mathbb{F}$, respectively. Then, for any bounded $X \subset \mathbb{F}$ we have $\chi_{\mathbb{E}}(X) \leq \chi_{\mathbb{F}}(X) \leq \sigma_{\mathbb{F}}(X) = \sigma_{\mathbb{E}}(X) \leq 2\chi_{\mathbb{E}}(X)$. The notion of a measure of weak compactness was introduced by De Blasi [16] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. Several authors have studied the measures of noncompactness in Banach spaces [17, 18, 19]. Motivated by [9, 15, 20, 21], in this paper, we study the existence results for quasilinear equation represented by first-order neutral integrodifferential equations using the semigroup theory and the measure of noncompactness.

2. **Preliminaries**

We consider the quasilinear integrodifferential equations with impulsive and nonlocal condition of the form

$$\begin{aligned} &\frac{d}{dt}\Big[x(t) + e\Big(t, x(t), \int_0^t k(t,s,x(s))ds\Big)\Big] + A(t,x(t))x(t) \\ &= f(t,x(t)) + \int_0^t g(t,s,x(s))ds, \ t \in [0,b], \ t \neq t_k, \qquad (1) \end{aligned}$$

$$x(0) + h(x) = x_0, \qquad (2)$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1,2,3,\dots,n, \qquad (3)$$

where $A : [0,b] \times X \to X$ is a continuous function in Banach space X, $x_0 \in X$, $f : [0,b] \times X \to X$, $g : \Lambda \times X \to X$, $h : \mathcal{PC}([0,b],X) \to X$, $e : [0,b] \times X \times X \to X$, $k : \Lambda \times X \to X$ and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, for all $k = 1,2,\dots,m$; $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = b$; constitutes an impulsive condition. Here $\Lambda = \{(t,s) : 0 \leq s \leq t \leq b\}$.

Let X be a Banach space with norm $||\cdot||$. Let $\mathcal{PC}([0,b],X)$ consist of functions u from $[0,b]$ into X, such that $x(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $x(t_i^+)$ exists, for $i = 1,2,3,\dots,n$. Evidently $\mathcal{PC}([0,b],X)$ is a Banach space with the norm

$$\|x\|_{\mathcal{PC}} = \sup_{t\in[0,b]} \|x(t)\|,$$

and denoted $\mathcal{L}([0,b],X)$ by the space of X-valued Bochner integrable functions on $[0,b]$ with the form

$$\|x\|_{\mathcal{L}} = \int_0^b \|x(t)\|dt.$$

The Hausdorff's measure of noncompactness χ_Y is defined by

$$\chi(\mathcal{B}) = \inf\{r > 0,\ \mathcal{B} \text{ can be covered by finite number of balls with radii r}\},$$

for bounded set $\mathcal{B}$ in a Banach space Y.

Lemma 2.1 [15]. Let Y be a real Banach space and $\mathcal{B}, E \subseteq Y$ be bounded, with the following properties:

(i) $\mathcal{B}$ is precompact if and only if $\chi_X(\mathcal{B}) = 0$.
(ii) $\chi_Y(\mathcal{B}) = \chi_Y(\bar{\mathcal{B}}) = \chi_Y(con\mathcal{B})$, where $\bar{\mathcal{B}}$ and con$B$ mean the closure and convex hull of B respectively.
(iii) $\chi_Y(\mathcal{B}) \le \chi_Y(E)$, where $\mathcal{B} \subseteq E$.
(iv) $\chi_Y(\mathcal{B} + E) \le \chi_Y(\mathcal{B}) + \chi_Y(E)$, where $\mathcal{B} + E = \{x + y :\ x \in \mathcal{B},\ y \in E\}$.
(v) $\chi_Y(\mathcal{B} \cup E) \le \max\{\chi_Y(\mathcal{B}),\ \chi_Y(E)\}$.
(vi) $\chi_Y(\lambda B) \le |\lambda|\chi_Y(B)$, for any $\lambda \in R$.
(vii) If the map $\mathcal{F} : D(\mathcal{F}) \subseteq Y \to Z$ is Lipschitz continuous with constant r, then $\chi_Z(\mathcal{F}\mathcal{B}) \le r\chi_Y(\mathcal{B})$, for any bounded subset $\mathcal{B} \subseteq D(\mathcal{F})$, where Z be a Banach space.
(viii) $\chi_Y(B) = \inf\{d_Y(\mathcal{B}, E); E \subseteq Y$ is precompact$\}$ $= \inf\{d_Y(B, E); E \subseteq Y$ is finite valued$\}$, where $d_Y(\mathcal{B}, E)$ means the non-symmetric (or symmetric) Hausdorff distance between $\mathcal{B}$ and E in Y.
(ix) If $\{\mathbb{W}_n\}_{n=1}^{+\infty}$ is decreasing sequence of bounded closed nonempty subsets of Y and $\lim\limits_{n\to\infty} \chi_Y(\mathbb{W}_n) = 0$, then $\bigcap\limits_{n=1}^{+\infty} \mathbb{W}_n$ is nonempty and compact in Y.

The map $\mathcal{F} : \mathbb{W} \subseteq Y \to Y$ is said to be a χ_Y-contraction if there exists a positive constant $r < 1$ such that $\chi_Y(\mathcal{F}(\mathcal{B})) \le r\chi_Y(\mathcal{B})$ for any bounded closed subset $\mathcal{B} \subseteq \mathbb{W}$, where Y is a Banach space.

Lemma 2.2 (Darbo-Sadovskii [15]). If $\mathbb{W} \subseteq Y$ is bounded closed and convex, the continuous map $\mathcal{F} : \mathbb{W} \to \mathbb{W}$ is a χ_Y-contraction, the map $\mathcal{F}$ has atleast one fixed point in $\mathbb{W}$.

We denote by χ the Hausdorff's measure of noncompactness of X and also denote χ_c by the Hausdorff's measure of noncompactness of $\mathcal{PC}([0,b],X)$.

Before we prove the existence results, we need the following Lemmas.

Lemma 2.3 [22] If $\mathbb{W} \subseteq \mathcal{PC}([0,b],X)$ is bounded, then $\chi(\mathbb{W}(t)) \le \chi_c(\mathbb{W})$, for all $t \in [0,b]$, where $\mathbb{W}(t) = \{u(t); u \in \mathbb{W}\} \subseteq X$. Furthermore if $\mathbb{W}$ is equicontinuous on $[0,b]$, then $\chi(\mathbb{W}(t))$ is continuous on $[0,b]$ and $\chi_c(\mathbb{W}) = \sup\{\chi(\mathbb{W}(t)),\ t \in [0,b]\}$.

Lemma 2.4 [22, 23]. If $\{u_n\}_{n=1}^{\infty} \subset \mathcal{L}^1([0,b],X)$ is uniformly integrable, then the function $\chi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and

$$\chi\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \le 2\int_0^t \chi(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{4}$$

Lemma 2.5 If $\mathbb{W} \subseteq \mathcal{PC}([0,b],X)$ is bounded and equicontinuous, then $\chi(\mathbb{W}(t))$ is continuous and

$$\chi\Big(\int_0^t \mathbb{W}(s)ds\Big) \leq \int_0^t \chi(\mathbb{W}(s))ds, \text{ for all } t \in [0,b], \tag{5}$$

where $\displaystyle\int_0^t \mathbb{W}(s)ds = \Big\{ \int_0^t u(s)ds : u \in \mathbb{W}\Big\}$.

The C_0 semigroup $U_u(t,s)$ is said to be equicontinuous if $(t,s) \to \{U_u(t,s)u(s) : u \in B\}$ is equicontinuous for $t > 0$, for all bounded set $\mathcal{B}$ in X. The following lemma is obvious.

Lemma 2.6 If the evolution family $\{U_u(t,s)\}_{0\leq s\leq t\leq b}$ is equicontinuous and $\eta \in \mathcal{L}([0,b],\mathcal{R}^+)$, then the set $\Big\{ \displaystyle\int_0^t U_u(t,s)u(s)ds,\ ||u(s)|| \leq \eta(s), \text{ for a.e } s \in [0,b]\Big\}$, is equicontinuous for $t \in [0,b]$.

We know that, for any fixed $u \in \mathcal{PC}([0,b],X)$ there exist a unique continuous function $U_u : [0,b] \times [0,b] \to \mathcal{B}(X)$ defined on $[0,b] \times [0,b]$ such that

$$U_u(t,s) = I + \int_s^t A_u(w)U_u(w,s)dw, \tag{6}$$

where $\mathcal{B}(X)$ denote the Banach space of bounded linear operators from X to X with the norm $||\mathcal{F}|| = \sup\{||\mathcal{F}u|| : ||u|| = 1\}$, and I stands for the identity operator on X, $A_u(t) = A(t,u(t))$, we have

$$U_u(t,t) = I,\ U_u(t,s)U_u(s,r) = U_u(t,r), \quad (t,s,r) \in [0,b] \times [0,b] \times [0,b],$$
$$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s), \quad \text{for almost all} \quad t,s \in [0,b].$$

3. The Existence of Mild Solution

Definition 3.1 A function $x \in \mathcal{PC}([0,b],X)$ is said to be a mild solution of $(1)-(3)$ if it satisfies the integral equation

$$\begin{aligned} x(t) &= U_x(t,0)x_0 - U_x(t,0)h(x) + U_x(t,0)e(0,x(0),0) - e\left(t,x(t),\int_0^t k(t,s,x(s))ds\right) \\ &+ \int_0^t A(s,x(s))U_x(t,s)e\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\ &+ \int_0^t U_x(t,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau\Big]ds + \sum_{0<t_k<t} U_x(t,t_k)I_k(x(t_k)),\ 0 \leq t \leq b. \end{aligned}$$

In this paper, we denote $\mathcal{M}_0 = \sup\{\|U_x(t,s)\| : (t,s) \in [0,b] \times [0,b]\}$, for all $x \in X$. Without loss of generality, we let $x_0 = 0$.

Assume the following conditions:

- (H_1) The evolution family $\{U_x(t,s)\}_{0\leq s\leq t\leq b}$ generated by $A(t,x(t))$ is equicontinuous and $||U_x(t,s)|| \leq \mathcal{M}_0$, for almost $t,s \in [0,b]$.
- (H_2) (a) The function $h : \mathcal{PC}([0,b] \times X) \to X$ is continuous and compact.
  (b) There exists $\mathcal{N}_0 > 0$ such that $||h(x)|| \leq \mathcal{N}_0$, for all $u \in \mathcal{PC}([0,b];X)$.

(H_3) (i) The nonlinear function $f : [0, b] \times X \to X$ satisfies the Carathèodory-type conditions; that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous, for a.e $t \in [a, b]$.

(ii) There exists a function $\alpha \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every $x \in X$, we have

$$\|f(t, x)\| \leq \alpha(t)(1 + \|x\|), \ a.e \ t \in [0, b].$$

(iii) There exists a function $m_f \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that, for every bounded $\mathcal{K} \subset X$, we have

$$\chi(f(t, \mathcal{K})) \leq m_f(t)\chi(\mathcal{K}), \ a.e \ t \in [0, b].$$

(H_4) (i) The nonlinear function $g : [0, b]\times[0, b]\times X \to X$ satisfies the Carathèodory-type conditions; i.e., $g(\cdot, \cdot, x)$ is measurable, for all $x \in X$ and $g(t, s, \cdot)$ is continuous for a.e $t \in [a, b]$.

(ii) There exist two functions $\beta_1 \in \mathcal{L}([0, b], \mathcal{R}^+)$ and $\beta_2 \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every $x \in X$, we have

$$\|g(t, s, x(s))\| \leq \beta_1(t)\beta_2(s)(1 + \|x(s)\|), \ a.e \ t \in [0, b].$$

(iii) There exist functions $m_g, n_g \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that, for every bounded $\mathcal{K} \subset X$, we have

$$\chi(g(t, s, \mathcal{K})) \leq m_g(t)n_g(s)\chi(\mathcal{K}), \ a.e \ t \in [0, b].$$

Assume that the finite bound of $\int_0^t m_g(s)ds$ is $\mathcal{G}_0$.

(H_5) (i) The function $e : [0, b] \times X \times X \to X$ satisfies the Carathèodory-type conditions; that is, $e(\cdot, x, x_1)$ is measurable, for all $x, x_1 \in X$ and $e(t, \cdot, \cdot)$ is continuous, for a.e $t \in [0, b]$.

(ii) There exists a function $\gamma \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every $x, x_1 \in X$, we have

$$\|e(t, x, x_1)\| \leq \gamma(t)(1 + \|x\|) + \|x_1\|, \ a.e \ t \in [0, b].$$

(iii) The nonlinear function $q : [0, b]\times[0, b]\times X \to X$ satisfies the Caratheodory-type conditions; i.e. $k(\cdot, \cdot, x)$ is measurable, for all $x \in X$ and $k(t, s, \cdot)$ is continuous, for a.e $t \in [0, b]$.

(iv) There exist two functions $\omega_1 \in \mathcal{L}([0, b], \mathcal{R}^+)$ and $\omega_2 \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every $x \in X$, we have

$$\|k(t, s, x(s))\| \leq \omega_1(t)\omega_2(s)(1 + \|x(s)\|), \ a.e \ t \in [0, b].$$

(v) There exists a function $\eta \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every $x, x_1 \in X$, we have

$$\|A(t, x(t))e(t, x, x_1)\| \leq \eta(t)\|e(t, x, x_1)\|, \ a.e \ t \in [0, b].$$

(vi) There exists a function $m_e \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that, for every bounded $\mathcal{K}, \ \mathcal{K}_1 \subset X$, we have

$$\chi(e(t, \mathcal{K}, \mathcal{K}_1)) \leq m_e(t)(\chi(\mathcal{K}) + \varphi(\mathcal{K}_1)), \ a.e \ t \in [0, b].$$

Assume that the finite bound of $\int_0^t m_e(s)ds$ is $\mathcal{G}_1$.

(vii) There exist functions $m_k, n_k \in \mathcal{L}([0, b], \mathcal{R}^+)$ such that for every bounded $\mathcal{K} \subset X$, we have

$$\chi(k(t, s, \mathcal{K})) \leq m_k(t)n_k(s)\chi(\mathcal{K}), \ a.e \ t \in [0, b].$$

Assume that the finite bound of $\int_0^t m_k(s)ds$ is $\mathcal{G}_2$.

(H_6) For every $t \in [0,b]$ and there exist positive constants $\mathcal{N}_1$ and $\mathcal{N}_2$, the scalar equation

$$\begin{aligned} m(t) &= \mathcal{M}_0\mathcal{N}_0 + \gamma_1(1+m(s)) + \mathcal{M}_0\gamma_0 + \mathcal{M}_0 C_1\omega(t)(1+m(s)) + \gamma(t)C_1\int_0^t \eta(t)\omega_1(s)ds \\ &\quad + \mathcal{M}_0\int_0^t \Big[\alpha(s)(1+m(s))ds + C_0\int_0^t \beta_1(s)(1+m(s))ds + \sum_{k=1}^n d_k\Big], \end{aligned}$$

where $C_0 = \displaystyle\int_0^s \beta(t)dt$.

(H_7) $I_k : X \to X$ is continuous. There exist constants $d_k > 0 \quad k = 1,2,3,\ldots,n$ such that

$$\|I_k(x(t_k))\| \le \sum_{k=1}^n d_k, \quad where, \ k = 1,2,3,\ldots,n.$$

For any bounded subset $\mathcal{K} \subset X$, and there is a constant $l_k > 0$ such that

$$\chi(I_k(\mathcal{K})) \le \sum_{k=1}^n l_i\chi(\mathcal{K}), \quad k = 1,2\ldots,n.$$

Theorem: 3.1. *If assumptions $(H_1) - (H_7)$ holds, then the quasilinear neutral impulsive problem $(1) - (3)$ has at least one mild solution.*

Proof. Let $m(t)$ be a solution of the scalar equation

$$\begin{aligned} m(t) &= \mathcal{M}_0\mathcal{N}_0 + \gamma_1(1+m(s)) + \mathcal{M}_0\gamma_0 + \mathcal{M}_0 C_1\omega(t)(1+m(s)) + \gamma(t)C_1\int_0^t \eta(t)\omega_1(s)ds \\ &\quad + \mathcal{M}_0\int_0^t \Big[\alpha(s)(1+m(s))ds + C_0\int_0^t \beta_1(s)(1+m(s))ds + \sum_{k=1}^n d_k\Big]. \end{aligned} \tag{7}$$

Let us assume that the finite bound of $\displaystyle\int_0^t \beta_2(s)ds$ is C_0, for $t \in [0,b]$. Consider the map $\mathcal{F} : \mathcal{PC}([0,b],X) \to \mathcal{PC}([0,b],X)$ defined by

$$\begin{aligned} (\mathcal{F}x)(t) &= U_x(t,0)h(x) + U_x(t,0)e(0,x(0),0) - e\left(t,x(t),\int_0^t k(t,s,x(s))ds\right) \\ &\quad + \int_0^t A(s,x(s))U_x(t,s)e\left(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\ &\quad + \int_0^t U_x(t,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\ &\quad + \sum_{0<t_k<t} U_x(t,t_k)I_k(x(t_k)), \quad 0 \le t \le b, \quad \text{for all } x \in \mathcal{PC}([0,b],X). \end{aligned} \tag{8}$$

Let us take $\mathbb{W}_0 = \{x \in \mathcal{PC}([0,b],X), \ ||x(t)|| \le m(t), \ \text{for all } t \in [0,b]\}$. Then $\mathbb{W}_0 \subseteq \mathcal{PC}([0,b];X)$ is bounded and convex. We define $\mathbb{W}_1 = \overline{con}\ K(\mathbb{W}_0)$, where $\overline{con}$ means the closure of the convex hull in $\mathcal{PC}([0,b],X)$. As $U_x(t,s)$ is equicontinuous, h is compact and $\mathbb{W}_0 \subseteq \mathcal{PC}([0,b],X)$ is bounded, due to Lemma 2.6 and using the assumptions, $\mathbb{W}_1 \subseteq \mathcal{PC}([0,b],X)$ is bounded closed convex nonempty and equicontinuous on $[0,b]$.

For any $x \in \mathcal{F}(\mathbb{W}_0)$, we know

$$\begin{aligned}
\|x(t)\| \;\le\; & \|U_x(t,0)h(x)\| + \|U_x(t,0)e(0,x(0),0)\| + \|e\left(t,x(t),\int_0^t k(t,s,x(s))ds\right)\| \\
& + \int_0^t \|A(t,x(t))U_x(t,s)e\left(t,x(t),\int_0^s k(s,\tau,x(\tau))d\tau\right)\|ds \\
& + \int_0^t \|U_x(t,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(s))d\tau\Big]\|ds + \sum_{0<t_k<t}\|U_x(t,t_k)I_i(x(t_k))\| \\
\le\; & \mathcal{M}_0\mathcal{N}_0 + \mathcal{M}_0\gamma_0 + \gamma_1(1+\|x\|) + \int_0^t k(t,s,x(s))ds \\
& + \mathcal{M}_0\eta(t)\int_0^t \|e\left(t,x(t),\int_0^s k(s,\tau,x(\tau))d\tau\right)\|ds \\
& + \mathcal{M}_0\Big[\int_0^t ||f(s,x(s))||ds + \int_0^t\int_0^s ||g(s,\tau,x(\tau))||d\tau ds\Big] + \mathcal{M}_0\sum_{k=1}^n \|I_k(x(t_k))\| \\
\le\; & \mathcal{M}_0\mathcal{N}_0 + \mathcal{M}_0\gamma_0 + \gamma_1(1+\|x\|) + \omega_1(t)\int_0^t \omega_2(s)(1+\|x\|)ds \\
& + \mathcal{M}_0\Big[\int_0^t \big[\eta(s)\gamma(s)ds + \int_0^s \omega_1(s)\omega_2(\tau)dsd\tau\big](1+\|x\|)\Big]
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{M}_0\int_0^t \alpha(s)(1+\|x(s)\|)ds + \mathcal{M}_0\int_0^t\int_0^s \beta_1(s)\beta_2(\tau)(1+\|x(\tau)\|)d\tau ds + \mathcal{M}_0\sum_{k=1}^n d_k \\
\le\; & \mathcal{M}_0\mathcal{N}_0 + \gamma_1(1+m(s)) + \mathcal{M}_0\gamma_0 + \mathcal{M}_0 C_1\omega(t)(1+m(s)) + \eta(t)\gamma(t)C_1\int_0^t \omega_1(s)ds \\
& + \mathcal{M}_0\int_0^t \Big[\alpha(s)(1+m(s))ds + C_0\int_0^t \beta_1(s)(1+m(s))ds + \sum_{k=1}^n d_k\Big] \\
=\; & m(t).
\end{aligned}$$

It follows that $\mathbb{W}_1 \subset \mathbb{W}_0$. We define $\mathbb{W}_{n+1} = \overline{con}\ \mathcal{F}(\mathbb{W}_n)$, for $n = 1, 2, 3, \cdots$. From above we know that $\{\mathbb{W}_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0,b]$ and nonempty subsets in $\mathcal{PC}([0,b],X)$.

Now for $n \ge 1$ and $t \in [0,b]$, $\mathbb{W}_n(t)$ and $\mathcal{F}(\mathbb{W}_n(t))$ are bounded subsets of X, hence, for any $\epsilon > 0$, there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{W}_n$, such that (see, e.g. [24], pp.125).

$$
\begin{aligned}
\chi(\mathbb{W}_{n+1}(t)) &= \chi(\mathcal{F}\mathbb{W}_n(t)) \\
&\le 2\chi\Big(e(t,\{x_k(s)\}_{k=1}^{\infty},\int_0^t k(t,s,\{x_k(t)\}_{k=1}^{\infty})ds)\Big) \\
&\quad +2\mathcal{M}_0\eta(t)\int_0^t \chi\Big(e(s,\{x_k(s)\}_{k=1}^{\infty},\int_0^s k(s,\tau,\{x_k(\tau)\}_{k=1}^{\infty})d\tau)\Big)ds \\
&\quad +2\mathcal{M}_0\int_0^t \chi\Big(f(s,\{x_k(s)\}_{k=1}^{\infty})\Big)ds + 4\mathcal{M}_0\int_0^t\int_0^s \chi\Big(g(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty})\Big)d\tau ds \\
&\quad +2\mathcal{M}_0\sum_{i=1}^{n}\chi\Big(I_k(\{u_k(t_k)\}_{k=1}^{\infty})\Big)+\epsilon \\
&\le 2m_e(t)\chi\{x_k(t)\}_{k=1}^{\infty} + 2m_k(t)\int_0^t m_k(s)\chi\{x_k(s)\}_{k=1}^{\infty}ds \\
&\quad +2\mathcal{M}_0\eta(t)\Big[\int_0^t m_e(s)\chi\{x_k(s)\}_{k=1}^{\infty}ds + 2\int_0^t\int_0^s m_k(s)m_k(\tau)\chi\{x_k(\tau)\}_{k=1}^{\infty}d\tau ds\Big] \\
&\quad +2\mathcal{M}_0\int_0^t m_f(s)\chi\Big(\{u_k(s)\}_{k=1}^{\infty}\Big)ds + 4\mathcal{M}_0\int_0^t\int_0^s m_g(s)n_g(\tau)\chi\Big(\{u_k(\tau)\}_{k=1}^{\infty}\Big)d\tau ds \\
&\quad +2\mathcal{M}_0\sum_{i=1}^{n} l_i\chi\Big(\{u_k(t_k)\}_{k=1}^{\infty}\Big)+\epsilon \\
&\le 2\Big[m_e(t)+m_k(t)\mathcal{G}_2+\mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\chi(\mathbb{W}_n(t)) + 2\mathcal{M}_0\Big[\int_0^t\{2\mathcal{G}_2 m_k(s) \\
&\quad +m_f(s)\}\chi(\mathbb{W}_n(s))ds + 2\mathcal{G}_0\int_0^t n_g(s)\chi(\mathbb{W}_n(s))ds\Big] + 2\mathcal{M}_0\sum_{k=1}^{n} l_k\chi(\mathbb{W}_n(t_k))+\epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, it follows that from the above inequality that

$$
\begin{aligned}
\chi(\mathbb{W}_{n+1}(t)) &\le 2\Big[m_e(t)+m_k(t)\mathcal{G}_2+\mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\chi(\mathbb{W}_n(t)) \\
&\quad +2\mathcal{M}_0\Big[\int_0^t [2\mathcal{G}_2 m_k(s)+m_f(s)+2\mathcal{G}_0 n_g(s)]\chi(\mathbb{W}_n(s))\Big]ds \\
&\quad +2\mathcal{M}_0\sum_{k=1}^{n} l_k\chi(\mathbb{W}_n(t_k)), \text{ for all } t\in[0,b].
\end{aligned} \tag{9}
$$

Because $\mathbb{W}_n$ is decreasing for n, we have

$$\lambda(t) = \lim_{n\to\infty}\chi(\mathbb{W}_n(t)),$$

for all $t \in [0, b]$. From (9), we have

$$
\begin{aligned}
\lambda(t) &\le 2\Big[m_e(t)+m_k(t)\mathcal{G}_2+\mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\lambda(t) \\
&\quad +2\mathcal{M}_0\Big[\int_0^t [2\mathcal{G}_2 m_k(s)+m_f(s)+2\mathcal{G}_0 n_g(s)]\lambda(s)ds+\sum_{k=1}^{n} l_k\lambda(t_k)\Big],
\end{aligned}
$$

for $t \in [0, b]$, which implies that $\lambda(t) = 0$, for all $t_i \in [0, b]$. By Lemma 2.3, we know that $\lim_{n\to\infty}\chi(\mathbb{W}_n(t)) = 0$. Using Lemma 2.1 we know that $\mathbb{W} = \bigcap_{n=1}^{\infty}\mathbb{W}_n$ is convex compact and nonempty in $\mathcal{PC}([0,b],X)$ and $\mathcal{F}(\mathbb{W}) \subset \mathbb{W}$. By the Schauder fixed point theorem, there exist at least one mild solution u of the initial value problem $(1)-(3)$, where $x \in \mathbb{W}$ is a fixed point of the continuous map $\mathcal{F}$. □

Remark 3.2. If the functions f, g and I_i are compact or Lipschitz continuous (see e.g [5, 7]), then $(H_3)-(H_7)$ is automatically satisfied.

In some of the early related results in references and above results, it is supposed that the map h is uniformly bounded. In fact, if h is compact, then it must be bounded on bounded set. Here we give an existence result under growth condition of f, g and I_i, when h is not uniformly bounded. Precisely, we replace the assumptions $(H_3)-(H_6)$ by

(H_8) There exists a function $p \in \mathcal{L}([0,b],\mathcal{R}^+)$ and a increasing function $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ such that

$$\|f(t,x)\| \leq L_f(t)\phi(\|x\|),$$

for a.e $t \in [0,b]$, for all $x \in \mathcal{PC}([0,b],X)$.

(H_9) There exist two functions $L_g \in \mathcal{L}([0,b],\mathcal{R}^+)$and $\widehat{L_g} \in \mathcal{L}([0,b],\mathcal{R}^+)$ and a increasing function $\Psi : \mathcal{R}^+ \to \mathcal{R}^+$ such that

$$\|g(t,s,x)\| \leq L_g(t)\widehat{L_g}(s)\Psi(\|x\|),$$

for a.e $t \in [0,b]$ and for all $L_g \in \mathcal{PC}([0,b],X)$. Assume that the finite bound of $\int_0^t L_g(s)ds$ is $\mathcal{G}_3$.

(H_{10}) There exists a function $L_e \in \mathcal{L}([0,b],\mathcal{R}^+)$ and a increasing function $\Gamma : \mathcal{R}^+ \to \mathcal{R}^+$ such that

$$\|e(t,x,x_1)\| \leq L_e(t)\Gamma(\|x\|) + \|x_1\|$$

for a.e $t \in [0,b]$ and for all $L_g \in \mathcal{PC}([0,b],X)$. Assume that the finite bound of $\int_0^t L_e(s)ds$ is G_5.

(H_{11}) There exist two functions $L_k \in \mathcal{L}([0,b],\mathcal{R}^+)$and $\widehat{L_k} \in \mathcal{L}([0,b],\mathcal{R}^+)$ and a increasing function $\Theta : \mathcal{R}^+ \to \mathcal{R}^+$ such that

$$\|k(t,s,x)\| \leq L_k(t)\widehat{L_k}(s)\Theta(\|x\|),$$

for a.e $t \in [0,b]$ and for all $L_k \in \mathcal{PC}([0,b],X)$. Assume that the finite bound of $\int_0^t L_k(s)ds$ is $\mathcal{G}_4$.

Theorem: 3.2. *Suppose that the assumptions $(H_1)-(H_2)$ and $(H_8)-(H_{11})$ are satisfied, then the equation $(1)-(3)$ has at least one mild solution if*

$$\begin{aligned}&\lim_{r\to\infty}\sup\frac{1}{r}\Big\{\mathcal{M}_0\Big[\varphi(r)+L_e(t)\Big]+L_e(t)(\Gamma\|x\|)\\&+\mathcal{G}_3L_k(t)\Theta(r)+\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r)+\mathcal{G}_3\Theta(r)\int_0^t\widehat{L_k}(s)ds\big]\\&+\mathcal{M}_0\Big[\phi(r)\int_0^t L_f(s)ds+\mathcal{G}_2\Psi(r)\int_0^t\widehat{L_g}(s)ds+\sum_{k=1}^n d_k\Big]\Big\}<1,\end{aligned} \tag{10}$$

where $\varphi(r) = \sup\{||h(x)||,\ ||x|| \leq r\}$.

Proof. The inequality (10) implies that there exist a constant $r > 0$ such that

$$\mathcal{M}_0\Big[\varphi(r) + L_e(t)\Big] + L_e(t)(\Gamma\|x\|) + \mathcal{G}_3 L_k(t)\Theta(r) + \eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r)$$
$$+\mathcal{G}_3\Theta(r)\int_0^t \widehat{L_k}(s)ds\big] + \mathcal{M}_0\Big[\phi(r)\int_0^t p_(s)ds + \mathcal{G}_2\Psi(r)\int_0^t \widehat{L_g}(s)ds + \sum_{k=1}^n d_k\Big] < r,$$

As in the proof of Theorem 3.1, let $\mathbb{W}_0 = \{x \in \mathcal{PC}([0,b],X),\ ||x(t)|| \le r\}$ and $\mathbb{W}_1 = \overline{con}\ \mathcal{F}\mathbb{W}_0$. Then for any $x \in \mathbb{W}_1$, we have

$$\begin{aligned}
\|x(t)\| \ \le\ & \|U_x(t,0)h(x)\| + \|U_x(t,0)e(0,x(0),0)\| + \|e\left(t, x(t), \int_0^t k(t,s,x(s))ds\right)\| \\
& + \int_0^t \|A(t,x(t))U_x(t,s)e\left(t, x(t), \int_0^s k(s,\tau,x(\tau))d\tau\right)\|ds \\
& + \int_0^t \|U_x(t,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(s))d\tau\Big]\|ds + \sum_{0<t_k<t}\|U_x(t,t_k)I_i(x(t_k))\| \\
\le\ & \mathcal{M}_0\Big[\varphi(r) + L_e(t)\Big] + L_e(t)(\Gamma\|x\|) + \int_0^t L_k(t)\widehat{L_k}(s)\Theta(\|x\|)ds \\
& +\eta(t)\mathcal{M}_0\int_0^t \big[p_(s)\Gamma(\|x\|) + \int_0^s L_k(s)\widehat{L_k}(\tau)\Theta(\|x\|)d\tau\big]ds \\
& +\mathcal{M}_0\Big[\int_0^t L_f(s)\phi(\|x(s)\|)ds + \int_0^t\int_0^s L_g(s)\widehat{L_g}(\tau)\Psi\|(x(\tau))||d\tau ds + \sum_{k=1}^n d_k\Big] \\
\le\ & \mathcal{M}_0\Big[\varphi(r) + L_e(t)\Big] + L_e(t)(\Gamma\|x\|) + \mathcal{G}_3 L_k(t)\Theta(\|x\|) \\
& +\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(\|x\|) + \mathcal{G}_3\int_0^t \widehat{L_k}(s)\Theta(\|x\|)ds\big] \\
& +\mathcal{M}_0\Big[\int_0^t p_(s)\phi(\|x(s)\|)ds + \mathcal{G}_2\int_0^t \widehat{L_g}(s)\Psi(\|x(s)\|)ds + \sum_{k=1}^n d_k\Big]
\end{aligned}$$

$$\begin{aligned}
\|x(t)\| \ \le\ & \mathcal{M}_0\Big[\varphi(r) + L_e(t)\Big] + L_e(t)(\Gamma\|x\|) + G_3 L_k(t)\Theta(r) \\
& +\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r) + \mathcal{G}_3\Theta(r)\int_0^t \widehat{L_k}(s)ds\big] \\
& +\mathcal{M}_0\Big[\phi(r)\int_0^t p_(s)ds + \mathcal{G}_2\Psi(r)\int_0^t \widehat{L_g}(s)ds + \sum_{k=1}^n d_k\Big] \\
<\ & r,
\end{aligned}$$

for $t \in [0,b]$. It means that $\mathbb{W}_1 \subset \mathbb{W}_0$. So we can complete the proof similarly to Theorem 3.1.

4. **When h is Lipschitz**

In this section, we discuss the equation $(1) - (3)$ when h is Lipschitz and f, g and I_k are not Lipschitz. Assume that

(H_{12}) The function h is a Lipschitz continuous in X, there exists a constant $L_0 > 0$ such that

$$\|h(x) - h(y)\| \leq L_0\|x - y\|, \quad x, y \in \mathcal{PC}([0, b], X).$$

Theorem: 4.1. *Suppose that the assumptions* $(H_1) - (H_{12})$ *are satisfied, then the equation* $(1) - (3)$ *has at least one mild solution provided that*

$$\begin{aligned} &\mathcal{M}_0[L_0 + h_4(t)]\chi_c(\mathcal{B}) + 2\Big[m_e(t) + m_k(t)\mathcal{G}_2 + \mathcal{M}_0\eta(t)\mathcal{G}_1\Big] \\ &+2\mathcal{M}_0\Big[\int_0^t \{2\mathcal{G}_2 m_k(s) + m_f(s) + 2\mathcal{G}_0 n_g(s)\}ds + \sum_{k=1}^{n} l_k\Big] < 1. \end{aligned} \tag{11}$$

Proof. Consider the map $\mathcal{F} : \mathcal{PC}([0, b], X) \to \mathcal{PC}([0, b], X)$ is defined by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where

$$(\mathcal{F}_1 x)(t) = U_x(t, 0)h(u) + U_x(t, 0)e(0, x(0), 0),$$

$$\begin{aligned} (\mathcal{F}_2 u)(t) \quad = \quad & \int_0^t A(t, x(t))U_x(t, s)e\left(t, x(t), \int_0^s k(s, \tau, x(\tau))d\tau\right) ds \\ & -e\left(t, x(t), \int_0^t k(t, s, x(s))ds\right) \\ & + \int_0^t U_x(t, s)\Big[f(s, x(s)) + \int_0^s g(s, \tau, x(\tau))d\tau\Big]ds \\ & + \sum_{0<t_k<t} U_x(t, t_k)I_k(x(t_k)), \end{aligned}$$

for $x \in \mathcal{PC}([0, b], X)$. As defined in the proof of Theorem 3.1. We define $\mathbb{W}_0 = \{x \in \mathcal{PC}([0, b], X) : ||x(t)|| \leq m(t), \text{ for all } t \in [0, b]\}$ and let $\mathbb{W} = \overline{con}\mathcal{F}\mathbb{W}_0$. Then from the proof of Theorem 3.1 we know that $\mathbb{W}$ is a bounded closed convex and equicontinuous subset of $\mathcal{PC}([0, b], X)$ and $\mathcal{F}\mathbb{W} \subset \mathbb{W}$. We shall prove that $\mathcal{F}$ is χ_c-contraction on $\mathbb{W}$. Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of $\mathcal{F}$ in $\mathbb{W}$, which is a mild solution of $(1) - (2)$. First, for every bounded subset $\mathcal{B} \subset \mathbb{W}$, from (H_{12}) and Lemma 2.1 we have

$$\begin{aligned} \chi_c(\mathcal{F}_1\mathcal{B}) &= \chi_c(U_{\mathcal{B}}(t, 0)h(\mathcal{B})) + U_{\mathcal{B}}(t, 0)e(0, \mathcal{B}(0), 0) \\ &\leq \mathcal{M}_0\chi_c\Big[(h(\mathcal{B})) + e(0, \mathcal{B}(0), 0)\Big] \\ &\leq \mathcal{M}_0[L_0 + h_4(t)]\chi_c(\mathcal{B}) \end{aligned} \tag{12}$$

Next, for every bounded subset $\mathcal{B} \subset \mathbb{W}$, for $t \in [0, b]$ and every $\epsilon > 0$, there is a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathcal{B}$ such that

$$\chi(\mathcal{F}_2(\mathcal{B}(t)) \leq 2\chi(\{\mathcal{F}_2 x_i(t)\}_{n=1}^{\infty} + \epsilon.$$

Note that $\mathcal{B}$ and $\mathcal{F}_2\mathcal{B}$ are equicontinuous, we can get from Lemma 2.1, Lemma 2.4, Lemma 2.5 and using the assumptions we get

$$
\begin{aligned}
\chi(\mathcal{F}_2\mathcal{B}(t)) \leq\ & 2\chi\Big(e(t,\{x_k(s)\}_{k=1}^{\infty}, \int_0^t k(t,s,\{x_k(t)\}_{k=1}^{\infty})ds)\Big) \\
& 2\mathcal{M}_0\eta(t)\int_0^t \chi\Big(e(s,\{x_k(s)\}_{k=1}^{\infty}, \int_0^s k(s,\tau,\{x_k(\tau)\}_{k=1}^{\infty})d\tau)\Big)ds \\
& +2\mathcal{M}_0\int_0^t \chi\Big(f(s,\{x_k(s)\}_{k=1}^{\infty})\Big)ds + 4\mathcal{M}_0\int_0^t\int_0^s \chi\Big(g(s,\tau,\{u_k(\tau)\}_{k=1}^{\infty})\Big)d\tau ds \\
& +2\mathcal{M}_0\sum_{i=1}^{n}\chi\Big(I_k(\{u_k(t_k)\}_{k=1}^{\infty})\Big) + \epsilon \\
\leq\ & 2m_e(t)\chi\{x_k(t)\}_{k=1}^{\infty} + 2m_k(t)\int_0^t m_k(s)\chi\{x_k(s)\}_{k=1}^{\infty}ds \\
& +2\mathcal{M}_0\eta(t)\Big[\int_0^t m_e(s)\chi\{x_k(s)\}_{k=1}^{\infty}ds + 2\int_0^t\int_0^s m_k(s)m_k(\tau)\chi\{x_k(\tau)\}_{k=1}^{\infty}d\tau ds\Big] \\
& +2\mathcal{M}_0\int_0^t m_f(s)\chi\Big(\{u_k(s)\}_{k=1}^{\infty}\Big)ds + 4\mathcal{M}_0\int_0^t\int_0^s m_g(s)n_g(\tau)\chi\Big(\{u_k(\tau)\}_{k=1}^{\infty}\Big)d\tau ds \\
& +2\mathcal{M}_0\sum_{i=1}^{n} l_i\chi\Big(\{u_k(t_k)\}_{k=1}^{\infty}\Big) + \epsilon \\
\leq\ & 2\Big[m_e(t) + m_k(t)\mathcal{G}_2 + \mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\chi(\mathcal{B}) \\
& +2\mathcal{M}_0\Big[\int_0^t\{2\mathcal{G}_2 m_k(s) + m_f(s)\}\chi(B))ds + 2\mathcal{G}_0\int_0^t n_g(s)\chi(B)ds\Big] \\
& +2\mathcal{M}_0\sum_{k=1}^{n} l_k\chi(\mathcal{B}) + \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, it follows that from the above inequality that

$$
\begin{aligned}
\chi_c(\mathcal{F}_2\mathcal{B}(t)) \leq\ & 2\Big[m_e(t) + m_k(t)\mathcal{G}_2 + \mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\chi_c(B) \\
& +2\mathcal{M}_0\Big[\int_0^t\{2\mathcal{G}_2 m_k(s) + m_f(s) + 2\mathcal{G}_0 n_g(s)\}ds + \sum_{k=1}^{n} l_k\Big]\chi_c(\mathcal{B})
\end{aligned}
\tag{13}
$$

for any bounded $\mathcal{B} \subset \mathbb{W}$.

Now, for any subset $\mathcal{B} \subset \mathbb{W}$, due to Lemma 2.1, (12) and (13) we have

$$
\begin{aligned}
\chi_c(\mathcal{F}\mathcal{B}) \leq\ & \chi_c(\mathcal{F}_1\mathcal{B}) + \chi_c(\mathcal{F}_2\mathcal{B}) \\
\leq\ & \mathcal{M}_0[L_0 + h_4(t)]\chi_c(\mathcal{B}) + 2\Big[m_e(t) + m_k(t)\mathcal{G}_2 + \mathcal{M}_0\eta(t)\mathcal{G}_1\Big]\chi_c(\mathcal{B}) \\
& +2\mathcal{M}_0\Big[\int_0^t\{2\mathcal{G}_2 m_k(s) + m_f(s) + 2\mathcal{G}_0 n_g(s)\}ds + \sum_{k=1}^{n} l_k\Big]\chi_c(\mathcal{B})
\end{aligned}
\tag{14}
$$

By (14) we know that $\mathcal{F}$ is a χ_c-contraction on $\mathbb{W}$. By Lemma 2.2, there is a fixed point x of $\mathcal{F}$ in $\mathbb{W}$, which is a solution of $(1)-(3)$. This completes the proof.

Theorem: 4.2. *Suppose that the assumptions $(H_1)-(H_{12})$ are satisfied, then the equation $(1)-(3)$ has at least one mild solution if (15) and the following condition* *are satisfied.*

$$\mathcal{M}_0 L_0 + \lim_{r\to\infty} \sup \frac{1}{r}\Big\{\mathcal{M}_0 L_e(t) + L_e(t)\Gamma(r) + \mathcal{G}_3 L_k(t)\Theta(r)$$
$$+\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r) + \mathcal{G}_3\Theta(r)\int_0^t \widehat{L_k}(s)ds\big]$$
$$+\mathcal{M}_0\Big[\phi(r)\int_0^t L_f(s)ds + \mathcal{G}_2\Psi(r)\int_0^t \widehat{L_g}(s)ds + \sum_{k=1}^n d_k\Big]\Big\} < 1. \tag{15}$$

Proof. From the equation (15) and fact that $L_0 < 1$, there exists a constant $r > 0$ such that

$$\mathcal{M}_0\Big(rL_0 + \|h(0)\| + L_e(t)\Big) + L_e(t)(\Gamma\|x\|) + \mathcal{G}_3 L_k(t)\Theta(r)$$
$$+\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r) + \mathcal{G}_3\Theta(r)\int_0^t \widehat{L_k}(s)ds\big]$$
$$+\mathcal{M}_0\Big[\phi(r)\int_0^t L_f(s)ds + \mathcal{G}_2\Psi(r)\int_0^t \widehat{L_g}(s)ds + \sum_{k=1}^n d_k\Big]\} < r.$$

We define $W_0 = \{x \in \mathcal{PC}([0,b],X),\ \|x(t)\| \leq r, \text{ for all } \ t \in [0,b]\}$. Then for every $x \in W_0$, we have

$$\begin{aligned}
\|\mathcal{F}x(t)\| \ &\leq \ \|U_x(t,0)h(u)\| + \|U_x(t,0)e(0,x(0),0)\| + \|e\left(t, x(t), \int_0^t k(t,s,x(s))ds\right)\| \\
&\quad + \int_0^t \|A(t,x(t))U_x(t,s)e\left(t, x(t), \int_0^s k(s,\tau,x(\tau))d\tau\right)\|ds \\
&\quad + \int_0^t \|U_x(t,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(s))d\tau\Big]\|ds + \sum_{0<t_k<t} \|U_x(t,t_k)I_i(x(t_k))\| \\
&\leq \ \mathcal{M}_0\Big(rL_0 + \|h(0)\| + L_e(t)\Big) + L_e(t)(\Gamma\|x\|) + \mathcal{G}_3 L_k(t)\Theta(r) \\
&\quad +\eta(t)\mathcal{M}_0\big[\mathcal{G}_4\Gamma(r) + \mathcal{G}_3\Theta(r)\int_0^t \widehat{L_k}(s)ds\big] \\
&\quad +\mathcal{M}_0\Big[\phi(r)\int_0^t L_f(s)ds + \mathcal{G}_2\Psi(r)\int_0^t \widehat{L_g}(s)ds + \sum_{k=1}^n d_k\Big]\} < r.
\end{aligned}$$

for $t \in [0,b]$. This means that $\mathcal{F}\mathbb{W}_0 \subset \mathbb{W}_0$. Define $\mathbb{W} = \overline{con}\mathcal{F}\mathbb{W}_0$. The above proof also implies that $\mathcal{F}\mathbb{W} \subset \mathbb{W}$. So we can prove the theorem similar with Theorem 4.1 and hence we omit it.

5. **Application**

The notion of controllability is of great importance in mathematical control theory. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the assumption that the system is controllable. It means that it is possible to steer any initial state of the system to any final state in some finite time using an admissible control. During the last few decades, several authors [25, 27] have discussed the existence, uniqueness, and asymptotic behavior of the solution of these systems. Apart from these, the study of controllability and observability properties of a system in control theory is certainly, at present, one of the most active interdisciplinary areas of research. Control theory arises in most modern applications. On the other hand, control theory has remained a discipline where many mathematical ideas and methods have fused to produce a new body of important mathematics. As an application of Theorem 3.1 we shall consider the system (1) – (3) with a control parameter such as

$$\frac{d}{dt}[x(t) + e(t, x(t), \int_0^t k(t, s, x(s))ds)] + A(t, x(t))x(t) = f(t, x(t)) + Cv(t) + \int_0^t g(t, s, x(s))ds, \ t \in [0, b], \ t \neq t_i, \tag{16}$$

$$x(0) + h(x) = x_0, \tag{17}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, 3, \ldots, n, \tag{18}$$

where A, f, g, h and I_k are as before and C is a bounded linear operator from a Banach space V into X and the control function $v \in L^2(J, V)$. The mild solution of (16) – (18) is given by

$$\begin{aligned} x(t) = {} & U_x(t,0)x_0 - U_x(t,0)h(x) + U_x(t,0)e(0, x(0), 0) - e\left(t, x(t), \int_0^t k(t, s, x(s))ds\right) \\ & + \int_0^t A(t, x(t))U_x(t, s)e\left(t, x(t), \int_0^s k(s, \tau, x(\tau))d\tau\right) ds \\ & + \int_0^t U_x(t, s)\Big[f(s, x(s)) + Cv(s) + \int_0^s g(s, \tau, x(\tau))d\tau\Big] ds \\ & + \sum_{0<t_k<t} U_x(t, t_k)I_k(x(t_k)), \quad 0 \leq t \leq b. \end{aligned}$$

Definition 5.1 ([26, 27]) System (16) – (18) is said to be controllable on the interval J, if for every $x_0, \ x_1 \in X$, there exists a control $v \in L^2(J, V)$ such that the mild solution $u(\cdot)$ of (16) – (18) satisfies $x(0) = x_0$ and $x(b) = x_1$.

To study the controllability result we need the following additional condition:

(H_{13}) The linear operator $W : L^2(J, V) \to X$, defined by

$$Wv = \int_0^b U_x(b, s)Cv(s)ds$$

induces an inverse operator W^{-1} defined an $L^2(J, V)/kerW$ and there exists a positive constant $\mathcal{M}_1 > 0$ such that $\|CW^{-1}\| \leq \mathcal{M}_1$.

Theorem: 5.1. *If the assumptions* $(H_1) - (H_{13})$ *are satisfied, then the system* (16) – (18) *is controllable on J.*

Proof. Using the assumption (H_{13}), for an arbitrary function $u(\cdot)$, define the control

$$\begin{aligned} v(t) &= W^{-1}\Big[u_1 - U_x(b,0)x_0 + U_x(b,0)h(x) - U_x(b,0)e(0,x(0),0) \\ &\quad + e\left(b, x(b), \int_0^b k(b,s,x(s))ds\right) \\ &\quad - \int_0^b A(b,x(b))U_x(b,s)e\left(b, x(b), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\ &\quad - \int_0^b U_x(b,s)\Big[f(s,x(s)) - \int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\ &\quad - \sum_{0<t_k<t} U_x(t,t_k)I_k(x(t_k))\Big](t). \end{aligned}$$

We shall show that when using this control, the operator $\mathcal{H} : Z \to Z$ defined by

$$\begin{aligned} (\mathcal{H}v)(t) &= U_x(t,0)x_0 - U_x(t,0)h(x) + U_x(t,0)e(0,x(0),0) - e\left(t, x(t), \int_0^t k(t,s,x(s))ds\right) \\ &\quad + \int_0^t A(t,x(t))U_x(t,s)e\left(t, x(t), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\ &\quad + \int_0^t U_x(t,s)\Big[f(s,x(s)) + CW^{-1}\Big[u_1 - U_x(b,0)x_0 + U_x(b,0)h(x) \\ &\quad - U_x(b,0)e(0,x(0),0) + e\left(b, x(b), \int_0^b k(b,s,x(s))ds\right) \\ &\quad + \int_0^b A(b,x(b))U_x(b,s)e\left(b, x(b), \int_0^s k(s,\tau,x(\tau))d\tau\right)ds \\ &\quad - \int_0^b U_x(b,s)\Big[f(s,x(s)) + \int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\ &\quad - \sum_{0<t_k<t} U_x(t,t_k)I_k(x(t_k))\Big](s) + \int_0^s g(s,\tau,x(\tau))d\tau\Big]ds \\ &\quad + \sum_{0<t_k<t} U_x(t,t_k)I_k(x(t_k)) \end{aligned}$$

has a fixed point. This fixed point is, then a solution of $(16) - (18)$. Clearly, $(\mathcal{H}v)(b) = x(b) = x_1$, which means that the control v steers the system $(16) - (18)$ from the initial state x_0 to the final state x_1 at time b, provided we can obtain a fixed point of the nonlinear operator $\mathcal{H}$. The remaining part of the proof is similar to Theorem 3.1 and hence, it is omitted.

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