Int. J. Anal. Appl. (2025), 23:175

International Journal of Analysis and Applications

Stability of an Additive-Quadratic Functional Equation in a Banach Space

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Abstract. Using the direct and fixed point methods, we obtain the Hyers-Ulam stability of the following additive-quadratic functional equation:

$$2h(p+q, r+s) + h(p+q, r-s) = 3\left[h(p,r) + h(p,s) + h(q,r) + h(q,s)\right] \tag{1}$$

in a Banach space.

1. Introduction and preliminaries

In 1940, Ulam [33] posed a query on the stability of (group) homomorphisms that prompted the investigation of stability issues in functional equations. Hyers [14] then provided a partial solution to the issue of additive mappings in Banach spaces. Hyers-Ulam stability has also been used to refer to the stability of functional equations.

Aoki [1] and Rassias [30] later expanded it to include additive mappings and linear mappings, respectively, by using an unbounded Cauchy difference. Găvruta [12] enhanced the Rassias theorem by substituting an unbounded Cauchy difference with a general control function. Hyers himself published noteworthy papers including several different homomorphisms as in [15–17].

Recent work by Park defined additive ρ-functional inequalities and demonstrated their Hyers-Ulam stability in Banach spaces using [24, 25, 27]. Extensive research has been conducted on the stability difficulties of several functional equations and functional inequalities (see [2, 6, 10, 11, 19–21, 23, 34]).

Received: Jun. 5, 2025.

2020 *Mathematics Subject Classification.* 39B52, 47H10, 39B62.

Key words and phrases. Hyers-Ulam stability; additive-quadratic functional equation; direct method; fixed point method.

https://doi.org/10.28924/2291-8639-23-2025-175
ISSN: 2291-8639

In 1996, Isac and Rassias [18] proposed applications for the stability of functional equations for proving fixed point theorems and applications in nonlinear analysis. Numerous scholarly works on the stability concerns of certain functional equations and the different definitions of stability using the fixed point method have been extensively researched by [4,5,8,9,26,28,29,31,32] and others.

In this paper, the sets of positive integers, real numbers, positive real numbers, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$, correspondingly. Also, let $\mathcal{X}$ be a (complex) normed space, $\mathcal{Y}$ a (complex) Banach space, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$ respectively.

We begin with a practical result from the theory of fixed points.

Theorem 1.1. [3,7] *Let $(\mathcal{Z}, d)$ be a complete generalized metric space and let $a \in \mathcal{Z}$. For a strict Lipschitz contraction $\mathcal{J} : \mathcal{Z} \to \mathcal{Z}$ with the Lipschitz constant $\alpha < 1$, either*

(1) $d(\mathcal{J}^n a, \mathcal{J}^{n+1} a) = \infty$ *for all* $n \in \mathbb{N}_0$ *or there exists* $n_0 \in \mathbb{N}$ *for which* $d(\mathcal{J}^n a, \mathcal{J}^{n+1} a) < \infty$ *for all* $n \geq n_0$*;*

(2) $\mathcal{J}^n a \to b^*$, *where* b^* *is a unique fixed point of* $\mathcal{J}$ *in* $\mathcal{Z}_{n_0} := \{b \in \mathcal{Z} : d(\mathcal{J}^{n_0} a, b) < \infty\}$*;*

(3) $d(b, b^*) \leq \frac{1}{1-\alpha} d(b, \mathcal{J} b)$ *for all* $b \in \mathcal{Z}_{n_0}$.

Next, we introduce the concept of additive-quadratic mapping.

Definition 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be vector spaces. A mapping $h : \mathcal{A}^2 \to \mathcal{B}$ is called additive-quadratic if h is additive in the first variable and quadratic in the second variable, that is, h satisfies the following system of equations*

$$h(p, r) + h(q, r) = h(p + q, r)$$

and

$$h(p, q + r) + h(p, q - r) = 2h(p, q) + 2h(p, r)$$

for all $p, q, r \in \mathcal{A}$. We denote the class of additive-quadratic mapping by $\mathcal{AQ}(\mathcal{A}, \mathcal{B})$.

For the function $h : \mathcal{X}^2 \to \mathcal{Y}$, the following functional equation was presented by Hwang and Park [13]:

$$h(p + q, r + s) + h(p - q, r - s) = 2h(p, r) + 2h(p, s) \tag{1.1}$$

for all $p, q, r, s \in \mathcal{X}$. Additionally, they demonstrated that every function satisfying (1.1), together with some additional conditions, is in $\mathcal{AQ}(\mathcal{X}, \mathcal{Y})$.

In this study, we first investigate the additive-quadratic functional equation (1). Second, we use the direct method to demonstrate the Hyers-Ulam stability of the functional equation (1). Using the fixed point method, we demonstrate the Hyers-Ulam stability of the functional equation (1).

2. Hyers-Ulam stability of the additive-quadratic functional equation: direct method

We prove the following lemma for obtaining the stability of the functional equation (1).

Lemma 2.1. *If a mapping $h : \mathcal{X}^2 \to \mathcal{Y}$ satisfies* (1)*, then the following are true:*

- (i) $h(p,0) = h(0,r) = 0$ *for all* $p, r \in \mathcal{X}$*;*
- (ii) h *is even in the second variable;*
- (iii) $h \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$.

Proof. (i) Putting $q = r = s = 0$ in the functional equation (1), we get $h(p,0) = 0$ for all $p \in \mathcal{X}$. Replacing (p,q,s) by $(0,0,0)$ in the equation (1), yields $h(0,r) = 0$ for all $r \in \mathcal{X}$.

(ii) Letting $q = r = 0$ in (1), we obtain $h(p,-s) = h(p,s)$ for all $p,s \in \mathcal{X}$, that is, h is even in the second variable.

(iii) If $s = 0$ in (1), then $2h(p+q,r) + h(p+q,r) = 3\left[h(p,r) + h(q,r)\right]$ and so $h(p+q,r) = h(p,r) + h(q,r)$ for all $p,q,r \in \mathcal{X}$. This entails that h is additive on its first variable. Next, replacing s by $-s$ in (1), we have $2h(p+q,r-s) + h(p+q,r+s) = 3\left[h(p,r) + h(p,-s) + h(q,r) + h(q,-s)\right]$ for all $p,q,r,s \in \mathcal{X}$. By using the evenness on its second variable of h, we find that

$$2h(p+q,r-s) + h(p+q,r+s) = 3\left[h(p,r) + h(p,s) + h(q,r) + h(q,s)\right]$$

for all $p,q,r,s \in \mathcal{X}$. Combining this to (1), we lead to

$$h(p+q,r+s) + h(p+q,r-s) = 2\left[h(p,r) + h(p,s) + h(q,r) + h(q,s)\right] \tag{2.1}$$

for all $p,q,r,s \in \mathcal{X}$. Now, setting $q = 0$ in (2.1), we get $h(p,r+s) + h(p,r-s) = 2h(p,r) + 2h(p,s)$ for all $p,r,s \in \mathcal{X}$. Therefore, h is quadratic on its second variable and consequently $h \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$. This completes the proof. □

For a given mapping $h : \mathcal{X}^2 \to \mathcal{Y}$, we define, for all $p,q,r,s \in \mathcal{X}$,

$$\delta h(p,q,r,s) := 2h(p+q,r+s) + h(p+q,r-s) - 3\left[h(p,r) + h(p,s) + h(q,r) + h(q,s)\right].$$

We also denote the class of mapping $\left\{g : \mathcal{X}^2 \to \mathcal{Y} : g(p,0) = g(0,q) = 0 \text{ for all } p,q \in \mathcal{X}\right\}$ by $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$. Now, we present our main results.

Theorem 2.1. *Let* $\omega : \mathcal{X}^2 \to \mathbb{R}_0^+$ *be a mapping such that*

$$\Lambda(p,q) := \sum_{j=1}^{\infty} 3^j \omega\left(\frac{p}{2^j}, \frac{q}{2^j}\right) < \infty \tag{2.2}$$

for all $p,q \in \mathcal{X}$*. If* $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ *and*

$$\|\delta h(p,q,r,s)\| \le \omega(p,q)\omega(r,s) \tag{2.3}$$

for all $p,q,r,s \in \mathcal{X}$*, then there exists a unique mapping* $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ *such that*

$$\|h(p,r) - H(p,r)\| \le \frac{1}{6} \min\left\{\omega(p,0)\Lambda(r,r), \omega(r,0)\tilde{\Lambda}(p,p)\right\} \tag{2.4}$$

for all $p,r \in \mathcal{X}$*, where*

$$\tilde{\Lambda}(p,q) := \sum_{j=1}^{\infty} 2^j \omega\left(\frac{p}{2^j}, \frac{q}{2^j}\right)$$

for all $p, q \in \mathcal{X}$.

Proof. Replacing (q, s) by $(0, r)$ in the inequality (2.3), we obtain

$$\|3h(p,r) - h(p,2r)\| \leq \frac{1}{2}\omega(p,0)\omega(r,r) \tag{2.5}$$

and so

$$\left\|3h\left(p,\frac{r}{2}\right) - h(p,r)\right\| \leq \frac{1}{2}\omega(p,0)\omega\left(\frac{r}{2},\frac{r}{2}\right)$$

for all $p, r \in \mathcal{X}$. Then, for each $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\begin{aligned}\left\|3^l h\left(p,\frac{r}{2^l}\right) - 3^m h\left(p,\frac{r}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|3^j h\left(p,\frac{r}{2^j}\right) - 3^{j+1}h\left(p,\frac{r}{2^{j+1}}\right)\right\| \\ &\leq \frac{1}{6}\omega(p,0)\sum_{j=l+1}^{m} 3^j\omega\left(\frac{r}{2^j},\frac{r}{2^j}\right) \end{aligned} \tag{2.6}$$

for all $p, r \in \mathcal{X}$. Thus $\{3^n h(p, 2^{-n}r)\}$ is a Cauchy sequence and so it is a convergent sequence in $\mathcal{Y}$ due to the completeness of $\mathcal{Y}$. Now, we define a mapping $R : \mathcal{X}^2 \to \mathcal{Y}$ by

$$R(p,r) := \lim_{n\to\infty} 3^n h\left(p,\frac{r}{2^n}\right)$$

for all $p, r \in \mathcal{X}$. Next, select $l = 0$ and let $m \to \infty$ in (2.6). Then we have

$$\|h(p,r) - R(p,r)\| \leq \frac{1}{6}\omega(p,0)\Lambda(r,r) \tag{2.7}$$

for all $p, r \in \mathcal{X}$. It implies by (2.2) and (2.3) that

$$\|\delta R(p,q,r,s)\| = \lim_{n\to\infty} 3^n\left\|\delta h\left(p,q,\frac{r}{2^n},\frac{s}{2^n}\right)\right\| \leq \omega(p,q)\lim_{n\to\infty} 3^n\omega\left(\frac{r}{2^n},\frac{s}{2^n}\right) = 0$$

for all $p, q, r, s \in \mathcal{X}$. Hence, by Lemma 2.1, $R \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$.

To prove the uniqueness property of R, let $\tilde{R}$ be another additive-quadratic mapping satisfying (2.7). Then

$$\begin{aligned}\|R(p,r) - \tilde{R}(p,r)\| &= 3^k\left\|R\left(p,\frac{r}{2^k}\right) - \tilde{R}\left(p,\frac{r}{2^k}\right)\right\| \\ &\leq 3^k\left\|R\left(p,\frac{r}{2^k}\right) - h\left(p,\frac{r}{2^k}\right)\right\| + 3^k\left\|h\left(p,\frac{r}{2^k}\right) - \tilde{R}\left(p,\frac{r}{2^k}\right)\right\| \\ &\leq 3^{k-1}\omega(p,0)\Lambda\left(\frac{r}{2^k},\frac{r}{2^k}\right)\end{aligned}$$

for all $p, r \in \mathcal{X}$. Therefore, $\|R(p,r) - \tilde{R}(p,r)\| \to 0$ when $k \to \infty$ and this confirms the uniqueness of R. Next, replacing (q, s) by $(p, 0)$ in (2.3), we obtain

$$\|h(2p,r) - 2h(p,r)\| \leq \frac{1}{3}\omega(p,p)\omega(r,0) \tag{2.8}$$

and so

$$\left\|h(p,r) - 2h\left(\frac{p}{2},r\right)\right\| \leq \frac{1}{3}\omega\left(\frac{p}{2},\frac{p}{2}\right)\omega(r,0)$$

for all $p, r \in \mathcal{X}$. Then, for each $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\left\| 2^l h\left(\frac{p}{2^l}, r\right) - 2^m h\left(\frac{p}{2^m}, r\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j h\left(\frac{p}{2^j}, r\right) - 2^{j+1} h\left(\frac{p}{2^{j+1}}, r\right) \right\|$$
$$\leq \frac{1}{6}\omega(r,0) \sum_{j=l+1}^{m} 2^j \omega\left(\frac{p}{2^j}, \frac{p}{2^j}\right) \tag{2.9}$$

for all $p, r \in \mathcal{X}$. Therefore, $\{2^n h(2^{-n}p, r)\}$ is a Cauchy sequence. By the completeness of $\mathcal{Y}$, the sequence $\{2^n h(2^{-n}p, r)\}$ converges. We define the mapping $S : \mathcal{X}^2 \to \mathcal{Y}$ by

$$S(p,r) := \lim_{n\to\infty} 2^n h\left(\frac{p}{2^n}, r\right)$$

for all $p, r \in \mathcal{X}$. Putting $l = 0$ and taking the limit as $m \to \infty$ in (2.9), yields

$$\|h(p,r) - S(p,r)\| \leq \frac{1}{6}\omega(r,0)\tilde{\Lambda}(p,p) \tag{2.10}$$

for all $p, r \in \mathcal{X}$. It follows from (2.2) and (2.3) that

$$\|\delta S(p,q,r,s)\| = \lim_{n\to\infty} 2^n \left\| \delta h\left(\frac{p}{2^n}, \frac{q}{2^n}, r, s\right) \right\| \leq \omega(r,s) \lim_{n\to\infty} 2^n \omega\left(\frac{p}{2^n}, \frac{q}{2^n}\right) = 0$$

for all $p, q, r, s \in \mathcal{X}$. By Lemma 2.1, we infer that $S \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$. Let $\tilde{S}$ be another additive-quadratic mapping satisfying (2.10). Then

$$\|S(p,r) - \tilde{S}(p,r)\| = 2^k \left\| S\left(\frac{p}{2^k}, r\right) - \tilde{S}\left(\frac{p}{2^k}, r\right) \right\|$$
$$\leq 2^k \left\| S\left(\frac{p}{2^k}, r\right) - h\left(\frac{p}{2^k}, r\right) \right\| + 2^k \left\| h\left(\frac{p}{2^k}, r\right) - \tilde{S}\left(\frac{p}{2^k}, r\right) \right\|$$
$$\leq \frac{2^k}{3}\omega(r,0)\tilde{\Lambda}\left(\frac{p}{2^k}, \frac{p}{2^k}\right),$$

which tends to zero as $k \to \infty$ for all $p, r \in \mathcal{X}$. This proves the uniqueness of S. It follows from (2.7) that

$$2^n \left\| h\left(\frac{p}{2^n}, r\right) - R\left(\frac{p}{2^n}, r\right) \right\| \leq \frac{2^n}{6}\omega\left(\frac{p}{2^n}, 0\right)\Lambda(r,r),$$

which tends to zero as $n \to \infty$ for all $p, r \in \mathcal{X}$. Since R is additive on its first variable, we have $\|S(p,r) - R(p,r)\| = 0$, i.e., $H(p,r) := R(p,r) = S(p,r)$ for all $p, r \in \mathcal{X}$. Therefore, there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ satisfying (2.4). This completes the proof. □

Theorem 2.2. *Let $\omega : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping satisfying*

$$\Psi(p,q) := \sum_{j=0}^{\infty} \frac{1}{2^j}\omega\left(2^j p, 2^j q\right) < \infty \tag{2.11}$$

for all $p, q \in \mathcal{X}$. Suppose that $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies (2.3)*. Then there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{1}{6}\min\left\{\omega(p,0)\tilde{\Psi}(r,r), \omega(r,0)\Psi(p,p)\right\} \tag{2.12}$$

for all $p, r \in \mathcal{X}$, *where*

$$\tilde{\Psi}(p,q) := \sum_{j=0}^{\infty} \frac{1}{3^j}\omega\left(2^j p, 2^j q\right)$$

for all $p, q \in \mathcal{X}$.

Proof. It follows from (2.5) that

$$\left\|h(p,r) - \frac{1}{3}h(p,2r)\right\| \leq \frac{1}{6}\omega(p,0)\omega(r,r)$$

for all $p, r \in \mathcal{X}$. Then, for all $m, l \in \mathbb{N}_0$ with $m > l$, we have

$$\begin{aligned}\left\|\frac{1}{3^l}h(p,2^l r) - \frac{1}{3^m}h(p,2^m r)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}h(p,2^j r) - \frac{1}{3^{j+1}}h(p,2^{j+1}r)\right\| \\ &\leq \frac{1}{6}\omega(p,0)\sum_{j=l}^{m-1}\frac{1}{3^j}\omega(2^j r, 2^j r) \qquad (2.13)\end{aligned}$$

for all $p, r \in \mathcal{X}$. Then the completeness of $\mathcal{Y}$ implies that $\{3^{-n}h(p,2^n r)\}$ is convergent for each $p, r \in \mathcal{X}$. Next, we define a mapping $R(p,r) : \mathcal{X}^2 \to \mathcal{Y}$ by

$$R(p,r) := \lim_{n\to\infty}\frac{1}{3^n}h(p,2^n r)$$

for all $p, r \in \mathcal{X}$. Choose $l = 0$ and let $m \to \infty$ in (2.13). Then we have

$$\|h(p,r) - R(p,r)\| \leq \frac{1}{6}\omega(p,0)\tilde{\Psi}(r,r) \qquad (2.14)$$

for all $p, r \in \mathcal{X}$. Thus it follows from (2.3) and (2.11) that

$$\|\delta R(p,q,r,s)\| = \lim_{n\to\infty}\frac{1}{3^n}\left\|\delta h\left(p,q,2^n r,2^n s\right)\right\| \leq \omega(p,q)\lim_{n\to\infty}\frac{1}{3^n}\omega\left(2^n r, 2^n s\right) = 0$$

for all $p, q, r, s \in \mathcal{X}$. By Lemma 2.1, we have $R \in \mathcal{AQ}(\mathcal{X},\mathcal{Y})$. Let $\tilde{R}$ be another mapping in $\mathcal{AQ}(\mathcal{X},\mathcal{Y})$ satisfying (2.14). Then we have

$$\begin{aligned}\|R(p,r) - \tilde{R}(p,r)\| &= \frac{1}{3^k}\left\|R(p,2^k r) - \tilde{R}(p,2^k r)\right\| \\ &\leq \frac{1}{3^k}\left\|R(p,2^k r) - h(p,2^k r)\right\| + \frac{1}{3^k}\left\|h(p,2^k r) - \tilde{R}(p,2^k r)\right\| \\ &\leq \frac{1}{3^{k+1}}\omega(p,0)\tilde{\Psi}(2^k r, 2^k r) \to 0 \text{ as } k \to \infty\end{aligned}$$

for all $p, r \in \mathcal{X}$ and so the uniqueness of R follows. Next, it implies by (2.8) that

$$\left\|\frac{1}{2}h(2p,r) - h(p,r)\right\| \leq \frac{1}{6}\omega(p,p)\omega(r,0)$$

for all $p,r\in\mathcal{X}$. Then, for each $m,l\in\mathbb{N}_0$ with $m>l$, we have

$$\left\|\frac{1}{2^l}h(2^lp,r)-\frac{1}{2^m}h(2^mp,r)\right\|\leq\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}h(2^jp,r)-\frac{1}{2^{j+1}}h(2^{j+1}p,r)\right\|$$
$$\leq\frac{1}{6}\omega(r,0)\sum_{j=l}^{m-1}\frac{1}{2^j}\omega(2^jp,2^jp) \tag{2.15}$$

for all $p,r\in\mathcal{X}$. Therefore, $\{2^{-n}h(2^np,r)\}$ is a Cauchy sequence. By the completeness of $\mathcal{Y}$, the sequence $\{2^{-n}h(2^np,r)\}$ converges. We define the mapping $S:\mathcal{X}^2\to\mathcal{Y}$ by

$$S(p,r):=\lim_{n\to\infty}\frac{1}{2^n}h(2^np,r)$$

for all $p,r\in\mathcal{X}$. Putting $l=0$ and taking the limit as $m\to\infty$ in (2.15), yields

$$\|h(p,r)-S(p,r)\|\leq\frac{1}{6}\omega(r,0)\Psi(p,p) \tag{2.16}$$

for all $p,r\in\mathcal{X}$. It follows from (2.3) and (2.11) that

$$\|\delta S(p,q,r,s)\|=\lim_{n\to\infty}\frac{1}{2^n}\left\|\delta h\left(2^np,2^nq,r,s\right)\right\|\leq\omega(r,s)\lim_{n\to\infty}\frac{1}{2^n}\omega\left(2^np,2^nq\right)=0$$

for all $p,q,r,s\in\mathcal{X}$. By Lemma 2.1, we infer that $S\in\mathcal{AQ}(\mathcal{X},\mathcal{Y})$. Let $\tilde{S}$ be another mapping in $\mathcal{AQ}(\mathcal{X},\mathcal{Y})$ satisfying (2.16). Then

$$\begin{aligned}\|S(p,r)-\tilde{S}(p,r)\|&=\frac{1}{2^k}\left\|S(2^kp,r)-\tilde{S}(2^kp,r)\right\|\\&\leq\frac{1}{2^k}\left\|S(2^kp,r)-h(2^kp,r)\right\|+\frac{1}{2^k}\left\|h(2^kp,r)-\tilde{S}(2^kp,r)\right\|\\&\leq\frac{1}{3\cdot2^k}\omega(r,0)\Psi(2^kp,2^kp)\to0\ \text{ as }\ k\to\infty\end{aligned}$$

for all $p,r\in\mathcal{X}$ and so the uniqueness of S follows. It follows from (2.14) that

$$\frac{1}{2^n}\left\|h\left(2^np,r\right)-R\left(2^np,r\right)\right\|\leq\frac{1}{6\cdot2^n}\omega(2^np,0)\tilde{\Psi}(r,r),$$

which tends to zero as $n\to\infty$ for all $p,r\in\mathcal{X}$. Since R is additive on its first variable, we have $\|S(p,r)-R(p,r)\|=0$, i.e., $H(p,r):=R(p,r)=S(p,r)$ for all $p,r\in\mathcal{X}$. Therefore, there exists a unique mapping $H\in\mathcal{AQ}(\mathcal{X},\mathcal{Y})$ satisfying (2.12). This completes the proof. □

If $\omega(p,q)=\sqrt{\theta}(\|p\|^t+\|q\|^t)$ for all $p,q\in\mathcal{X}$, then we obtain the following corollaries:

Corollary 2.1. *Let $t,\theta\in\mathbb{R}_0^+$ with $t>1$, let $h\in\mathcal{F}_0(\mathcal{X},\mathcal{Y})$ and*

$$\|\delta h(p,q,r,s)\|\leq\theta(\|p\|^t+\|q\|^t)(\|r\|^t+\|s\|^t) \tag{2.17}$$

for all $p,q,r,s\in\mathcal{X}$. Then there exists a unique mapping $H\in\mathcal{AQ}(\mathcal{X},\mathcal{Y})$ such that

$$\|h(p,r)-H(p,r)\|\leq\frac{2\theta}{3(2^t-2)}\|p\|^t\|r\|^t$$

for all $p,r\in\mathcal{X}$.

Corollary 2.2. *Let $t, \theta \in \mathbb{R}_0^+$ with $t < \frac{\ln 3}{\ln 2}$. If $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies* (2.17)*, then there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{\theta}{3-2^t}\|p\|^t\|r\|^t$$

for all $p, r \in \mathcal{X}$.

3. Hyers-Ulam stability of the additive-quadratic functional equation: fixed point method

In this section, we use the fixed point method to prove the Hyers-Ulam stability of the additive-quadratic functional equation (1).

Theorem 3.1. *Let $\omega : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\omega\left(\frac{p}{2}, \frac{q}{2}\right) \leq \frac{L}{3}\omega(p,q) \leq \frac{L}{2}\omega(p,q) \tag{3.1}$$

for all $p, q \in \mathcal{X}$. Then, for a mapping $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfying (2.3)*, there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{L}{6(1-L)}\min\{\omega(p,0)\omega(r,r), \omega(r,0)\omega(p,p)\} \tag{3.2}$$

for all $p, r \in \mathcal{X}$.

Proof. Consider the set $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with the generalized metric d defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}_0^+ : \|f(p,r) - g(p,r)\| \leq \mu\omega(p,0)\,\omega(r,r),\ \forall p, r \in \mathcal{X}\right\},$$

where $\inf\emptyset = +\infty$ as usual. Then $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), d)$ is complete, see [22]. Define a mapping $\mathcal{J} : \mathcal{F}_0(\mathcal{X}, \mathcal{Y}) \to \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ by

$$\mathcal{J}f(p,r) := 3f\left(p, \frac{r}{2}\right)$$

for all $p, r \in \mathcal{X}$. For all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with $d(f,g) = \varepsilon$, we have

$$\|f(p,r) - g(p,r)\| \leq \varepsilon\omega(p,0)\,\omega(r,r)$$

for all $p, r \in \mathcal{X}$. Consequently, from (3.1), we have

$$\begin{aligned}
\|\mathcal{J}f(p,r) - \mathcal{J}g(p,r)\| &= \left\|3f\left(p, \frac{r}{2}\right) - 3g\left(p, \frac{r}{2}\right)\right\| \\
&\leq 3\varepsilon\omega(p,0)\,\omega\left(\frac{r}{2}, \frac{r}{2}\right) \\
&\leq 3\varepsilon\frac{L}{3}\omega(p,0)\,\omega(r,r) \\
&= L\varepsilon\omega(p,0)\,\omega(r,r)
\end{aligned}$$

for all $p, r \in \mathcal{X}$. Then we have $d(\mathcal{J}f, \mathcal{J}g) \leq L\varepsilon$, which means that

$$d(\mathcal{J}f, \mathcal{J}g) \leq Ld(f,g)$$

for all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$. It follows from (2.5) that

$$\left\|3h\left(p, \frac{r}{2}\right) - h(p,r)\right\| \le \frac{1}{2}\omega(p,0)\omega\left(\frac{r}{2}, \frac{r}{2}\right) \le \frac{L}{6}\omega(p,0)\omega(r,r)$$

for all $p, r \in \mathcal{X}$ and so

$$d(h, \mathcal{J}h) \le \frac{L}{6}.$$

From Theorem 1.1, there exists $R : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the following:

(1) R is a unique fixed point of $\mathcal{J}$, i.e.,

$$R(p,r) = 3R\left(p, \frac{r}{2}\right)$$

for all $p, r \in \mathcal{X}$. Thus there exists $\mu \in (0, \infty)$ satisfying

$$\|h(p,r) - R(p,r)\| \le \mu\omega(p,0)\,\omega(r,r)$$

for all $p, r \in \mathcal{X}$;

(2) $d(\mathcal{J}^l h, R) \to 0$ as $l \to \infty$, which implies that

$$\lim_{l\to\infty} 3^l h\left(p, \frac{r}{2^l}\right) = R(p,r)$$

for all $p, r \in \mathcal{X}$;

(3) $d(h, R) \le \frac{1}{1-L} d(h, \mathcal{J}h)$, which implies that

$$\left\|h(p,r) - R(p,r)\right\| \le \frac{L}{6(1-L)}\omega(p,0)\,\omega(r,r)$$

for all $p, r \in \mathcal{X}$.

From (3.1) and for all $p, q \in \mathcal{X}$, we have $3^n\omega\left(\frac{p}{2^n}, \frac{q}{2^n}\right) \le L^n\omega(p,q)$ tends to zero as $n \to \infty$. As in the proof of Theorem 2.1, we can show that $R \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$. Next, consider another generalized metric $\tilde{d}$ on $\mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ given by

$$\tilde{d}(f,g) = \inf\left\{\mu \in \mathbb{R}_0^+ : \|f(p,r) - g(p,r)\| \le \mu\omega(r,0)\omega(p,p),\ \forall p, r \in \mathcal{X}\right\},$$

where $\inf\emptyset = +\infty$ as usual. Then $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), \tilde{d})$ is complete, see [22]. We define the mapping by

$$\tilde{\mathcal{J}}f(p,r) := 2f\left(\frac{p}{2}, r\right)$$

for all $p, r \in \mathcal{X}$. Let $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ with $\tilde{d}(f,g) = \varepsilon$. Then from (3.1), we have

$$\begin{aligned}
\|\tilde{\mathcal{J}}f(p,r) - \tilde{\mathcal{J}}g(p,r)\| &= \left\|2f\left(\frac{p}{2}, r\right) - 2g\left(\frac{p}{2}, r\right)\right\| \\
&\le 2\varepsilon\omega(r,0)\,\omega\left(\frac{p}{2}, \frac{p}{2}\right) \\
&\le 2\varepsilon\frac{L}{2}\omega(r,0)\,\omega(p,p) \\
&= L\varepsilon\omega(r,0)\,\omega(p,p)
\end{aligned}$$

for all $p, r \in \mathcal{X}$. Thus $\tilde{d}(\tilde{\mathcal{J}}f, \tilde{\mathcal{J}}g) \le L\varepsilon$ and so

$$\tilde{d}(\tilde{\mathcal{J}}f, \tilde{\mathcal{J}}g) \le L\tilde{d}(f,g)$$

for all $f, g \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$. It implies by (2.8) that

$$\left\|h(p,r) - 2h\left(\frac{p}{2}, r\right)\right\| \leq \frac{1}{3}\omega\left(\frac{p}{2}, \frac{p}{2}\right)\omega(r,0) \leq \frac{L}{6}\omega(r,0)\omega(p,p)$$

for all $p, r \in \mathcal{X}$. Thus

$$\tilde{d}(h, \tilde{\mathcal{J}}h) \leq \frac{L}{6}.$$

It follows from Theorem 1.1 that there exists a mapping $S : \mathcal{X}^2 \to \mathcal{Y}$ satisfying the following:

(1) S is a unique fixed point of $\tilde{\mathcal{J}}$, i.e.,

$$S\left(p, r\right) = 2S\left(\frac{p}{2}, r\right)$$

for all $p, r \in \mathcal{X}$. Thus there exists $\mu \in (0, \infty)$ satisfying

$$\|h\left(p, r\right) - S\left(p, r\right)\| \leq \mu\omega\left(r, 0\right)\omega(p, p)$$

for all $p, r \in \mathcal{X}$;

(2) $\tilde{d}(\tilde{\mathcal{J}}^l h, S) \to 0$ as $l \to \infty$, which implies that

$$\lim_{l\to\infty} 2^l h\left(\frac{p}{2^l}, r\right) = S\left(p, r\right)$$

for all $p, r \in \mathcal{X}$;

(3) $\tilde{d}(h, S) \leq \frac{1}{1-L}\tilde{d}(h, \tilde{\mathcal{J}}h)$, which implies that

$$\left\|h(p,r) - S(p,r)\right\| \leq \frac{L}{6(1-L)}\omega\left(r, 0\right)\omega(p,p)$$

for all $p, r \in \mathcal{X}$.

From (3.1) and for all $p, q \in \mathcal{X}$, we have $2^n\omega\left(\frac{p}{2^n}, \frac{q}{2^n}\right) \leq L^n\omega\left(p, q\right)$ tends to zero as $n \to \infty$. As in the proof of Theorem 2.1, we can show that $S \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$. We can also obtain that $H(p, r) := R(p, r) = S(p, r)$ for all $p, r \in \mathcal{X}$. Therefore, we can conclude that there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ which satisfies (3.2). This completes the proof. □

Theorem 3.2. *Let $\omega : \mathcal{X}^2 \to \mathbb{R}_0^+$ be a mapping such that there exists $L \in \mathbb{R}_0^+$ with $L < 1$ satisfying*

$$\omega\left(p, q\right) \leq 2L\omega\left(\frac{p}{2}, \frac{q}{2}\right) \leq 3L\omega\left(\frac{p}{2}, \frac{q}{2}\right)$$

for all $p, q \in \mathcal{X}$. Then, for a mapping $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfying (2.3)*, there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{1}{6(1-L)}\min\left\{\omega(p,0)\omega(r,r), \omega(r,0)\omega(p,p)\right\}$$

for all $p, r \in \mathcal{X}$.

Proof. Consider the complete metric spaces $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), d)$ and $(\mathcal{F}_0(\mathcal{X}, \mathcal{Y}), \tilde{d})$ given in the proof of Theorem 3.1. If we define a mapping $\mathcal{J} : \mathcal{F}_0(\mathcal{X}, \mathcal{Y}) \to \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ by

$$\mathcal{J}f(p, r) := \frac{1}{3}f\left(p, 2r\right)$$

for all $p, r \in \mathcal{X}$, then it follows from (2.5) that

$$\left\| h(p,r) - \frac{1}{3}h(p,2r) \right\| \leq \frac{1}{6}\omega(p,0)\omega(r,r)$$

for all $p, r \in \mathcal{X}$. By using the same technique as in the proof of Theorems 2.2 and 3.1, there exists a unique mapping $R \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that

$$\left\| h(p,r) - R(p,r) \right\| \leq \frac{1}{6(1-L)}\omega\left(p,0\right)\omega(r,r)$$

for all $p, r \in \mathcal{X}$. Next, we consider the mapping defined $\tilde{\mathcal{J}} : \mathcal{F}_0(\mathcal{X}, \mathcal{Y}) \to \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ by

$$\tilde{\mathcal{J}} f(p,r) := \frac{1}{2} f\left(2p, r\right)$$

for all $p, r \in \mathcal{X}$, then it implies by (2.8) that

$$\left\| \frac{1}{2}h(2p,r) - h(p,r) \right\| \leq \frac{1}{6}\omega(p,p)\omega(r,0)$$

for all $p, r \in \mathcal{X}$. As in the proof of Theorems 2.2 and 3.1, there exists a unique mapping $S \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that

$$\left\| h(p,r) - S(p,r) \right\| \leq \frac{1}{6(1-L)}\omega\left(r,0\right)\omega(p,p)$$

for all $p, r \in \mathcal{X}$. The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

By taking $L = 2^{1-t}$ and $\omega(p,q) = \sqrt{\theta}(\|p\|^t + \|q\|^t)$ for all $p, q \in \mathcal{X}$ in Theorem 3.1, we have the following:

Corollary 3.1. *Let $t, \theta \in \mathbb{R}_0^+$ with $t > 1$. If $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ satisfies* (2.17)*, then there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{2\theta}{3(2^t - 2)}\|p\|^t\|r\|^t$$

for all $p, r \in \mathcal{X}$.

By taking $L = \frac{2^t}{3}$ and $\omega(p,q) = \sqrt{\theta}(\|p\|^t + \|q\|^t)$ for all $p, q \in \mathcal{X}$ in Theorem 3.2, we have the following:

Corollary 3.2. *Let $t, \theta \in \mathbb{R}_0^+$ with $t < \frac{\ln 3}{\ln 2}$ and let $h \in \mathcal{F}_0(\mathcal{X}, \mathcal{Y})$ be a mapping satisfying* (2.17)*. Then there exists a unique mapping $H \in \mathcal{AQ}(\mathcal{X}, \mathcal{Y})$ such that*

$$\|h(p,r) - H(p,r)\| \leq \frac{\theta}{3 - 2^t}\|p\|^t\|r\|^t$$

for all $p, r \in \mathcal{X}$.

4. Conclusion

We have proven the Hyers-Ulam stability results for the additive-quadratic functional equation (1) in Banach spaces using the direct and fixed point techniques.

Acknowledgments: This research was supported by Lampang Rajabhat University and Thailand Science Research and Innovation Fund (LPRU Fundamental Fund 2025, 030/2568); and the University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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