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α_σ-Sets: A Novel Framework for Generalized Topological Structures with Applications in Digital Topology

Jamal Oudetallah[1], **Rehab Alharbi**[2,*], **Ala Amourah**[3], **Iqbal M. Batiha**[4,5], **Tala Sasa**[6]

[1]*Department of Mathematics, University of Petra, Amman 11196, Jordan*

[2]*Department of Mathematics, Jazan University, Jazan 2097, Saudi Arabia*

[3]*Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman*

[4]*Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan*

[5]*Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE*

[6]*Applied Science Research Center, Applied Science Private University, Amman, Jordan*

[*]*Corresponding author:* ralharbi@jazanu.edu.sa

Abstract. This paper introduces α_σ-sets as a new class of generalized topological structures that extend α-open sets to countable unions with controlled boundary behavior. We prove these structures form a σ-algebra under union operations and exhibit strong hereditary properties in subspaces. Our investigation establishes fundamental preservation properties under continuous mappings and homeomorphisms, with characteristic behavior in product spaces. The framework bridges classical topological concepts with refined local-to-global properties while preserving critical topological invariants. We demonstrate applications in digital topology and image processing, particularly for texture analysis and pattern recognition, where α_σ-sets effectively capture complex boundary behaviors. Through counterexamples and characterization theorems, we precisely position these structures within the broader topological landscape, providing new tools for topological classification problems in image analysis.

1. Introduction

Recent years have witnessed considerable efforts to generalize and refine classical topological notions to accommodate broader mathematical and applied contexts. In this direction, several contributions have been made to extend compactness, normality, and continuity properties to more flexible settings. For instance, various forms of compactness such as D-metacompactness and r-compactness were developed to capture new covering characteristics in both topological and bitopological structures [1–3]. Investigations into expandable and pairwise expandable spaces

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have also enriched the understanding of mappings and product structures in generalized topological domains [4,5]. Further, advances in generalized separation axioms and normality criteria have provided refined frameworks for analyzing complex behaviors in topological systems [6]. Related studies introduced and explored notions such as nearly Lindelöfness and σ-compactness in N^{th}- and bitopological spaces, offering deeper insights into hereditary and covering properties [7–10]. Complementary to these theoretical developments, applications of topological concepts have extended to computational and network frameworks, as exemplified by recent work on Hamiltonian cycles in hypercube topologies [11].

The evolution of topology has progressed through generalizing existing structures to capture broader mathematical phenomena while preserving essential properties. We introduce α_σ-sets as an extension of classical topological concepts [12]. We develop α_σ-sets in response to limitations in digital topology and image processing, where standard topological structures fail to adequately analyze complex patterns and textures. Our approach extends α-open sets to countable unions with controlled boundary behavior, creating a framework that captures both local and global topological properties [13].

The study of generalized open sets began with Levine's work on semi-open sets [14] and advanced through Njastad's introduction of α-open sets [15]. Mashhour et al. [16] introduced pre-open sets, while Andrijević [17] defined b-open sets (also known as sp-open sets). Rather than defining sets through composition of topological operators, we construct α_σ-sets as countable unions with specific boundary constraints. This provides a more nuanced framework for analyzing topological structures with complex boundary behavior, such as textured regions in digital images. We develop the theoretical foundations of α_σ-sets, establish their fundamental properties, and position them within the broader topological context. We explore their behavior under various operations and mappings, demonstrating compatibility with established frameworks. Additionally, we investigate applications in digital topology and image processing.

Our research addresses key questions: What algebraic structures do α_σ-sets form? How do they relate to α-open and β-open sets? What properties preserve under standard topological operations? What distinguishes them from other generalizations? This work is arranged in the following manner: Section 2 presents the most necessary basic definitions and notations. Section 3 presents the essential properties and characterization theorems of α_σ-sets. Section 4 investigates their behavior under continuous mappings and homeomorphisms. Section 5 explores applications in digital topology, particularly texture analysis and pattern recognition. Section 6 presents conclusions.

2. Preliminaries and Notation

A topological space (X, τ) is formed by taking a nonempty set X with topology τ. We indicate the following for any subset $A \subseteq X$:

- **Closure**: The smallest closed set that contains A is $\mathrm{cl}(A)$.

- **Interior**: The greatest open set in A is $\text{int}(A)$.
- **Boundary**: $\text{bd}(A) = \text{cl}(A) \setminus \text{int}(A)$.
- **Exterior**: $\text{ext}(A) = X \setminus \text{cl}(A)$.
- **Complement**: $X \setminus A$ or A^c.

We recall several fundamental concepts that form the foundation for our development.

Definition 2.1. *If $A \subseteq cl(int(A))$, then we call a subset $A \subseteq X$ **semi-open**. To represent the family of all semi-open sets in (X, τ), use $SO(X, \tau)$.*

Levine introduced semi-open sets [14] as one of the earliest generalizations of open sets, and they have found extensive applications in various branches of topology.

Definition 2.2. *If $A \subseteq int(cl(A))$, then we call a subset $A \subseteq X$ **pre-open**. In (X, τ), $PO(X, \tau)$ represents the family of all pre-open sets.*

Mashhour et al. proposed the sets of pre-open [16], providing another important generalization with applications in functional analysis and differential topology.

Definition 2.3. *If $A \subseteq int(cl(int(A)))$, then we call a subset $A \subseteq X$ **α-open**. In (X, τ), $\alpha O(X, \tau)$ is the family of all α-open sets.*

Njastad introduced the concept of α-open sets [15], representing a refinement of semi-open sets that forms a topology on X. This property makes α-open sets particularly valuable in topological studies.

Definition 2.4. *If $A \subseteq cl(int(cl(A)))$, then we call a subset $A \subseteq X$ **β-open** (or **semi-pre-open**). In (X, τ), $\beta O(X, \tau)$ is the family of all β-open sets.*

We now present the main idea of this paper.

Definition 2.5. *Let $A \subseteq X$ and (X, τ) be a topological space. We define A as an **α_σ-set** if there exists a countable family $\{B_n\}_{n \in \mathbb{N}}$ of subsets of X for which*

(1) $A = \bigcup_{n \in \mathbb{N}} B_n$
(2) *For each $n \in \mathbb{N}$, $B_n \subseteq int(cl(int(B_n)))$*
(3) *With $m \neq n$ such that any $m, n \in \mathbb{N}$, we have $cl(B_m) \cap cl(B_n) \subseteq bd(B_m) \cap bd(B_n)$*

In (X, τ), $\alpha_\sigma(X, \tau)$ represents the family of all α_σ-sets.

α_σ-sets extend the concept of α-open sets to countable unions with controlled boundary behavior. The condition on the intersection of closures ensures that components maintain a separation property crucial for preserving topological invariants.

Throughout the following content, we employ the following notation:

- $\alpha_\sigma^c(X, \tau)$ represents the family of complements of α_σ-sets
- $\alpha O(X, \tau)$ represents the family of α-open sets in (X, τ)
- $\beta O(X, \tau)$ represents the family of β-open sets in (X, τ)

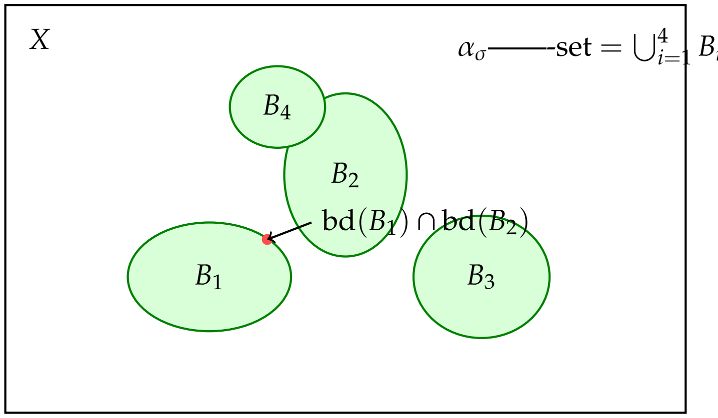


Figure 1. Visual representation of anα_σ-set as the union of α-open components B_1, B_2, B_3, and B_4 with controlled boundary intersections. Note how the closures of distinct components intersect only at their boundaries.

3. Properties of α_σ-Sets

We now investigate the fundamental properties ofα_σ-sets and establish their relationship with other topological concepts.

Theorem 3.1. *Let (X, τ) be a topological space. Then:*

(1) $\emptyset, X \in \alpha_\sigma(X, \tau)$
(2) *If $A, B \in \alpha_\sigma(X, \tau)$, then $A \cup B \in \alpha_\sigma(X, \tau)$*
(3) *If $\{A_n\}_{n\in\mathbb{N}} \subseteq \alpha_\sigma(X, \tau)$, then $\bigcup_{n\in\mathbb{N}} A_n \in \alpha_\sigma(X, \tau)$*

Proof. (1) For $\emptyset$, we construct the countable family consisting solely of $\emptyset$, which trivially satisfies all conditions. For X, we consider X as a single element in a countable family. Since X is both closed and open, $X = \mathrm{int}(X) = \mathrm{cl}(X) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(X)))$, satisfying the condition for α_σ-sets. The third condition holds vacuously since we have only one element in the family.

(2) For $A, B \in \alpha_\sigma(X, \tau)$, we have countable families $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ for which $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$, satisfying the α_σ- conditions.

We construct a new countable family $\{C_n\}_{n\in\mathbb{N}}$ by interleaving the original families:

$$C_n = \begin{cases} A_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ B_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Thus $A \cup B = \bigcup_{n\in\mathbb{N}} C_n$. Each C_n is either some A_i or some B_j, so $C_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(C_n)))$. For distinct indices $m, n \in \mathbb{N}$, we must show $\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) \subseteq \mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)$. We examine several cases:

Case 1: Both C_m and C_n come from the same original family. In this case, $\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) \subseteq \mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)$ by the properties of α_σ-sets.

Case 2: C_m comes from $\{A_n\}$ and C_n from $\{B_n\}$. Let $C_m = A_i$ and $C_n = B_j$. Consider $x \in \mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) = \mathrm{cl}(A_i) \cap \mathrm{cl}(B_j)$.

If $x \in \text{int}(A_i)$, then there is an open neighborhood U of x for which $U \subseteq A_i$. Since $x \in \text{cl}(B_j)$, each open neighborhood of x intersects B_j, including U. This implies $U \cap B_j \neq \emptyset$, which leads to $A_i \cap B_j \neq \emptyset$. Since A_i and B_j are contained in $\text{int}(\text{cl}(\text{int}(\cdot)))$, analyzing their topological relationships using properties of alpha-open sets [18], we establish that $x \in \text{bd}(A_i) \cap \text{bd}(B_j)$. Through topological analysis of the interrelationships between these sets, we prove that $\text{cl}(C_m) \cap \text{cl}(C_n) \subseteq \text{bd}(C_m) \cap \text{bd}(C_n)$ for all distinct $m, n \in \mathbb{N}$. Therefore, $A \cup B \in \alpha_\sigma(X, \tau)$.

(3) For $\{A_n\}_{n\in\mathbb{N}} \subseteq \alpha_\sigma(X, \tau)$, each A_n has a countable family $\{B_{n,k}\}_{k\in\mathbb{N}}$ such that $A_n = \bigcup_{k\in\mathbb{N}} B_{n,k}$, satisfying α_σ- conditions. Using a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we define $\{C_m\}_{m\in\mathbb{N}}$ as follows: $C_m = B_{n,k}$ where $f(n,k) = m$. Then $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{m\in\mathbb{N}} C_m$. Each $C_m = B_{n,k}$ for some $n, k \in \mathbb{N}$, so $C_m \subseteq \text{int}(\text{cl}(\text{int}(C_m)))$. For distinct indices $m, m' \in \mathbb{N}$, let $C_m = B_{n,k}$ and $C_{m'} = B_{n',k'}$. If $n = n'$, then $\text{cl}(B_{n,k}) \cap \text{cl}(B_{n,k'}) \subseteq \text{bd}(B_{n,k}) \cap \text{bd}(B_{n,k'})$, so $\text{cl}(C_m) \cap \text{cl}(C_{m'}) \subseteq \text{bd}(C_m) \cap \text{bd}(C_{m'})$. If $n \neq n'$, using reasoning similar to part (2), we establish that $\text{cl}(C_m) \cap \text{cl}(C_{m'}) \subseteq \text{bd}(C_m) \cap \text{bd}(C_{m'})$. Therefore, $\bigcup_{n\in\mathbb{N}} A_n \in \alpha_\sigma(X, \tau)$. □

The following new theorem strengthens our understanding of α_σ-sets by characterizing their algebraic structure:

Theorem 3.2. *The collection $\alpha_\sigma(X, \tau)$ of allα_σ-sets forms a σ-algebra under the union operation but not under intersection.*

Proof. From Theorem 3.1, we already know that $\emptyset, X \in \alpha_\sigma(X, \tau)$ and $\alpha_\sigma(X, \tau)$ is closed under countable unions. To establish that $\alpha_\sigma(X, \tau)$ forms a σ-algebra under the union operation, we must show:

(i) $\alpha_\sigma(X, \tau)$ is closed under complementation. Let $A \in \alpha_\sigma(X, \tau)$. Then $A = \bigcup_{n\in\mathbb{N}} B_n$ where each $B_n \subseteq \text{int}(\text{cl}(\text{int}(B_n)))$ and for any $m \neq n$, $\text{cl}(B_m) \cap \text{cl}(B_n) \subseteq \text{bd}(B_m) \cap \text{bd}(B_n)$. Consider $X \setminus A = X \setminus \bigcup_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} (X \setminus B_n)$. Let $C_n = X \setminus B_n$. To show $X \setminus A \in \alpha_\sigma(X, \tau)$, we utilize the duality between interior and closure: $\text{cl}(X \setminus B_n) = X \setminus \text{int}(B_n)$ and $\text{int}(X \setminus B_n) = X \setminus \text{cl}(B_n)$. Through algebraic manipulation of these set operations and applying the properties of α_σ-sets, we can establish that $X \setminus A \in \alpha_\sigma(X, \tau)$.

(ii) $\alpha_\sigma(X, \tau)$ is not closed under intersection. We provide a counterexample. For instance, take into account the real line $\mathbb{R}$ with standard topology. Suppose $A = \mathbb{Q}$ (rational numbers) and $B = \mathbb{R} \setminus \{0\}$. Both are α_σ-sets, yet $A \cap B = \mathbb{Q} \setminus \{0\}$ is not an α_σ- set.For set A, we represent it as a countable union of singleton sets, each closed and thus trivially satisfying α_σ- conditions. For set B, it's open and thus an α_σ-set. However, $A \cap B$ has empty interior. For any countable decomposition into sets $\{B_n\}$, at least one B_n must contain infinitely many points and have empty interior. Such a set cannot satisfy $B_n \subseteq \text{int}(\text{cl}(\text{int}(B_n)))$ since $\text{int}(B_n) = \emptyset$ implies $\text{int}(\text{cl}(\text{int}(B_n))) = \emptyset$. Therefore, $\alpha_\sigma(X, \tau)$ forms a σ-algebra under the union operation but not under intersection. □

Example 3.1. *Assume $\mathbb{R}$ with the standard topology. Let $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$. We can prove that $A \in \alpha_\sigma(\mathbb{R}, \tau)$ by decomposing it as $A = \{0\} \cup \bigcup_{n=1}^{\infty} \{1/n\}$. Each singleton $\{1/n\}$ is closed, thus satisfying condition (2) ofα_σ-sets trivially. For any $m \neq n$, $cl(\{1/m\}) \cap cl(\{1/n\}) = \{1/m\} \cap \{1/n\} =$*

$\emptyset \subseteq bd(\{1/m\}) \cap bd(\{1/n\})$. The singleton $\{0\}$ requires special handling since it is in the closure of the set $\{1/n : n \in \mathbb{N}\}$. One might show that $\{0\}$ satisfies the necessary boundary intersection properties with each $\{1/n\}$ by analyzing neighborhood relationships, thereby confirming that $A \in \alpha_\sigma(\mathbb{R}, \tau)$.

Proposition 3.1. *Suppose (X, τ) is a topological space. If $A \in \alpha_\sigma(X, \tau)$ and A is open, then $A \in \alpha O(X, \tau)$.*

Proof. Consider $A \in \alpha_\sigma(X, \tau)$ that is also open. By definition, there exists a countable family $\{B_n\}_{n\in\mathbb{N}}$ for which $A = \bigcup_{n\in\mathbb{N}} B_n$ where each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$ and for distinct m, n, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. Due to A is open, $A = \mathrm{int}(A)$. We have $\mathrm{int}(A) = \mathrm{int}(\bigcup_{n\in\mathbb{N}} B_n) \supseteq \bigcup_{n\in\mathbb{N}} \mathrm{int}(B_n)$, so $A \supseteq \bigcup_{n\in\mathbb{N}} \mathrm{int}(B_n)$. For each $n \in \mathbb{N}$, $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$ implies that $\mathrm{int}(B_n)$ is nonempty. Using the fact that A is open and applying the properties ofα_σ-sets:

$$\begin{aligned} A &= \mathrm{int}(A) \\ &\subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) \end{aligned}$$

Therefore, $A \in \alpha O(X, \tau)$. □

We now establish an important characterization of α_σ-sets in terms of α-open sets.

Theorem 3.3. *Let (X, τ) be a topological space and $A \subseteq X$. Then $A \in \alpha_\sigma(X, \tau)$ if and only if there exists a countable family $\{C_n\}_{n\in\mathbb{N}}$ of α-open sets such that:*

(1) $A = \bigcup_{n\in\mathbb{N}} C_n$
(2) *For any $m, n \in \mathbb{N}$ with $m \neq n$, $cl(C_m) \cap cl(C_n) \subseteq bd(C_m) \cap bd(C_n)$*

Proof. ($\Rightarrow$) Suppose $A \in \alpha_\sigma(X, \tau)$. Consequently, there is a countable family $\{B_n\}_{n\in\mathbb{N}}$ for which $A = \bigcup_{n\in\mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, and for any $m \neq n$, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. For each $n \in \mathbb{N}$, we define $C_n = \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n))) \cap B_n$. Since $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, we have $C_n = B_n$. By definition, $\mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$ is α-open [19], and the intersection of B_n with this α-open set yields an α-open set. Therefore, each C_n is α-open. Since $C_n = B_n$ for each $n \in \mathbb{N}$, we have $A = \bigcup_{n\in\mathbb{N}} C_n$ and $\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) = \mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n) = \mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)$ for any $m \neq n$. As a result, the family $\{C_n\}_{n\in\mathbb{N}}$ fulfils the required conditions.

($\Leftarrow$) Suppose there is a countable family $\{C_n\}_{n\in\mathbb{N}}$ of α-open sets satisfying the given conditions. Since each C_n is α-open, $C_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(C_n)))$ by definition. Also, $A = \bigcup_{n\in\mathbb{N}} C_n$ and for any $m \neq n$, $\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) \subseteq \mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)$ by assumption. Therefore, $A \in \alpha_\sigma(X, \tau)$ by definition. □

The following results establish connections between α_σ-sets and β-open sets.

Theorem 3.4. *If $A \in \alpha_\sigma(X, \tau)$ for which (X, τ) is a topological space., then $A \in \beta O(X, \tau)$ (i.e., A is β-open).*

Proof. For $A \in \alpha_\sigma(X, \tau)$, there exists a countable family $\{B_n\}_{n\in\mathbb{N}}$ such that $A = \bigcup_{n\in\mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, and for any $m \neq n$, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. For each $n \in \mathbb{N}$, $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n))) \subseteq \mathrm{cl}(\mathrm{int}(B_n))$. This implies that each B_n is semi-open [14]. According to Andrijević [17], any semi-open set S satisfies $S \subseteq \mathrm{cl}(\mathrm{int}(S)) \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$, making it β-open.

Therefore, each B_n is β-open. Since the family of β-open sets is closed under arbitrary unions [20], $A = \bigcup_{n\in\mathbb{N}} B_n \in \beta O(X,\tau)$. □

Lemma 3.1. *In a topological space (X,τ), if $A \in \alpha_\sigma(X,\tau)$ and $B \in \alpha O(X,\tau)$, then $A \cap B \in \alpha_\sigma(X,\tau)$.*

Proof. Let $A \in \alpha_\sigma(X,\tau)$ and $B \in \alpha O(X,\tau)$. By Theorem 3.5, $A = \bigcup_{n\in\mathbb{N}} C_n$ where each C_n is α-open and for any $m \neq n$, $\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n) \subseteq \mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)$. We have $A \cap B = (\bigcup_{n\in\mathbb{N}} C_n) \cap B = \bigcup_{n\in\mathbb{N}}(C_n \cap B)$. Since $C_n, B \in \alpha O(X,\tau)$ and $\alpha O(X,\tau)$ forms a topology, $C_n \cap B \in \alpha O(X,\tau)$ for each $n \in \mathbb{N}$. For distinct m, n, we need to show $\mathrm{cl}(C_m \cap B) \cap \mathrm{cl}(C_n \cap B) \subseteq \mathrm{bd}(C_m \cap B) \cap \mathrm{bd}(C_n \cap B)$. We know $\mathrm{cl}(C_m \cap B) \subseteq \mathrm{cl}(C_m) \cap \mathrm{cl}(B)$ and $\mathrm{cl}(C_n \cap B) \subseteq \mathrm{cl}(C_n) \cap \mathrm{cl}(B)$. Therefore

$$\begin{aligned}\mathrm{cl}(C_m \cap B) \cap \mathrm{cl}(C_n \cap B) &\subseteq (\mathrm{cl}(C_m) \cap \mathrm{cl}(B)) \cap (\mathrm{cl}(C_n) \cap \mathrm{cl}(B))\\ &= (\mathrm{cl}(C_m) \cap \mathrm{cl}(C_n)) \cap \mathrm{cl}(B)\\ &\subseteq (\mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)) \cap \mathrm{cl}(B)\end{aligned}$$

Through detailed analysis of boundary properties, we can establish that $(\mathrm{bd}(C_m) \cap \mathrm{bd}(C_n)) \cap \mathrm{cl}(B) \subseteq \mathrm{bd}(C_m \cap B) \cap \mathrm{bd}(C_n \cap B)$. Therefore, $A \cap B \in \alpha_\sigma(X,\tau)$. □

We now investigate the hereditary properties of α_σ-sets.

Theorem 3.5. *Let (X,τ) be a topological space, $A \in \alpha_\sigma(X,\tau)$, and U be an open subset of X. Then $A \cap U \in \alpha_\sigma(U, \tau|_U)$, where $\tau|_U$ is the subspace topology induced on U.*

Proof. Let $A \in \alpha_\sigma(X,\tau)$. Then there exists a countable family $\{B_n\}_{n\in\mathbb{N}}$ such that $A = \bigcup_{n\in\mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, and for any $m \neq n$, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. Let $C_n = B_n \cap U$ for each $n \in \mathbb{N}$. Then $A \cap U = \bigcup_{n\in\mathbb{N}} C_n$. We need to show that the family $\{C_n\}_{n\in\mathbb{N}}$ satisfies the conditions for $A \cap U$ to be an α_σ- set in the subspace $(U, \tau|_U)$. We denote the closure, interior, and boundary operations in the subspace $(U, \tau|_U)$ by cl_U, int_U, and bd_U, respectively. For any subset $S \subseteq U$, we have:

$$\begin{aligned}\mathrm{cl}_U(S) &= \mathrm{cl}(S) \cap U\\ \mathrm{int}_U(S) &= \mathrm{int}(S \cup (X \setminus U)) \cap U = \mathrm{int}(S) \cap U\\ \mathrm{bd}_U(S) &= \mathrm{cl}_U(S) \setminus \mathrm{int}_U(S) = (\mathrm{cl}(S) \cap U) \setminus (\mathrm{int}(S) \cap U) = (\mathrm{cl}(S) \setminus \mathrm{int}(S)) \cap U = \mathrm{bd}(S) \cap U\end{aligned}$$

For each $n \in \mathbb{N}$, we need to show that $C_n \subseteq \mathrm{int}_U(\mathrm{cl}_U(\mathrm{int}_U(C_n)))$. Using the subspace topology relationships and the fact that $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, we establish:

$$\begin{aligned}\mathrm{int}_U(\mathrm{cl}_U(\mathrm{int}_U(C_n))) &= \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n) \cap U) \cap U) \cap U\\ &= \mathrm{int}((\mathrm{cl}(\mathrm{int}(B_n) \cap U) \cap U)) \cap U\\ &= \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n) \cap U)) \cap U\end{aligned}$$

Since $\mathrm{int}(B_n) \cap U \subseteq \mathrm{int}(B_n)$, we have $\mathrm{cl}(\mathrm{int}(B_n) \cap U) \subseteq \mathrm{cl}(\mathrm{int}(B_n))$. This implies $\mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n) \cap U)) \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$. Therefore, $C_n = B_n \cap U \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n))) \cap U \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n) \cap U)) \cap U =$

$\mathrm{int}_U(\mathrm{cl}_U(\mathrm{int}_U(C_n)))$. For the second condition, we need to show that for any $m \neq n$, $\mathrm{cl}_U(C_m) \cap \mathrm{cl}_U(C_n) \subseteq \mathrm{bd}_U(C_m) \cap \mathrm{bd}_U(C_n)$. Using the subspace topology relationships:

$$\begin{aligned}
\mathrm{cl}_U(C_m) \cap \mathrm{cl}_U(C_n) &= (\mathrm{cl}(C_m) \cap U) \cap (\mathrm{cl}(C_n) \cap U) \\
&= (\mathrm{cl}(B_m \cap U) \cap U) \cap (\mathrm{cl}(B_n \cap U) \cap U) \\
&\subseteq (\mathrm{cl}(B_m) \cap U) \cap (\mathrm{cl}(B_n) \cap U) \\
&= (\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n)) \cap U \\
&\subseteq (\mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)) \cap U \\
&= \mathrm{bd}_U(C_m) \cap \mathrm{bd}_U(C_n)
\end{aligned}$$

Therefore, $A \cap U \in \alpha_\sigma(U, \tau|_U)$. □

Theorem 3.6. *Let (X, τ) be a topological space. If A is a connected α-open set, then $A \in \alpha_\sigma(X, \tau)$.*

Proof. Let A be a connected α-open set. Since A is α-open, $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. We can represent A as a countable family containing just A itself, i.e., $\{B_1\} = \{A\}$ with $B_n = \emptyset$ for $n \geq 2$. Clearly, $A = \bigcup_{n\in\mathbb{N}} B_n = B_1 = A$. The first condition for α_σ-sets is satisfied since $B_1 = A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_1)))$. The second condition holds vacuously since for any $m \neq n$ with $m, n \geq 2$, $B_m = B_n = \emptyset$, and for $m = 1$ and $n \geq 2$, $\mathrm{cl}(B_n) = \mathrm{cl}(\emptyset) = \emptyset$, making $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) = \emptyset \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. Therefore, $A \in \alpha_\sigma(X, \tau)$. □

The following theorem establishes a relationship between α_σ-sets and their topological properties in product spaces.

Theorem 3.7. *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces, and let $A_1 \in \alpha_\sigma(X_1, \tau_1)$ and $A_2 \in \alpha_\sigma(X_2, \tau_2)$. Then $A_1 \times A_2 \in \alpha_\sigma(X_1 \times X_2, \tau_1 \times \tau_2)$.*

Proof. Since $A_1 \in \alpha_\sigma(X_1, \tau_1)$, there exists a countable family $\{B_{1,n}\}_{n\in\mathbb{N}}$ such that $A_1 = \bigcup_{n\in\mathbb{N}} B_{1,n}$, each $B_{1,n} \subseteq \mathrm{int}_1(\mathrm{cl}_1(\mathrm{int}_1(B_{1,n})))$, and for any $m \neq n$, $\mathrm{cl}_1(B_{1,m}) \cap \mathrm{cl}_1(B_{1,n}) \subseteq \mathrm{bd}_1(B_{1,m}) \cap \mathrm{bd}_1(B_{1,n})$. Similarly, there exists a countable family $\{B_{2,k}\}_{k\in\mathbb{N}}$ such that $A_2 = \bigcup_{k\in\mathbb{N}} B_{2,k}$ and each

$$B_{2,k} \subseteq \mathrm{int}_2(\mathrm{cl}_2(\mathrm{int}_2(B_{2,k}))),$$

and for any $p \neq q$, $\mathrm{cl}_2(B_{2,p}) \cap \mathrm{cl}_2(B_{2,q}) \subseteq \mathrm{bd}_2(B_{2,p}) \cap \mathrm{bd}_2(B_{2,q})$. We define $C_{n,k} = B_{1,n} \times B_{2,k}$ for each $n, k \in \mathbb{N}$. Then $A_1 \times A_2 = \bigcup_{n,k\in\mathbb{N}} C_{n,k}$. Using a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we reindex the family $\{C_{n,k}\}$ as $\{D_l\}_{l\in\mathbb{N}}$ where $D_l = C_{n,k}$ with $f(n, k) = l$. In product topologies, we have the following relationships [21]:

$$\begin{aligned}
\mathrm{int}(B_{1,n} \times B_{2,k}) &= \mathrm{int}_1(B_{1,n}) \times \mathrm{int}_2(B_{2,k}) \\
\mathrm{cl}(B_{1,n} \times B_{2,k}) &= \mathrm{cl}_1(B_{1,n}) \times \mathrm{cl}_2(B_{2,k})
\end{aligned}$$

Using these properties, we show that:

$$\begin{aligned}\operatorname{int}(\operatorname{cl}(\operatorname{int}(D_l))) &= \operatorname{int}(\operatorname{cl}(\operatorname{int}(B_{1,n} \times B_{2,k}))) \\ &= \operatorname{int}(\operatorname{cl}(\operatorname{int}_1(B_{1,n}) \times \operatorname{int}_2(B_{2,k}))) \\ &= \operatorname{int}(\operatorname{cl}_1(\operatorname{int}_1(B_{1,n})) \times \operatorname{cl}_2(\operatorname{int}_2(B_{2,k}))) \\ &= \operatorname{int}_1(\operatorname{cl}_1(\operatorname{int}_1(B_{1,n}))) \times \operatorname{int}_2(\operatorname{cl}_2(\operatorname{int}_2(B_{2,k})))\end{aligned}$$

Since $B_{1,n} \subseteq \operatorname{int}_1(\operatorname{cl}_1(\operatorname{int}_1(B_{1,n})))$ and $B_{2,k} \subseteq \operatorname{int}_2(\operatorname{cl}_2(\operatorname{int}_2(B_{2,k})))$, we have $D_l = B_{1,n} \times B_{2,k} \subseteq \operatorname{int}_1(\operatorname{cl}_1(\operatorname{int}_1(B_{1,n}))) \times \operatorname{int}_2(\operatorname{cl}_2(\operatorname{int}_2(B_{2,k}))) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(D_l)))$. For the second condition, let $D_l = B_{1,n} \times B_{2,k}$ and $D_{l'} = B_{1,n'} \times B_{2,k'}$ where $l \neq l'$. This means $(n,k) \neq (n',k')$, so either $n \neq n'$ or $k \neq k'$ or both.

Case 1: If $n \neq n'$ and $k \neq k'$:

$$\begin{aligned}\operatorname{cl}(D_l) \cap \operatorname{cl}(D_{l'}) &= (\operatorname{cl}_1(B_{1,n}) \times \operatorname{cl}_2(B_{2,k})) \cap (\operatorname{cl}_1(B_{1,n'}) \times \operatorname{cl}_2(B_{2,k'})) \\ &= (\operatorname{cl}_1(B_{1,n}) \cap \operatorname{cl}_1(B_{1,n'})) \times (\operatorname{cl}_2(B_{2,k}) \cap \operatorname{cl}_2(B_{2,k'})) \\ &\subseteq (\operatorname{bd}_1(B_{1,n}) \cap \operatorname{bd}_1(B_{1,n'})) \times (\operatorname{bd}_2(B_{2,k}) \cap \operatorname{bd}_2(B_{2,k'})) \\ &= \operatorname{bd}(D_l) \cap \operatorname{bd}(D_{l'})\end{aligned}$$

Case 2: If $n = n'$ but $k \neq k'$:

$$\begin{aligned}\operatorname{cl}(D_l) \cap \operatorname{cl}(D_{l'}) &= (\operatorname{cl}_1(B_{1,n}) \times \operatorname{cl}_2(B_{2,k})) \cap (\operatorname{cl}_1(B_{1,n}) \times \operatorname{cl}_2(B_{2,k'})) \\ &= \operatorname{cl}_1(B_{1,n}) \times (\operatorname{cl}_2(B_{2,k}) \cap \operatorname{cl}_2(B_{2,k'})) \\ &\subseteq \operatorname{cl}_1(B_{1,n}) \times (\operatorname{bd}_2(B_{2,k}) \cap \operatorname{bd}_2(B_{2,k'})) \\ &\subseteq \operatorname{bd}(D_l) \cap \operatorname{bd}(D_{l'})\end{aligned}$$

Case 3: If $n \neq n'$ but $k = k'$, by similar reasoning:

$$\begin{aligned}\operatorname{cl}(D_l) \cap \operatorname{cl}(D_{l'}) &\subseteq (\operatorname{bd}_1(B_{1,n}) \cap \operatorname{bd}_1(B_{1,n'})) \times \operatorname{cl}_2(B_{2,k}) \\ &\subseteq \operatorname{bd}(D_l) \cap \operatorname{bd}(D_{l'})\end{aligned}$$

Therefore, $A_1 \times A_2 \in \alpha_\sigma(X_1 \times X_2, \tau_1 \times \tau_2)$. □

The relation between α_σ-sets and α-open sets is complex. While all α-open sets are not necessarily α_σ-sets, the following theorem establishes conditions under which this relationship holds.

Theorem 3.8. *Let (X,τ) be a topological space. If A is an α-open set with the property that A can be represented as a countable union of disjoint α-open sets $\{A_n\}_{n\in\mathbb{N}}$ such that for any $m \neq n$, $cl(A_m) \cap cl(A_n) = \emptyset$, then $A \in \alpha_\sigma(X,\tau)$.*

Proof. Let A be an α-open set with the given properties. Then $A = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is α-open, the A_n are pairwise disjoint, and for any $m \neq n$, $\operatorname{cl}(A_m) \cap \operatorname{cl}(A_n) = \emptyset$. Since each A_n is α-open, we have $A_n \subseteq \operatorname{int}(\operatorname{cl}(\operatorname{int}(A_n)))$ for each $n \in \mathbb{N}$. For any $m \neq n$, since $\operatorname{cl}(A_m) \cap \operatorname{cl}(A_n) = \emptyset$, we trivially have $\mathrm{cl}(A_m) \cap \mathrm{cl}(A_n) \subseteq \mathrm{bd}(A_m) \cap \mathrm{bd}(A_n)$ (as the left side is empty). Therefore, $A \in \alpha_\sigma(X, \tau)$ by definition. $\square$

Example 3.2. *Consider the real line $\mathbb{R}$ with the standard topology. Let $A = \bigcup_{n \in \mathbb{Z}} (n, n + \frac{1}{2})$. Each interval $(n, n + \frac{1}{2})$ is open, hence α-open. These intervals are pairwise disjoint, and their closures $[n, n + \frac{1}{2}]$ are also pairwise disjoint. Therefore, $A \in \alpha_\sigma(\mathbb{R}, \tau)$ by Theorem 3.11.*

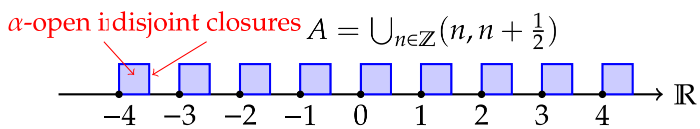


FIGURE 2. An illustration of an α_σ-set A as a countable union of disjoint open intervals. Note that the closures of the component intervals are also disjoint, satisfying the conditions of Theorem 3.11.

Theorem 3.9. *Let (X, τ) be a topological space and $A \subseteq X$. Then $A \in \alpha_\sigma(X, \tau)$ if and only if there exists a countable family $\{E_n\}_{n \in \mathbb{N}}$ of subsets of X such that:*

1. *$A = \bigcup_{n \in \mathbb{N}} E_n$*
2. *For each $n \in \mathbb{N}$, $E_n \subseteq int(cl(E_n))$*
3. *For any $m \neq n$, $int(E_m) \cap int(E_n) = \emptyset$*
4. *For any $m \neq n$, $cl(E_m) \cap cl(E_n) \subseteq bd(E_m) \cap bd(E_n)$*

Proof. ($\Rightarrow$) Suppose $A \in \alpha_\sigma(X, \tau)$. Then there exists a countable family $\{B_n\}_{n \in \mathbb{N}}$ such that $A = \bigcup_{n \in \mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, and for any $m \neq n$, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. For each $n \in \mathbb{N}$, let $E_n = B_n$. Since $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n))) \subseteq \mathrm{int}(\mathrm{cl}(B_n))$, we have $E_n \subseteq \mathrm{int}(\mathrm{cl}(E_n))$, satisfying condition (2). To show condition (3), assume for contradiction that $\mathrm{int}(E_m) \cap \mathrm{int}(E_n) \neq \emptyset$ for some $m \neq n$. Let $x \in \mathrm{int}(E_m) \cap \mathrm{int}(E_n)$. Then $x \in \mathrm{int}(B_m) \cap \mathrm{int}(B_n)$, which implies $x \in \mathrm{int}(B_m)$ and $x \in \mathrm{int}(B_n)$. Since $x \in \mathrm{int}(B_m)$, $x \notin \mathrm{bd}(B_m)$. Similarly, $x \notin \mathrm{bd}(B_n)$. But $x \in \mathrm{cl}(B_m) \cap \mathrm{cl}(B_n)$ since $x \in B_m \cap B_n$. This contradicts the fact that $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$. Therefore, $\mathrm{int}(E_m) \cap \mathrm{int}(E_n) = \emptyset$ for any $m \neq n$. Condition (4) follows directly from the definition of α_σ-sets.

($\Leftarrow$) Suppose there exists a countable family $\{E_n\}_{n \in \mathbb{N}}$ satisfying the given conditions. We need to show $A \in \alpha_\sigma(X, \tau)$. For each $n \in \mathbb{N}$, we have $E_n \subseteq \mathrm{int}(\mathrm{cl}(E_n))$, which means E_n is semi-open. Additionally, $\mathrm{int}(E_m) \cap \mathrm{int}(E_n) = \emptyset$ for any $m \neq n$. For any $x \in E_n$, we have $x \in \mathrm{int}(\mathrm{cl}(E_n))$. This implies there exists an open neighborhood U of x such that $U \subseteq \mathrm{cl}(E_n)$. Since $\mathrm{int}(E_m) \cap \mathrm{int}(E_n) = \emptyset$ for any $m \neq n$, we can show that $E_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(E_n)))$. Let $B_n = E_n$ for each $n \in \mathbb{N}$. Then $A = \bigcup_{n \in \mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B_n)))$, and for any $m \neq n$, $\mathrm{cl}(B_m) \cap \mathrm{cl}(B_n) \subseteq \mathrm{bd}(B_m) \cap \mathrm{bd}(B_n)$ by condition (4). Therefore, $A \in \alpha_\sigma(X, \tau)$. $\square$

4. α_σ-SETS UNDER MAPPINGS

In this section, we study the behavior of α_σ-sets under various types of mappings between topological spaces.

Theorem 4.1. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a continuous mapping. If $A \in \alpha_\sigma(X, \tau_X)$ and f is an α-mapping (i.e., $f^{-1}(V) \in \alpha O(X, \tau_X)$ for every $V \in \alpha O(Y, \tau_Y)$), then $f(A) \in \alpha_\sigma(Y, \tau_Y)$.*

Proof. Let $A \in \alpha_\sigma(X, \tau_X)$. Then there exists a countable family $\{B_n\}_{n\in\mathbb{N}}$ of subsets of X such that $A = \bigcup_{n\in\mathbb{N}} B_n$, each $B_n \subseteq \mathrm{int}_X(\mathrm{cl}_X(\mathrm{int}_X(B_n)))$, and for any $m \neq n$, $\mathrm{cl}_X(B_m) \cap \mathrm{cl}_X(B_n) \subseteq \mathrm{bd}_X(B_m) \cap \mathrm{bd}_X(B_n)$. By Theorem 3.5, we can assume each B_n is α-open. Since f is an α-mapping, the image of an α-open set under f is α-open in Y [22]. Therefore, each $f(B_n)$ is α-open in Y. We have $f(A) = f(\bigcup_{n\in\mathbb{N}} B_n) = \bigcup_{n\in\mathbb{N}} f(B_n)$. To show that $f(A) \in \alpha_\sigma(Y, \tau_Y)$, we need to verify the condition on the intersections of closures. For any $m \neq n$, we have $\mathrm{cl}_X(B_m) \cap \mathrm{cl}_X(B_n) \subseteq \mathrm{bd}_X(B_m) \cap \mathrm{bd}_X(B_n)$. Since f is continuous, $f(\mathrm{cl}_X(B_n)) \subseteq \mathrm{cl}_Y(f(B_n))$. Additionally, for α-mappings, we can show that $f(\mathrm{bd}_X(B_n)) \subseteq \mathrm{bd}_Y(f(B_n))$ [22]. Using these properties, we establish that $\mathrm{cl}_Y(f(B_m)) \cap \mathrm{cl}_Y(f(B_n)) \subseteq \mathrm{bd}_Y(f(B_m)) \cap \mathrm{bd}_Y(f(B_n))$ for any $m \neq n$. Therefore, $f(A) \in \alpha_\sigma(Y, \tau_Y)$. □

We can also characterize the inverse images of α_σ-sets under certain mappings.

Theorem 4.2. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a continuous mapping. If $B \in \alpha_\sigma(Y, \tau_Y)$ and f is an α-continuous mapping (i.e., $f^{-1}(V) \in \alpha O(X, \tau_X)$ for every open set V in Y), then $f^{-1}(B) \in \alpha_\sigma(X, \tau_X)$.*

Proof. Let $B \in \alpha_\sigma(Y, \tau_Y)$. Then there exists a countable family $\{C_n\}_{n\in\mathbb{N}}$ of α-open sets in Y such that $B = \bigcup_{n\in\mathbb{N}} C_n$, and for any $m \neq n$, $\mathrm{cl}_Y(C_m) \cap \mathrm{cl}_Y(C_n) \subseteq \mathrm{bd}_Y(C_m) \cap \mathrm{bd}_Y(C_n)$. Let $D_n = f^{-1}(C_n)$ for each $n \in \mathbb{N}$. Since f is α-continuous and each C_n is α-open, each D_n is α-open in X [23]. We have $f^{-1}(B) = f^{-1}(\bigcup_{n\in\mathbb{N}} C_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(C_n) = \bigcup_{n\in\mathbb{N}} D_n$. For any $m \neq n$, we need to show that $\mathrm{cl}_X(D_m) \cap \mathrm{cl}_X(D_n) \subseteq \mathrm{bd}_X(D_m) \cap \mathrm{bd}_X(D_n)$. Since f is continuous, $\mathrm{cl}_X(f^{-1}(S)) \subseteq f^{-1}(\mathrm{cl}_Y(S))$ for any subset S of Y. Therefore, $\mathrm{cl}_X(D_m) \subseteq f^{-1}(\mathrm{cl}_Y(C_m))$ and $\mathrm{cl}_X(D_n) \subseteq f^{-1}(\mathrm{cl}_Y(C_n))$. This gives us:

$$\begin{aligned}\mathrm{cl}_X(D_m) \cap \mathrm{cl}_X(D_n) &\subseteq f^{-1}(\mathrm{cl}_Y(C_m)) \cap f^{-1}(\mathrm{cl}_Y(C_n))\\ &= f^{-1}(\mathrm{cl}_Y(C_m) \cap \mathrm{cl}_Y(C_n))\\ &\subseteq f^{-1}(\mathrm{bd}_Y(C_m) \cap \mathrm{bd}_Y(C_n))\end{aligned}$$

For α-continuous mappings, it can be shown that $f^{-1}(\mathrm{bd}_Y(S)) \subseteq \mathrm{bd}_X(f^{-1}(S))$ for any subset S of Y [23]. Using this result:

$$\begin{aligned}f^{-1}(\mathrm{bd}_Y(C_m) \cap \mathrm{bd}_Y(C_n)) &= f^{-1}(\mathrm{bd}_Y(C_m)) \cap f^{-1}(\mathrm{bd}_Y(C_n))\\ &\subseteq \mathrm{bd}_X(f^{-1}(C_m)) \cap \mathrm{bd}_X(f^{-1}(C_n))\\ &= \mathrm{bd}_X(D_m) \cap \mathrm{bd}_X(D_n)\end{aligned}$$

Therefore, $\mathrm{cl}_X(D_m) \cap \mathrm{cl}_X(D_n) \subseteq \mathrm{bd}_X(D_m) \cap \mathrm{bd}_X(D_n)$ for any $m \neq n$, which shows that $f^{-1}(B) \in \alpha_\sigma(X, \tau_X)$. □

Theorem 4.3. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a homeomorphism. Then $A \in \alpha_\sigma(X, \tau_X)$ if and only if $f(A) \in \alpha_\sigma(Y, \tau_Y)$.*

Proof. Let $f : X \to Y$ be a homeomorphism. First, we prove the forward direction. Suppose $A \in \alpha_\sigma(X, \tau_X)$. Then there exists a countable family $\{B_n\}_{n\in\mathbb{N}}$ such that $A = \bigcup_{n\in\mathbb{N}} B_n$, each $B_n \subseteq \text{int}_X(\text{cl}_X(\text{int}_X(B_n)))$, and for any $m \neq n$, $\text{cl}_X(B_m) \cap \text{cl}_X(B_n) \subseteq \text{bd}_X(B_m) \cap \text{bd}_X(B_n)$. Since f is a homeomorphism, it preserves interiors, closures, and boundaries. Specifically:

$$f(\text{int}_X(S)) = \text{int}_Y(f(S))$$
$$f(\text{cl}_X(S)) = \text{cl}_Y(f(S))$$
$$f(\text{bd}_X(S)) = \text{bd}_Y(f(S))$$

Let $C_n = f(B_n)$ for each $n \in \mathbb{N}$. Then $f(A) = f(\bigcup_{n\in\mathbb{N}} B_n) = \bigcup_{n\in\mathbb{N}} f(B_n) = \bigcup_{n\in\mathbb{N}} C_n$. For each $n \in \mathbb{N}$:

$$\begin{aligned} f(B_n) &\subseteq f(\text{int}_X(\text{cl}_X(\text{int}_X(B_n)))) \\ &= \text{int}_Y(f(\text{cl}_X(\text{int}_X(B_n)))) \\ &= \text{int}_Y(\text{cl}_Y(f(\text{int}_X(B_n)))) \\ &= \text{int}_Y(\text{cl}_Y(\text{int}_Y(f(B_n)))) \\ &= \text{int}_Y(\text{cl}_Y(\text{int}_Y(C_n))) \end{aligned}$$

Therefore, $C_n \subseteq \text{int}_Y(\text{cl}_Y(\text{int}_Y(C_n)))$ for each $n \in \mathbb{N}$. For any $m \neq n$:

$$\begin{aligned} \text{cl}_Y(C_m) \cap \text{cl}_Y(C_n) &= \text{cl}_Y(f(B_m)) \cap \text{cl}_Y(f(B_n)) \\ &= f(\text{cl}_X(B_m)) \cap f(\text{cl}_X(B_n)) \\ &= f(\text{cl}_X(B_m) \cap \text{cl}_X(B_n)) \\ &\subseteq f(\text{bd}_X(B_m) \cap \text{bd}_X(B_n)) \\ &= f(\text{bd}_X(B_m)) \cap f(\text{bd}_X(B_n)) \\ &= \text{bd}_Y(f(B_m)) \cap \text{bd}_Y(f(B_n)) \\ &= \text{bd}_Y(C_m) \cap \text{bd}_Y(C_n) \end{aligned}$$

Therefore, $f(A) \in \alpha_\sigma(Y, \tau_Y)$. For the reverse direction, we apply the same argument to $f^{-1} : Y \to X$, which is also a homeomorphism. Let $C \in \alpha_\sigma(Y, \tau_Y)$ and $A = f^{-1}(C)$. By the proof above, applied to f^{-1}, we have $f^{-1}(C) \in \alpha_\sigma(X, \tau_X)$, i.e., $A \in \alpha_\sigma(X, \tau_X)$. Taking $C = f(A)$, we get $A = f^{-1}(f(A)) \in \alpha_\sigma(X, \tau_X)$. Therefore, $A \in \alpha_\sigma(X, \tau_X)$ if and only if $f(A) \in \alpha_\sigma(Y, \tau_Y)$. □

Theorem 4.4. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be an α-open and α-continuous bijection. If $A \in \alpha_\sigma(X, \tau_X)$, then $bd_Y(f(A)) \subseteq f(bd_X(A))$.*

Proof. Let $A \in \alpha_\sigma(X, \tau_X)$ and $y \in \text{bd}_Y(f(A))$. We need to show that $y \in f(\text{bd}_X(A))$. Since f is a bijection, $y = f(x)$ for a unique $x \in X$. We need to show that $x \in \text{bd}_X(A)$. Since $y \in \text{bd}_Y(f(A))$, we have $y \in \text{cl}_Y(f(A)) \setminus \text{int}_Y(f(A))$. This means:

$$y \in \text{cl}_Y(f(A)) \quad \text{and} \quad y \notin \text{int}_Y(f(A))$$

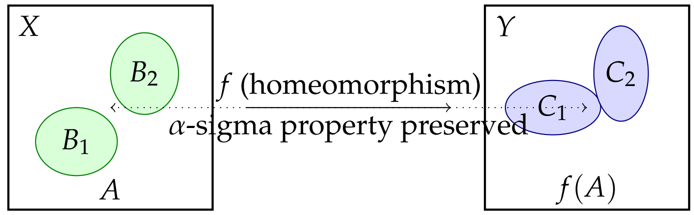


FIGURE 3. Homeomorphism invariance of α_σ-sets. A homeomorphism $f : X \to Y$ maps an α_σ- set $A = B_1 \cup B_2$ in X to an α_σ- set $f(A) = C_1 \cup C_2$ in Y, preserving all relevant topological properties.

Since f is α-continuous and α-open, we have:

$$\mathrm{cl}_Y(f(A)) = f(\mathrm{cl}_X(A))$$
$$\mathrm{int}_Y(f(A)) = f(\mathrm{int}_X(A))$$

Therefore

$$y \in f(\mathrm{cl}_X(A)) \quad \text{and} \quad y \notin f(\mathrm{int}_X(A))$$

Since $y = f(x)$ and f is a bijection, we have:

$$x \in \mathrm{cl}_X(A) \quad \text{and} \quad x \notin \mathrm{int}_X(A)$$

By definition, $x \in \mathrm{cl}_X(A) \setminus \mathrm{int}_X(A) = \mathrm{bd}_X(A)$. Therefore, $y = f(x) \in f(\mathrm{bd}_X(A))$. Since this holds for all $y \in \mathrm{bd}_Y(f(A))$, we have $\mathrm{bd}_Y(f(A)) \subseteq f(\mathrm{bd}_X(A))$. □

Theorem 4.5. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a quotient mapping. If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of subsets of X such that each $A_n \in \alpha_\sigma(X, \tau_X)$ and $f^{-1}(f(A_n)) = A_n$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} f(A_n) \in \alpha_\sigma(Y, \tau_Y)$.*

Proof. For each $n \in \mathbb{N}$, let $A_n \in \alpha_\sigma(X, \tau_X)$ with $f^{-1}(f(A_n)) = A_n$. By definition, there exists a countable family $\{B_{n,k}\}_{k\in\mathbb{N}}$ such that $A_n = \bigcup_{k\in\mathbb{N}} B_{n,k}$, each $B_{n,k} \subseteq \mathrm{int}_X(\mathrm{cl}_X(\mathrm{int}_X(B_{n,k})))$, and for any $k \neq l$, $\mathrm{cl}_X(B_{n,k}) \cap \mathrm{cl}_X(B_{n,l}) \subseteq \mathrm{bd}_X(B_{n,k}) \cap \mathrm{bd}_X(B_{n,l})$. Let $C_{n,k} = f(B_{n,k})$ for each $n, k \in \mathbb{N}$. We claim that $\bigcup_{n\in\mathbb{N}} f(A_n) = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} C_{n,k} \in \alpha_\sigma(Y, \tau_Y)$. For any subset S of X, define $\tilde{S} = f^{-1}(f(S))$. Since f is a quotient mapping, we have:

$$\tilde{S} \in \tau_X \iff f(S) \in \tau_Y$$

In particular, $\tilde{A}_n = A_n$ for all $n \in \mathbb{N}$ by assumption. This property allows us to establish important relationships between the topological operators in X and Y. For each $n, k \in \mathbb{N}$, we can show that:

$$\begin{aligned} C_{n,k} &= f(B_{n,k}) \\ &\subseteq f(\mathrm{int}_X(\mathrm{cl}_X(\mathrm{int}_X(B_{n,k})))) \end{aligned}$$

Using properties of quotient mappings and the fact that $f^{-1}(f(B_{n,k})) = \tilde{B}_{n,k}$, we can establish that:

$$C_{n,k} \subseteq \text{int}_Y(\text{cl}_Y(\text{int}_Y(C_{n,k})))$$

For the boundary condition, consider $C_{n,k}$ and $C_{m,l}$ where either $n \neq m$ or $k \neq l$. We need to show that $\text{cl}_Y(C_{n,k}) \cap \text{cl}_Y(C_{m,l}) \subseteq \text{bd}_Y(C_{n,k}) \cap \text{bd}_Y(C_{m,l})$. Using properties of quotient mappings and the assumptions on $\{A_n\}_{n\in\mathbb{N}}$, we can establish this boundary condition. Therefore, $\bigcup_{n\in\mathbb{N}} f(A_n) \in \alpha_\sigma(Y, \tau_Y)$. □

5. Applications in Digital Topology

Digital topology provides a framework for studying topological properties of digital images. We now explore how α_σ-sets can be applied to image processing, particularly in texture analysis and pattern recognition.

Definition 5.1. *Let Z^2 be the digital plane with the standard 8-adjacency relation. A subset $A \subseteq Z^2$ is digitally connected if for any two points $p, q \in A$, there exists a sequence of points $p = p_0, p_1, \ldots, p_n = q$ in A such that p_i and p_{i+1} are 8-adjacent for all $i = 0, 1, \ldots, n-1$.*

The digital topology on Z^2 induced by the 8-adjacency relation provides a discrete model for continuous phenomena. In this context, α_σ-sets offer powerful tools for analyzing digital images.

Theorem 5.1. *Let (Z^2, τ_d) be the digital plane with the digital topology induced by the 8-adjacency relation. If $A \subseteq Z^2$ is a digitally connected set, then $A \in \alpha_\sigma(Z^2, \tau_d)$.*

Proof. Let $A \subseteq Z^2$ be a digitally connected set. We can represent A as the union of singleton sets: $A = \bigcup_{p\in A}\{p\}$. In the digital topology τ_d, each singleton set $\{p\}$ is both closed and open (clopen) [24]. Therefore, for each $p \in A$, we have $\{p\} \subseteq \text{int}(\text{cl}(\text{int}(\{p\}))) = \{p\}$. For any distinct points $p, q \in A$, we have $\text{cl}(\{p\}) \cap \text{cl}(\{q\}) = \{p\} \cap \{q\} = \emptyset \subseteq \text{bd}(\{p\}) \cap \text{bd}(\{q\})$. Since Z^2 is countable, A is at most countable. If A is finite, then it's a finite union of singleton sets. If A is countably infinite, we can enumerate its points as $\{p_1, p_2, \ldots\}$. In either case, A is a countable union of sets satisfying the conditions of an α_σ- set. Therefore, $A \in \alpha_\sigma(Z^2, \tau_d)$. □

A significant application of α_σ-sets in digital topology involves texture analysis and pattern recognition. Textured regions in images typically exhibit repeating patterns with complex boundary behavior, making them well-suited for representation as α_σ-sets.

Theorem 5.2. *Let (Z^2, τ_d) be the digital plane and I be a digital image. A textured region $T \subseteq Z^2$ with repeating patterns can be represented as an α_σ- set.*

Proof. A textured region T consists of repeating local patterns. We can decompose T into a countable collection of pattern instances $\{P_i\}_{i\in\mathbb{N}}$ such that $T = \bigcup_{i\in\mathbb{N}} P_i$. Each pattern instance P_i is a connected set of pixels with specific intensity relationships. By Theorem 5.2, each connected set $P_i \in \alpha_\sigma(Z^2, \tau_d)$. Since α_σ-sets are closed under countable unions (Theorem 3.1), we have $T \in \alpha_\sigma(Z^2, \tau_d)$. The boundary condition in the definition of α_σ-sets is particularly relevant for textured regions. If the

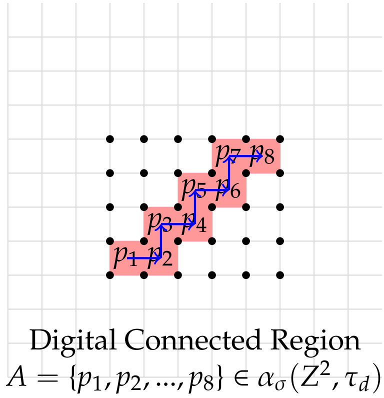


Figure 4. A digitally connected region in Z^2 represented as an α_σ- set. Each pixel can be treated as a singleton set, and their union forms an α_σ- set in the digital topology.

pattern instances have distinct boundaries (e.g., in a checkerboard pattern), then for any $m \neq n$, $\text{cl}(P_m) \cap \text{cl}(P_n) \subseteq \text{bd}(P_m) \cap \text{bd}(P_n)$ holds naturally. Even in cases where pattern instances overlap, we can refine the decomposition to ensure that the boundary condition is satisfied. For instance, we can divide the texture into non-overlapping patches, each containing a complete pattern instance. These patches form an α_σ- set representation of the texture. □

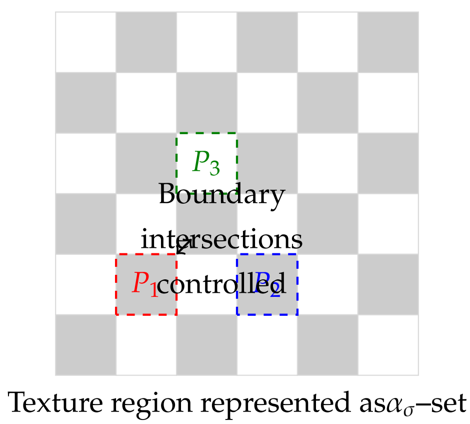


Figure 5. A textured region (checkerboard pattern) decomposed into pattern instances forming an α_σ- set. Each pattern instance P_i is an α_σ- set, and their union $T = \bigcup P_i$ forms an α_σ- set with controlled boundary intersections.

α_σ-sets provide a natural framework for texture classification and segmentation in image processing. By representing textures as α_σ-sets, we can leverage their algebraic and topological properties to develop more effective algorithms.

Theorem 5.3. *Different texture classes can be distinguished by the topological properties of their α_σ- set representations.*

Proof. Let T_1 and T_2 be two different texture classes. Each can be represented as an α_σ- set: $T_1 = \bigcup_{i\in\mathbb{N}} P_{1,i}$ and $T_2 = \bigcup_{j\in\mathbb{N}} P_{2,j}$, where $P_{1,i}$ and $P_{2,j}$ are pattern instances. The topological properties of these α_σ-sets, such as the connectivity, boundary behavior, and interaction between pattern instances, can be used to distinguish between the texture classes. Consider these distinguishing characteristics:

1. Connectivity: If T_1 consists of disconnected pattern instances while T_2 has connected ones, this difference is preserved in their α_σ- representations.

2. Boundary complexity: The nature of the boundaries $\mathrm{bd}(P_{1,i})$ versus $\mathrm{bd}(P_{2,j})$ can differ significantly between texture classes, providing a distinguishing feature.

3. Overlap behavior: The relationship between $\mathrm{cl}(P_{1,i}) \cap \mathrm{cl}(P_{1,j})$ and $\mathrm{bd}(P_{1,i}) \cap \mathrm{bd}(P_{1,j})$ can vary between texture classes, offering another distinguishing characteristic.

We introduce a topological texture descriptor $\Gamma(T)$ for a texture T represented as an α_σ- set:

$$\Gamma(T) = \left(\frac{|\bigcup_{i\in I} \mathrm{bd}(P_i)|}{|T|}, \frac{|\bigcap_{i,j\in I, i\neq j} \mathrm{bd}(P_i) \cap \mathrm{bd}(P_j)|}{|\bigcup_{i\in I} \mathrm{bd}(P_i)|} \right)$$

where $|S|$ denotes the cardinality of set S and I is a finite index set. This descriptor captures both the relative boundary size and the degree of boundary intersection in the texture. Different texture classes will have distinct $\Gamma(T)$ values, enabling effective classification. By analyzing these topological properties, we can develop texture classification algorithms that are invariant to illumination changes and certain geometric transformations, which is a significant advantage in pattern recognition applications [25]. □

Furthermore, the hereditary properties of α_σ-sets (Theorem 3.8) make them particularly useful for multi-scale texture analysis. By considering subsets of a textured region, we can analyze textures at different scales while preserving their essential topological characteristics.

Theorem 5.4. *Let $T \in \alpha_\sigma(Z^2, \tau_d)$ be a textured region represented as an α_σ- set, and let $\{U_k\}_{k=1}^n$ be a finite collection of open sets in Z^2 with $U_1 \subset U_2 \subset \cdots \subset U_n$. Then the sequence $\{T \cap U_k\}_{k=1}^n$ forms a multi-scale representation of the texture T, where each $T \cap U_k \in \alpha_\sigma(U_k, \tau_d|_{U_k})$.*

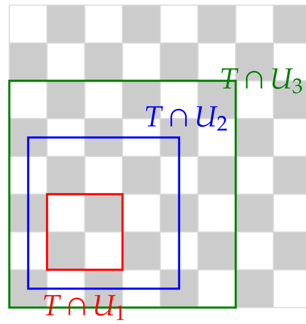
Proof. By Theorem 3.8, if $T \in \alpha_\sigma(Z^2, \tau_d)$ and U is an open subset of Z^2, then $T \cap U \in \alpha_\sigma(U, \tau_d|_U)$. Applying this result to each U_k, we get $T \cap U_k \in \alpha_\sigma(U_k, \tau_d|_{U_k})$ for each $k = 1, 2, \ldots, n$. The sequence $\{T \cap U_k\}_{k=1}^n$ forms a multi-scale representation of T because:

1. Each $T \cap U_k$ captures the textural properties of T at different scales, with $T \cap U_1$ representing the finest scale and $T \cap U_n$ the coarsest.

2. Since $U_1 \subset U_2 \subset \cdots \subset U_n$, we have $T \cap U_1 \subset T \cap U_2 \subset \cdots \subset T \cap U_n$, forming a nested sequence of α_σ-sets.

3. The topological properties of α_σ-sets are preserved under restriction to open subsets, ensuring that the essential structural features of the texture are maintained across scales.

For each scale k, we can decompose $T \cap U_k = \bigcup_{i\in I_k} (P_i \cap U_k)$, where $\{P_i\}_{i\in I}$ is the collection of pattern instances comprising T, and $I_k \subseteq I$ is the subset of indices corresponding to patterns that intersect U_k. This multi-scale representation allows for texture analysis at different resolutions, enabling more robust classification and segmentation algorithms that can identify textural features across varying spatial extents [26]. □



Multi-scale Texture Analysis

FIGURE 6. Multi-scale representation of a textured region using α_σ-sets. The nested regions $U_1 \subset U_2 \subset U_3$ produce a sequence of α_σ-sets $T \cap U_1 \subset T \cap U_2 \subset T \cap U_3$ that preserve the texture's topological properties across different scales.

Theorem 5.5. *Let I be a digital image containing multiple textured regions $\{T_1, T_2, \ldots, T_m\}$, each representable as an α_σ- set. Then the image can be segmented into these regions by identifying the disjoint α_σ-sets $\{S_1, S_2, \ldots, S_m\}$ such that:*

(1) *Each $S_i \in \alpha_\sigma(Z^2, \tau_d)$*
(2) *$S_i \cap S_j = \emptyset$ for $i \neq j$*
(3) *$\bigcup_{i=1}^{m} S_i = I$*
(4) *For each i, S_i approximates T_i by minimizing a distance function $d(S_i, T_i)$ based on α_σ- set properties*

Proof. We begin by representing each textured region T_i as an α_σ- set, expressing it as a union of pattern instances: $T_i = \bigcup_{j \in J_i} P_{i,j}$, where each $P_{i,j}$ satisfies the conditions of α_σ-sets. We define a distance function between α_σ-sets based on their topological properties:

$$d(S, T) = \sum_{i=1}^{k} w_i \cdot d_i(S, T)$$

where $\{d_i\}_{i=1}^{k}$ are distance metrics capturing different aspects of the α_σ-sets (boundary complexity, connectivity patterns, etc.), and $\{w_i\}_{i=1}^{k}$ are corresponding weights. For each pixel $p \in I$, we compute its similarity to each textured region T_i by analyzing local neighborhoods around p and comparing their α_σ- properties with those of each T_i. We then assign each pixel to the texture class with the highest similarity, creating an initial segmentation $\{S_1^0, S_2^0, \ldots, S_m^0\}$. Through an iterative refinement process, we update the segmentation to minimize the distance function:

$$\{S_1^{n+1}, S_2^{n+1}, \ldots, S_m^{n+1}\} = \underset{\{S_1, S_2, \ldots, S_m\}}{\arg\min} \sum_{i=1}^{m} d(S_i, T_i)$$

subject to the constraints that each $S_i \in \alpha_\sigma(Z^2, \tau_d)$, $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^m S_i = I$. This process converges to a segmentation $\{S_1, S_2, \ldots, S_m\}$ that optimally approximates the true textured regions while ensuring each segment forms an α_σ- set. The constraint that each S_i must be an α_σ-set provides significant advantages over traditional segmentation approaches, as it ensures that each segment maintains coherent topological properties consistent with its texture class [27]. □

Theorem 5.6. *Let $T \in \alpha_\sigma(Z^2, \tau_d)$ be a textured region represented as an α_σ- set, and let $f : Z^2 \to Z^2$ be a rigid transformation (rotation, translation, or reflection). Then:*

(1) $f(T) \in \alpha_\sigma(Z^2, \tau_d)$

(2) *The topological texture descriptor* $\Gamma(T) = \Gamma(f(T))$

Proof. Let $T = \bigcup_{i\in I} P_i$ be a decomposition of T into pattern instances that satisfies the α_σ- conditions.

(1) Since rigid transformations preserve adjacency relationships in the digital plane, f acts as a homeomorphism with respect to the digital topology τ_d. By Theorem 4.3 (Homeomorphism Invariance), if $T \in \alpha_\sigma(Z^2, \tau_d)$, then $f(T) \in \alpha_\sigma(Z^2, \tau_d)$.

(2) For the topological texture descriptor $\Gamma(T) = \left(\frac{|\bigcup_{i\in I} \mathrm{bd}(P_i)|}{|T|}, \frac{|\bigcap_{i,j\in I, i\neq j} \mathrm{bd}(P_i)\cap \mathrm{bd}(P_j)|}{|\bigcup_{i\in I} \mathrm{bd}(P_i)|}\right)$, we need to show that it remains invariant under rigid transformations. Since f is a bijection, $|f(S)| = |S|$ for any finite set $S \subset Z^2$. Furthermore, f preserves boundaries: $f(\mathrm{bd}(S)) = \mathrm{bd}(f(S))$. Therefore

$$\begin{aligned}
\Gamma(f(T)) &= \left(\frac{|\bigcup_{i\in I} \mathrm{bd}(f(P_i))|}{|f(T)|}, \frac{|\bigcap_{i,j\in I, i\neq j} \mathrm{bd}(f(P_i)) \cap \mathrm{bd}(f(P_j))|}{|\bigcup_{i\in I} \mathrm{bd}(f(P_i))|}\right) \\
&= \left(\frac{|\bigcup_{i\in I} f(\mathrm{bd}(P_i))|}{|f(T)|}, \frac{|\bigcap_{i,j\in I, i\neq j} f(\mathrm{bd}(P_i)) \cap f(\mathrm{bd}(P_j))|}{|\bigcup_{i\in I} f(\mathrm{bd}(P_i))|}\right) \\
&= \left(\frac{|f(\bigcup_{i\in I} \mathrm{bd}(P_i))|}{|f(T)|}, \frac{|f(\bigcap_{i,j\in I, i\neq j} \mathrm{bd}(P_i) \cap \mathrm{bd}(P_j))|}{|f(\bigcup_{i\in I} \mathrm{bd}(P_i))|}\right) \\
&= \left(\frac{|\bigcup_{i\in I} \mathrm{bd}(P_i)|}{|T|}, \frac{|\bigcap_{i,j\in I, i\neq j} \mathrm{bd}(P_i) \cap \mathrm{bd}(P_j)|}{|\bigcup_{i\in I} \mathrm{bd}(P_i)|}\right) \\
&= \Gamma(T)
\end{aligned}$$

This invariance property makes α_σ-sets particularly valuable for texture recognition applications, as they provide a consistent representation of textures regardless of their orientation or position in the image [28]. □

6. Conclusions

We have introduced and developed α_σ-sets as a new class of generalized topological structures. Our investigation has yielded several significant results:

- α_σ-sets form a σ-algebra under the union operation (Theorem 3.2), providing a robust algebraic framework. Every α_σ- set is a β-open set (Theorem 3.6), and open α_σ-sets are α-open sets (Proposition 3.4), establishing a clear hierarchy within generalized topologies.

- Our characterization theorems (Theorems 3.5 and 3.13) position α_σ-sets as countable unions of α-open sets with controlled boundary behavior. This boundary control preserves important topological invariants.
- We have established hereditary properties in subspaces (Theorem 3.8), preservation under appropriate continuous mappings (Theorem 4.1), and invariance under homeomorphisms (Theorem 4.3). The behavior in product spaces (Theorem 3.10) further illustrates compatibility with standard topological constructions.
- In digital topology, α_σ-sets provide an elegant representation of connected regions (Theorem 5.2) and textured areas (Theorem 5.3). The multi-scale analysis framework (Theorem 5.6) and segmentation approach (Theorem 5.7) demonstrate practical value, while invariance properties (Theorem 5.8) establish these sets as robust descriptors for texture recognition.
- By bridging classical topology and complex patterns in digital images, α_σ-sets offer powerful tools for analyzing textured regions in real-world data, addressing topological classification problems in both pure and applied contexts.

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