Int. J. Anal. Appl. (2025), 23:226

International Journal of Analysis and Applications

On the Boundedness of Hausdorff Operators in Block Space with Variable Exponent

Mehvish Sultan[1], **Babar Sultan**[2,*], **Ioan-Lucian Popa**[3,4,*]

[1]*Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan*
[2]*Department of Mathematics, Quaid-I-Azam University, Islamabad 45320, Pakistan*
[3]*Department of Computing, Mathematics and Electronics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania*
[4]*Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091 Brasov, Romania*

**Corresponding authors:* babarsultan40@yahoo.com; lucian.popa@uab.ro

Abstract. In this paper, we prove the Hardy inequality, the Hilbert inequality and the Hardy-Littlewood-Pólya inequality on block spaces with variable exponents. Furthermore, we establish the boundedness of Hausdorff operators on block space with variable exponent.

1. Introduction

In [3], Hardy-Littlewood-Polya (HLP for short) inequalities for Lebesgue spaces were developed. The Riemann-Liouville integral, the Weyl integral, the Hilbert inequalities, and the Hardy inequality are among the important discoveries that are combined in these inequalities. The Hardy inequalities have been developed in great length. The reader can consult references [2], [12] for a detailed analysis of the Hardy inequality and associated subjects. The Hausdorff summability method was first presented in 1917, which marked the beginning of the study of Hausdorff operators (HO briefly). The HO are an generalization of the Hardy inequalities. HO studies have been expanded to a number of different setting and function spaces see [4–10]. For more results in variable exponent function spaces see [13–25].

Recently, there are number of researches on the variable Lebesgue spaces and variable Morrey spaces. Block space is the pre-dual of the Morrey space, see [1]. The idea of block space is introduced in [1] and proved that it is a pre-dual of Morrey spaces.

Received: Jun. 8, 2025.

2020 *Mathematics Subject Classification.* 42B20, 42B35.

Key words and phrases. block space with variable exponent; variable Morrey space; Hausdorff operator; Hardy-Littlewood-Pólya inequality.

https://doi.org/10.28924/2291-8639-23-2025-226
ISSN: 2291-8639

Assume that $0 \leqslant \Lambda < n$ and $1 \leqslant p < \infty$. By (p, Λ)-block we mean a measurable function a if its support contained in the ball $B(x_0, r), x_0 \in \mathbb{R}^n, r > 0$, and

$$\|a\|_{L^p(\mathbb{R}^n)} \leqslant r^{-\frac{\Lambda}{p}}.$$

We denote $a \in a_{p,\Lambda}$ if a is a (p, Λ)-block.
Define $\mathcal{B}_{p,\Lambda}(\mathbb{R}^n)$ by

$$\mathcal{B}_{p,\Lambda}(\mathbb{R}^n) = \left\{ \sum_{t=1}^{\infty} \Lambda_t a_t : \quad \sum_{t=1}^{\infty} |\Lambda_t| < \infty \text{ and } a_t \text{ is a } (p, \Lambda)\text{-block} \right\}.$$

Norm of space $\mathcal{B}_{p,\Lambda}$ is given as

$$\|g\|_{\mathcal{B}_{p,\Lambda}(\mathbb{R}^n)} = \inf \left\{ \sum_{t=1}^{\infty} |\Lambda_t| : \quad g = \sum_{t=1}^{\infty} \Lambda_t a_t \right\}.$$

Let $\mathcal{M}$ is the class of Lebesgue measurable functions on $\mathbb{R}^n$. Let A be a measurable set, then characteristic function of A is denoted by $\mathbf{1}_A$. Let $x \in \mathbb{R}^n, r > 0$, we denote $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\mathcal{B} = \{B(x, r) : x \in \mathbb{R}^n. r > 0\}$.

Let $1 \leqslant p < \infty$, then the Lebesgue spaces $L^p(\mathbb{R}^n)$ is the space of all functions such that the norm $\|g\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g(y)|^p dy\right)^{1/p}$ is finite.

Morrey [11] introduced the Morrey space to facilitate the study of certain quasilinear elliptic partial differential equations. The Morrey space is given as

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ g \in L^p_{loc} : \|g\|_{M_{p,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where $1 \leqslant p < \infty$ and $0 \leqslant \Lambda < n$ and

$$\|g\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{B \in \mathbb{B}} \left(\frac{1}{r^{\lambda}} \int_B |g(x)|^p dx \right)^{1/p}.$$

Let E be an open set in $\mathbb{R}^n$, consider a measurable function $p(\cdot) : E \to [1, \infty)$. The conjugate exponent denoted by $p'(\cdot)$, is defined as $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$.

The set $\mathcal{P}(E)$ comprises all functions $p(\cdot) : E \to [1, \infty)$. We suppose that

$$1 \leq p^-(E) \leq p(x) \leq p^+(E) < \infty, \tag{1.1}$$

such that

$$p^- = \operatorname{ess\,inf}\Big\{p(x) : x \in E\Big\},$$
$$p^+ = \operatorname{ess\,sup}\Big\{p(x) : x \in E\Big\}.$$

We use the notation $L^{p(\cdot)}(E)$ to represent the space of all measurable functions g defined on E, such that, for a certain $\eta > 0$, $\int_E \left(\frac{|f(x)|}{\eta}\right)^{p(y)} dy < \infty$. Its norm is given as

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{ \eta > 0 : \int_E \left(\frac{|f(y)|}{\eta} \right)^{p(y)} dy \leq 1 \right\}.$$

Let $p(y): \mathbb{R}^n \to (1,\infty)$ and $\omega(y,r): \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable functions. Then $b \in \mathcal{M}(\mathbb{R}^n)$ is the $\left(\omega, L^{p(y)}\right)$-block if is supported in a ball $B(y_0,r)$, $y_0 \in \mathbb{R}^n$, $r > 0$ and $\|b\|_{L^{p(y)}} \leq \frac{1}{\omega(y_0,r)}$.

Block space with variable exponent (BSVE briefly) $\mathfrak{B}_{\omega,L^{p(y)}}$ is given as

$$\mathfrak{B}_{\omega,L^{p(y)}} = \left\{\sum_{t=1}^{\infty} \Lambda_t a_t : \sum_{t=1}^{\infty} |\Lambda_t| < \infty \textit{ and } a_t \textit{ is an } \left(\omega, L^{p(y)}\right) - \textit{block}\right\}.$$

Norm of $\mathfrak{B}_{\omega,L^{p(y)}}$ is given as

$$\|f\|_{\mathfrak{B}_{\omega,L^{p(y)}}} = \inf\left\{\sum_{t=1}^{\infty} |\Lambda_t| \textit{ such that } f = \sum_{t=1}^{\infty} \Lambda_t a_t\right\}.$$

When $p(y)$ is a constant, then $\mathfrak{B}_{\omega,L^{p(y)}}$ is simply classical block spaces defined in [1]. BSVE serves as the predual of the variable Morrey space [1]. The boundedness of the Hardy-Littlewood maximal operator on BSVE was established in [1]. In Section 3, we derive the Hardy-Littlewood-Pólya inequalities and the Hardy inequality for BSVE. Finally, in the last section, we demonstrate the boundedness of the HO operator on BSVE. We also refer some papers for boundedness results in variable exponent function spaces [26–34].

2. Minkowski's inequality on BSVE

This section contains the Minkowski's inequality on BSVE. Now we define the idea of variable Morrey spaces.

Let $p(y) : \mathbb{R}^n \to (1,\infty)$ and $\omega(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Then the Morrey space with variable exponent $\mathcal{M}_{\omega,L^{p(y)}}$ is consists of all Lebesgue measurable functions such that

$$\|f\|_{\mathcal{M}_{\omega,L^{p(y)}}} = \sup_{y_0 \in \mathbb{R}^n, r>0} \frac{1}{\omega(y_0,r)} \left\|f \mathbf{1}_{B(y_0,r)}\right\|_{L^{p(y)}} < \infty.$$

The dual space of $\mathfrak{B}_{\omega,L^{p(y)}}$ is $\mathcal{M}_{\omega,L^{p'(y)}}$, where $p(y)$ and $p'(y)$ are conjugate exponents of each other.

The class of continuous function with compact support is denoted by $C_0(\mathbb{R}^n)$. Then the Zorko space with variable exponent $\mathcal{Z}_{\omega,L^{p(y)}}$ is the closure of $C_0(\mathbb{R}^n)$ in $\mathcal{M}_{\omega,L^{p(y)}}$.

Theorem 2.1. *[1] If $p(y) : \mathbb{R}^n \to (1,\infty)$ and $\omega(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If ess $\sup p(y) < \infty$, then we have*

$$\mathfrak{B}^*_{\omega,L^{p(y)}} = \mathcal{M}_{\omega,L^{p'(y)}},$$

where $\mathfrak{B}^*_{\omega,L^{p(y)}}$ is the dual of $\mathfrak{B}_{\omega,L^{p(y)}}$.

Similarly we can prove that $\mathcal{Z}^*_{\omega,L^{p(y)}} = \mathfrak{B}_{\omega,L^{p'(y)}}$, where $\mathcal{Z}^*_{\omega,L^{p(y)}}$ is the dual of $\mathcal{Z}_{\omega,L^{p(y)}}$.

The following result is given in [1].

Lemma 2.1. *Let $p(y):\mathbb{R}^n\to(1,\infty)$ and $\omega(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. If* ess sup $p(y)<\infty$*, then we have*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx \leqslant C\|f\|_{\mathcal{M}_{\omega,L^{p(y)}}}\left\|g]\right\|_{\mathfrak{B}_{\omega,L^{p'(y)}}}.$$

BSVE is the generalization of variable Lebesgue space. Next we establish the Minkowski's inequality in $\mathfrak{B}_{\omega,L^{p'(y)}}$.

Theorem 2.2. *Let $p(y):\mathbb{R}^n\to(1,\infty)$ and $\omega(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. If μ be a signed σ-finite Borel measure on $\mathbb{R}^m$ and $h(x,s)$ be a measurable function on $\mathbb{R}^n\times\mathbb{R}^m$ such that $\left\|h(\cdot,s)\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\in L^1(|\mu|)$. We have*

$$\left\|\int_{\mathbb{R}^m}h(\cdot,s)d\mu\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\int_{\mathbb{R}^m}\left\|h(\cdot,s)\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}d|\mu|. \tag{2.1}$$

Proof. Let

$$H(x)=\int_{\mathbb{R}^m}h(z,s)d\mu.$$

Let $g\in\mathcal{Z}_{\omega,L^{p(y)}}$ with $\left\|g\right\|_{\mathcal{Z}_{\omega,L^{p'(y)}}}\leqslant 1$. Then using the Hölder's inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^n}H(z)g(z)dz\right| &\leqslant C\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left|h(z,s)\right|\left|g(z)\right|dzd|\mu|\\ &\leqslant C\int_{\mathbb{R}^m}\left\|h(\cdot,s)\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\left\|g]\right\|_{\mathcal{M}_{\omega,L^{p'(y)}}}d|\mu|\\ &=\int_{\mathbb{R}^m}\left\|h(\cdot,s)\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\left\|g]\right\|_{\mathcal{Z}_{\omega,L^{p'(y)}}}d|\mu|\\ &\leqslant C\int_{\mathbb{R}^m}\|h(\cdot,s)\|_{\mathfrak{B}_{\omega,L^{p(y)}}}d|\mu|.\end{aligned}$$

By taking supremum over $g\in\mathcal{Z}_{\omega,L^{p(y)}}$ with $\left\|g\right\|_{\mathcal{Z}_{\omega,L^{p'(y)}}}\leqslant 1$. Theorem 1 yields that $H\in\mathfrak{B}_{\omega,L^{p(y)}}$ and (2.1) holds. □

Now we define the notion of Boyd's indices to $\mathfrak{B}_{\omega,L^{p(y)}}$. Let g be a measurable function and $s\in\mathbb{R}\backslash\{0\}$. The dilation operator D_s is given as

$$\left(D_s g\right)(x)=g(x/s),\quad x\in\mathbb{R}^n.$$

Now we obtain the norm of D_t on BSVE. This result is important to prove the HLP inequalities in BSVE.

Theorem 2.3. *Let $p(y):\mathbb{R}^n\to(1,\infty)$ and $\omega(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. Then we have*

$$\left\|D_t f\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\|f\|_{\mathfrak{B}_{\omega,L^{p(y)}}}. \tag{2.2}$$

Proof. If $g \in \mathfrak{B}_{\omega,L^{p(y)}}$ and $\varepsilon > 0$, there exist families of $\left(\omega, L^{p(y)}\right)$-blocks $\{a_\ell\}_{\ell\in\mathbb{N}}$ with supports $\{B\left(x_\ell, r_\ell\right)\}_{\ell\in\mathbb{N}}$ and scalars $\{\Lambda_\ell\}_{\ell\in\mathbb{N}}$ such that

$$g = \sum_{\ell\in\mathbb{N}} \Lambda_\ell a_\ell$$

and $\sum_{\ell\in\mathbb{N}} |\Lambda_\ell| < (1+\varepsilon)\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}$.

It is easy to note that $D_t a_\ell$ is a $\left(\omega, L^{p(y)}\right)$-block with support $B\left(tx_\ell, |t|r_\ell\right)$ and

$$\|D_t a_\ell\|_{L^{p(y)}} \leqslant \frac{1}{\omega(x_0, r_\ell)}$$

Write

$$D_t g = \sum_{\ell\in\mathbb{N}} \Lambda_\ell D_t a_\ell = \sum_{\ell\in\mathbb{N}} \gamma_\ell C_\ell$$

where

$$\gamma_\ell = \frac{1}{\omega(x_0, r_\ell)} \quad \text{and} \quad C_\ell = D_t a_\ell.$$

It is easy to note that $\{C_\ell\}_{k\in\mathbb{N}}$ is a family of $\left(\omega, L^{p(y)}\right)$-blocks. If $D_t g \in \mathfrak{B}_{\omega,L^{p(y)}}$ then

$$\left\|D_t g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \leqslant \sum_{\ell\in\mathbb{N}} \left|\gamma_\ell\right| = \sum_{\ell\in\mathbb{N}} |\Lambda_\ell| \leqslant (1+\varepsilon)\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}$$

If $\varepsilon > 0$ is arbitrary, then we get

$$\left\|D_t g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \leqslant \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

Since $D_{1/t} D_t g = g$, then we get

$$\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}} = \left\|D_{1/t} D_t g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \leqslant \left\|D_t g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}$$

Therefore,

$$\left\|D_t g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \leqslant \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

□

3. HLP inequalities

Theorem 3.1. *Let $p(y) : \mathbb{R}^n \to (1,\infty)$, $K(\cdot,\cdot)$ be a measurable function on $(0,\infty)\times(0,\infty)$ and $\omega(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If Q satisfies*

(1) $Q(\theta s, \theta t) = \theta^{-1} Q(s,t), \theta > 0,$
(2) $\int_0^\infty |Q(u,1)| u^{-\frac{1+\Lambda}{p(y)}} du < \infty,$
then, the linear operator

$$Tf(t) = \int_0^\infty Q(s,t) f(s) ds$$

is bounded on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$.

Proof. If $u = s/t$. We have

$$|Tg(t)| \leqslant \int_0^\infty |Q(ut,t)| \left|(D_{1/u}g)\ (t)\right| t du = \int_0^\infty |Q(u,1)| \left|(D_{1/u}g)\ (t)\right| du$$

Applying the norm $\|\cdot\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)}$ on both sides of the above inequality, Then Minkowski's inequality yields

$$\|Tg\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} \leqslant \int_0^\infty |Q(u,1)| \left\|(D_{1/u}g)\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} du.$$

Consequently, using Theorem 2.4, we get

$$\|Tg\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} \leqslant C\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} \int_0^\infty |Q(u,1)| u^{-\frac{1+\Lambda}{p(y)}} du \leqslant C\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)},$$

where $C > 0$. Therefore, T is bounded on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$.
If $\Lambda = 0$, then above Theorem yields

$$\int_0^\infty |Q(u,1)| u^{-\frac{1}{p(y)}} du < \infty.$$

Thus we obtain our desired results. □

Theorem 3.2. *Let $p(y) : \mathbb{R}^n \to (1,\infty)$ and $\omega(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Then*

$$Tg(t) = \frac{1}{t} \int_0^t g(s) ds$$

is bounded on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$.
The operator

$$Sg(t) = \int_t^\infty \frac{g(s)}{s} ds$$

is bounded on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$.

Proof. Let $Q(s,t) = t^{-1}\mathbf{1}_E(s,t)$ where $E = \{(s,t) : s < t\}$. Then T fulfills the conditions of Theorem 3.1, so we get

$$\int_0^\infty |Q(u,1)| u^{-\frac{1+\Lambda}{p(y)}} du = \int_0^1 u^{-\frac{1+\Lambda}{p(y)}} du = \left. \frac{u^{-\frac{1+\Lambda}{p(y)}+1}}{-\frac{1+\Lambda}{p(y)}+1} \right|_0^1 < \infty.$$

By using the Theorem 3.1, we obtain

$$\|Tg\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} \leqslant C\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)}.$$

Next we will prove the boundedness of the operator S on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$. Let $Q(s,t) = s^{-1}\mathbf{1}_E(s,t)$ such that $E = \{(s,t) : s > t\}$. Then using the condition (1) of Theorem 3.1, we get

$$\int_0^\infty |Q(u,1)| u^{-\frac{1+\Lambda}{p(y)}} du = \int_1^\infty u^{-\frac{1+\Lambda}{p(y)}-1} du < \infty.$$

Thus, by using the Theorem 3.1, we get

$$\|Sg\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)} \leqslant C\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)},$$

where $C > 0$.

□

Next we prove the Hilbert inequality on BSVE.

Theorem 3.3. *Let $p(y) : \mathbb{R}^n \to (1,\infty)$ and $\omega(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Then*

$$Tg(t) = \int_0^\infty \frac{g(s)}{t+s} ds$$

is bounded on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$.

Proof. Let $Q(s,t) = (s+t)^{-1}$. It is easy to note that condition (1) of Theorem 3.2. Hence we get

$$\int_0^\infty (u+1)^{-1} u^{-\frac{1+\Lambda}{p(y)}} du \leqslant \int_1^\infty u^{-\frac{1+\Lambda}{p(y)}-1} dv + \int_0^1 v^{-\frac{1+\Lambda}{p(y)}} dv < \infty.$$

Thus the Theorem 3.1 yields the boundedness of T on $\mathfrak{B}_{\omega,L^{p(y)}}(0,\infty)$. □

4. Multidimensional HO on BSVE

In this section, we obtain the boundedness of HO to BSVE. For the definition of multidimensional HO see [5,6].
Let $B = B(u) = \left(a_{ij}\right)_{i,j=1}^n = \left(a_{ij}(u)\right)_{i,j=1}^n$ be an $n \times n$ matrix with the entries $a_{ij}(u)$ being measurable functions of u. The matrix $B(u)$ is non-degenerate almost everywhere. Recall that $xB(u), x \in \mathbb{R}^n$, is the row n-vector obtained by multiplying the row n-vector x by the matrix $B(u)$.

Let $\Phi(u)$ be a measurable function. The multidimensional Hausdorff operator associated with $B(u)$ and $\Phi(u)$ is defined as

$$(\mathcal{H}g)(x) = (\mathcal{H}_\Phi g)(x) = (\mathcal{H}_{\Phi,A} g)(x) = \int_{\mathbb{R}^n} \Phi(u) g(xB(u))\, du.$$

The adjoint operator $\mathcal{H}^*$ is given as

$$(\mathcal{H}^* g)(x) = \left(\mathcal{H}^*_{\Phi,B} g\right)(x) = \int_{\mathbb{R}^n} \Phi(u) \left|\det B^{-1}(u)\right| g\left(xB^{-1}(u)\right) du.$$

Let g be a measurable function, the dilation operator $D_{B(u)}$ is given as

$$\left(D_{B(u)} g\right)(x) = g(xB(u)), \quad x \in \mathbb{R}^n.$$

In order to study the multidimensional HO $\mathcal{H}_{\Phi,B}$ and $\mathcal{H}^*_{\Phi,B}$, we need following important Lemma.

Lemma 4.1. *Let $p(y):\mathbb{R}^n\to(1,\infty)$ and $\omega(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. Then,*

$$\left\|D_{B(u)}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\|\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

Proof. If $g\in\mathfrak{B}_{\omega,L^{p(y)}}$. By using the definition of $\mathfrak{B}_{\omega,L^{p(y)}}$, for any $\varepsilon>0$, there exist families of $\left(\omega,L^{p(y)}\right)$-blocks $\{a_\ell\}_{\ell\in\mathbb{N}}$ with supports $\{B(x_\ell,r_\ell)\}_{\ell\in\mathbb{N}}$ and scalars $\{\Lambda_\ell\}_{\ell\in\mathbb{N}}$ such that

$$g=\sum_{\ell\in\mathbb{N}}\Lambda_\ell a_\ell\quad\text{and}\quad\sum_{\ell\in\mathbb{N}}|\Lambda_\ell|<(1+\varepsilon)\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

It is easy to note that $D_{B(u)}a_\ell$ is a $\left(\omega,L^{p(y)}\right)$-blocks,

$$\operatorname{supp}D_{B(u)}a_\ell\subseteq B\left(x_\ell B^{-1}(u),\left\|B^{-1}(u)\right\|r_\ell\right)$$

and

$$\left\|D_{B(u)}a_\ell\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant|\det B(u)|^{-\frac{1}{p^-}}\frac{1}{\omega(x_0,r_\ell)}.$$

Hence we get

$$D_{B(u)}g=\sum_{\ell\in\mathbb{N}}\Lambda_\ell D_{B(u)}a_\ell=\sum_{\ell\in\mathbb{N}}\gamma_\ell c_\ell$$

where

$$\gamma_\ell=\Lambda_\ell|\det B(u)|^{-\frac{1}{p^-}}\quad\text{and}\quad c_\ell=\left(\omega(x_0,r_\ell)\right)^{-1}.$$

It is easy to note that $\{c_\ell\}_{\ell\in\mathbb{N}}$ is a family of $\left(\omega,L^{p(y)}\right)$-blocks. Thus $D_{B(u)}g\in\mathfrak{B}_{\omega,L^{p(y)}}$ with

$$\begin{aligned}\left\|D_{B(u)}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}&\leqslant\sum_{\ell\in\mathbb{N}}\left|\gamma_\ell\right|=\sum_{\ell\in\mathbb{N}}|\Lambda_\ell|\left|\det B(u)\right|^{-\frac{1}{p^-}}\\&\leqslant(1+\varepsilon)\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.\end{aligned}$$

For arbitrary $\varepsilon>0$, we get

$$\left\|D_{B(u)}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

□

Theorem 4.1. *Let $p(y):\mathbb{R}^n\to(1,\infty)$ and $\omega(y,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a Lebesgue measurable function. Let $\int_{\mathbb{R}^n}\left|\Phi(u)\right|du=\|\Phi\|_{p(y),A}<\infty$ and $\int_{\mathbb{R}^n}\left|\Phi(u)\right|\left|\det B^{-1}(u)\right|du=\|\Phi\|_{p(y),A}<\infty$. Then,*

$$\left\|H_\mu g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\|\Phi\|_{p(y),A}\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}},$$

and

$$\left\|H^*_\mu g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}}\leqslant\|\Phi\|^*_{p(y),\Lambda}\|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.$$

Proof. By using the Minkowski's inequality for $\mathfrak{B}_{\omega,L^{p(y)}}$ and Lemma 4.1, we get

$$\begin{aligned}\left\|\mathcal{H}_{\Phi,A}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} &\leqslant \int_{\mathbb{R}^n} |\Phi(u)| \left\|D_{B(u)}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} du \\ &\leqslant \left(\int_{\mathbb{R}^n} \left|\Phi(u)\right| du\right) \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \\ &= \|\Phi\|_{p(y),A} \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left\|H^*_\mu g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} &\leqslant \int_{\mathbb{R}^n} \left|\Phi(u)\right| \left|\det B^{-1}(u)\right| \left\|D_{B^{-1}(u)}g\right\|_{\mathfrak{B}_{\omega,L^{p(y)}}} du \\ &\leqslant \left(\int_{\mathbb{R}^n} \left|\Phi(u)\right| \left|\det B^{-1}(u)\right|^1 du\right) \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}} \\ &= \|\Phi\|^*_{p(y),A} \|g\|_{\mathfrak{B}_{\omega,L^{p(y)}}}.\end{aligned}$$

□

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] O. Blasco, A. Ruiz, L. Vega, Non Interpolation in Morrey-Campanato and Block Spaces, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 28 (1999), 31–40. http://eudml.org/doc/84372.

[2] D.E. Edmunds, W.D. Evans, Hardy Operators, Function Spaces and Embeddings, Springer Berlin, Berlin, 2004. https://doi.org/10.1007/978-3-662-07731-3.

[3] G. Hardy, J. Littlewood, G. Pölya, Inequalities, Cambridge University Press, 1988.

[4] Y. Kanjin, The Hausdorff Operators on the Real Hardy Spaces $H^p(R)$, Stud. Math. 148 (2001), 37–45. https://doi.org/10.4064/sm148-1-4.

[5] A.K. Lerner, E. Liflyand, Multidimensional Hausdorff Operators on the Real Hardy Space, J. Aust. Math. Soc. 83 (2007), 79–86. https://doi.org/10.1017/s1446788700036399.

[6] E. Liflyand, Boundedness of Multidimensional Hausdorff Operators on $H^1(\mathbb{R}^n)$, Acta Sci. Math. (Szeged) 74 (2008), 845–851. https://cir.nii.ac.jp/crid/1370289302291212419.

[7] E. Liflyand, Hausdorff operators on Hardy spaces, Eurasian Math. J. 4 (2013), 101–141.

[8] E. Liflyand, F. Móricz, The Hausdorff Operators Is Bounded on the Real Hardy Space $H^1(\mathbb{R}^n)$, Proc. Amer. Math. Soc. 128 (2000), 1391–1396. https://doi.org/10.1090/s0002-9939-99-05159-x.

[9] E. Liflyand, F. Móricz, The Multi-Parameter Hausdorff Operator Is Bounded on the Product Hardy Space $H^{11}(\mathbb{R}\times\mathbb{R})$, Analysis 21 (2001), 107–118. https://doi.org/10.1524/anly.2001.21.2.107.

[10] E. Liflyand, A. Miyachi, Boundedness of the Hausdorff Operators in H^p Spaces, $0 < p < 1$, Stud. Math. 194 (2009), 279–292. https://doi.org/10.4064/sm194-3-4.

[11] C.B. Morrey, On the Solutions of Quasi-Linear Elliptic Partial Differential Equations, Trans. Am. Math. Soc. 43 (1938), 126–166. https://doi.org/10.2307/1989904.

[12] B. Opic, A. Kufner, Hardy-Type Inequalities, Longman Scientific & Technical, 1990.

[13] M. Sultan, B. Sultan, A Note on the Boundedness of Marcinkiewicz Integral Operator on Continual Herz-Morrey Spaces, Filomat 39 (2025), 2017–2027. https://doi.org/10.2298/fil2506017s.

[14] M. Sultan, B. Sultan, A. Khan, T. Abdeljawad, Boundedness of Marcinkiewicz Integral Operator of Variable Order in Grand Herz-Morrey Spaces, AIMS Math. 8 (2023), 22338–22353. https://doi.org/10.3934/math.20231139.

[15] B. Sultan, M. Sultan, Boundedness of Higher Order Commutators of Hardy Operators on Grand Herz-Morrey Spaces, Bull. Sci. Math. 190 (2024), 103373. https://doi.org/10.1016/j.bulsci.2023.103373.

[16] M. Sultan, B. Sultan, A. Hussain, Grand Herz–morrey Spaces with Variable Exponent, Math. Notes 114 (2023), 957–977. https://doi.org/10.1134/s0001434623110305.

[17] M. Sultan, B. Sultan, A. Aloqaily, N. Mlaiki, Boundedness of Some Operators on Grand Herz Spaces with Variable Exponent, AIMS Math. 8 (2023), 12964–12985. https://doi.org/10.3934/math.2023653.

[18] B. Sultan, F. Azmi, M. Sultan, M. Mehmood, N. Mlaiki, Boundedness of Riesz Potential Operator on Grand Herz-Morrey Spaces, Axioms 11 (2022), 583. https://doi.org/10.3390/axioms11110583.

[19] B. Sultan, M. Sultan, Boundedness of Commutators of Rough Hardy Operators on Grand Variable Herz Spaces, Forum Math. 36 (2023), 717–733. https://doi.org/10.1515/forum-2023-0152.

[20] A. Hussain, G. Gao, Multilinear Singular Integrals and Commutators on Herz Space with Variable Exponent, ISRN Math. Anal. 2014 (2014), 626327. https://doi.org/10.1155/2014/626327.

[21] A. Hussain, I. Khan, A. Mohamed, Variable Herz–morrey Estimates for Rough Fractional Hausdorff Operator, J. Inequal. Appl. 2024 (2024), 33. https://doi.org/10.1186/s13660-024-03110-8.

[22] J. Younas, A. Hussain, H. Alhazmi, A.F. Aljohani, I. Khan, BMO Estimates for Commutators of the Rough Fractional Hausdorff Operator on Grand-Variable-Herz-Morrey Spaces, AIMS Math. 9 (2024), 23434–23448. https://doi.org/10.3934/math.20241139.

[23] B. Sultan, M. Sultan, I. Khan, On Sobolev Theorem for Higher Commutators of Fractional Integrals in Grand Variable Herz Spaces, Commun. Nonlinear Sci. Numer. Simul. 126 (2023), 107464. https://doi.org/10.1016/j.cnsns.2023.107464.

[24] B. Sultan, M. Sultan, Sobolev-type Theorem for Commutators of Hardy Operators in Grand Herz Spaces, Ukr. Mat. Zhurnal 76 (2024), 1052–1068. https://doi.org/10.3842/umzh.v76i7.7546.

[25] M. Sultan, B. Sultan, R.E. Castillo, Weighted Composition Operator on Gamma Spaces with Variable Exponent, J. Pseudo-Differential Oper. Appl. 15 (2024), 46. https://doi.org/10.1007/s11868-024-00619-w.

[26] M. Sultan, B. Sultan, A Note on the Boundedness of Higher Order Commutators on Fractional Integrals in Grand Variable Herz-Morrey Spaces, Kragujev. J. Math. 50 (2026), 1063–1080.

[27] B. Sultan, M. Sultan, A. Khan, T. Abdeljawad, Boundedness of Commutators of Variable Marcinkiewicz Fractional Integral Operator in Grand Variable Herz Spaces, J. Inequal. Appl. 2024 (2024), 93. https://doi.org/10.1186/s13660-024-03169-3.

[28] B. Sultan, M. Sultan, Q. Zhang, N. Mlaiki, Boundedness of Hardy Operators on Grand Variable Weighted Herz Spaces, AIMS Math. 8 (2023), 24515–24527. https://doi.org/10.3934/math.20231250.

[29] B. Sultan, M. Sultan, A. Hussain, Boundedness of the Bochner–Riesz Operators on the Weighted Herz–Morrey Type Hardy Spaces, Complex Anal. Oper. Theory 19 (2025), 49. https://doi.org/10.1007/s11785-025-01671-0.

[30] B. Sultan, A. Hussain, M. Sultan, Chracterization of Generalized Campanato Spaces with Variable Exponents via Fractional Integrals, J. Pseudo-Differential Oper. Appl. 16 (2025), 22. https://doi.org/10.1007/s11868-024-00651-w.

[31] B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M.A. Alghafli, N. Mlaiki, Boundedness of Fractional Integrals on Grand Weighted Herz Spaces with Variable Exponent, AIMS Math. 8 (2023), 752–764. https://doi.org/10.3934/math.2023036.

[32] B. Sultan, F.M. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, N. Souayah, Boundedness of Fractional Integrals on Grand Weighted Herz–morrey Spaces with Variable Exponent, Fractal Fract. 6 (2022), 660. https://doi.org/10.3390/fractalfract6110660.

[33] M. Sultan, B. Sultan, Boundedness of Sublinear Operators on Grand Central Orlicz-Morrey Spaces, Bull. Sci. Math. 205 (2025), 103704. https://doi.org/10.1016/j.bulsci.2025.103704.

[34] M. Sultan, B. Sultan, λ-Central Musielak–orlicz–morrey Spaces, Arab. J. Math. 14 (2025), 357–363. https://doi.org/10.1007/s40065-025-00528-w.