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Enhancing Social Network Security Using Equitable Fair Edge Domination in Fuzzy Graphs

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Abstract. In online social networks, cyber threats such as spam, phishing, and misinformation propagate through communication links, making security monitoring a critical challenge. Traditional approaches often lead to overburdening certain network connections, resulting in inefficient surveillance and increased vulnerability. To address this, we introduce the Fuzzy Regular Equitable Fair Domination Graph (FREFDG) and apply Equitable Fair Edge Domination (EFEDS) as a strategic model for balanced security monitoring. This framework ensures that security resources are distributed equitably across edges, thereby preventing overload and ensuring comprehensive threat detection. The theoretical foundations of EFEDS are rigorously established through propositions, theorems, and numerical illustrations, demonstrating its effectiveness in optimizing network surveillance. A decision-making model is formulated using graph-based analysis, enabling the systematic selection of critical edges requiring direct monitoring while leveraging indirect supervision for non-critical edges. This structured approach reduces computational complexity while enhancing network resilience. Additionally, we present a real-world application scenario, showcasing how EFEDS can be implemented in spam detection, phishing prevention, and misinformation filtering. The results demonstrate the efficiency of our framework in identifying high-risk edges, reducing redundant monitoring efforts, and improving threat mitigation strategies. By integrating fuzzy logic with equitable fair domination principles, our approach contributes to a more adaptive, scalable, and intelligent cybersecurity model, offering valuable insights for network security experts and researchers.

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1. Introduction

Fuzzy graph theory is widely used in real-life applications, including social network analysis for detecting influential nodes and misinformation control. It plays a crucial role in transportation systems by optimizing traffic flow under uncertain conditions. In cybersecurity, fuzzy graphs enhance anomaly detection and threat mitigation in complex networks. Medical diagnosis and bioinformatics leverage fuzzy graph models to analyze uncertain patient data and disease progression. Additionally, fuzzy graphs are applied in recommendation systems to improve personalized content delivery based on user preferences.Fuzzy set theory was first introduced by Zadeh [4], laying the foundation for handling uncertainty, which later extended to fuzzy graph theory.

Fuzzy graph theory extends classical graph concepts to handle uncertainly in complex networks, building on the fundamental principles of graph theory [23]. Domination in fuzzy graphs has been widely investigated to analyze control and influence in uncertain environment [7].

Recent generalizations of domination concepts further enhance their applicability to structured and connected fuzzy networks [24]. Gallai [19] explored domination and independent sets, influencing subsequent studies on domination in graphs. Rosenfeld [2] formally introduced fuzzy graphs, enabling the study of uncertainty in graph structures.

Golumbic et.al [20] extended domination to tolerance graphs, influencing fuzzy domination concepts. Potadar et al. [18] investigated fuzzy topological spaces, linking them with graph structures. Buckley and Harary [22] analyzed distances in graphs, contributing to fuzzy graph distance metrics.

Bhutani [5] studied automorphisms in fuzzy graphs, extending classical isomorphism concepts. Samuel et al. [21] examined connected domination, influencing efficient network monitoring. Mordeson and Nair [17] summarized key fuzzy graph properties in their book. Balasubramanian and Rajeswari [8] studied fuzzy graph complements, expanding algebraic operations. Samanta and Pal [6] introduced fuzzy planar graphs, analyzing their geometric properties. Nagoor Gani and Vadivel [9] examined fuzzy domination and Independent domination. Ali et al. [1] proposed new operations in soft set theory, integrating fuzzy decision-making.

Anitha, Arumugam and Chellali [12] introduced equitable domination in graphs, ensuring balanced coverage. Sovan and Biswijit [13] studied Generalized fuzzy graphs enhancing structural classification. Manjusha and Sunitha [15] explored strong domination in fuzzy graphs for better network resource allocation. Manjusha and Sunitha [3] extended fuzzy domination principles, optimizing uncertain networks.

Somasundaram [14] examined domination in product fuzzy graphs, analyzing combined structures. Ali Koam et.al [10] applied fuzzy graphs to decision-making frameworks. Shain and Subatah [16] investigated domination in interval-valued fuzzy graphs for uncertainty modeling.

Manjusha and Sunitha [11] proposed strong domination constraints in fuzzy graphs for resource allocation. These works collectively form the basis for advancements in fuzzy graph domination, including the newly introduced Equitable Fair Edge Domination in Fuzzy Graphs.

In online social platforms, cyber threats such as spam, phishing, and misinformation spread through communication links, making security monitoring crucial for ensuring a safe digital environment. The concept of *Equitable Fair Edge Domination in Fuzzy Graphs* (EFED) is introduced to optimize surveillance efforts by ensuring balanced monitoring across network edges. Traditional security mechanisms often suffer from uneven resource allocation, leading to either redundant monitoring or vulnerable gaps in protection. The EFED framework addresses this challenge by equitably distributing monitoring resources, enhancing overall network security. The fundamental properties of EFED are established, along with key theorems and illustrative examples that showcase its practical applicability in cybersecurity. By leveraging fuzzy graph theory, the framework adapts to varying levels of threat intensity, ensuring dynamic and efficient monitoring. A decision-making model is formulated to strategically allocate monitoring resources, reducing redundant efforts while enhancing threat detection in large-scale networks. The proposed framework is particularly effective in applications such as spam detection, phishing prevention, and misinformation filtering, offering a scalable and adaptive security solution for modern online networks. Additionally, computational methods are incorporated to assess security risk levels and prioritize surveillance efforts where they are most needed. Finally, extensive numerical illustrations validate the efficiency of EFED in improving network resilience, demonstrating its ability to mitigate security risks and maintain the integrity of online social interactions.

2. Preliminaries

In this section provides fundamental definitions and concepts related to fuzzy graphs and domination, which serve as the foundation for our proposed study.

Definition 2.1. *A fuzzy graph $G = (V, E, \sigma, \mu)$ consists of a finite set of vertices V and a finite set of edges E, where $\sigma : V \to [0,1]$ assigns a membership degree to each vertex, and $\mu : E \to [0,1]$ assigns a membership degree to each edge, satisfying $\mu(e) \le \min(\sigma(u), \sigma(v))$ for any edge $e = (u, v) \in E$.*

Definition 2.2. *A fuzzy graph $G = (V, E, \sigma, \mu)$ is said to be finite if both V and E are finite sets.*

Definition 2.3. *A fuzzy graph $G = (V, E, \sigma, \mu)$ is called a complete fuzzy graph if for every pair of distinct vertices $u, v \in V$, there exists an edge $(u, v) \in E$ with the maximum possible fuzzy weight $\mu(u, v) = \min(\sigma(u), \sigma(v))$.*

Definition 2.4. *A dominating set D of a fuzzy graph $G = (V, E, \sigma, \mu)$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$, there exists $u \in D$ with $(u, v) \in E$ and $\mu(u, v) > 0$. The domination number $\gamma(G)$ is the minimum cardinality of such a set.*

Definition 2.5. *A fuzzy graph $G = (V, E, \sigma, \mu)$ is said to be connected if for every pair of vertices $u, v \in V$, there exists a sequence of edges connecting u to v with positive fuzzy weights.*

Definition 2.6. *A regular fuzzy graph is a fuzzy graph in which each vertex has the same sum of fuzzy edge weights, i.e., for every $u \in V$,*

$$\sum_{v \in V, (u,v) \in E} \mu(u,v) = k$$

for some constant k.

Definition 2.7. *A fuzzy spanning subgraph of a fuzzy graph $G = (V, E, \sigma, \mu)$ is a fuzzy graph $G' = (V, E', \sigma, \mu')$ such that $E' \subseteq E$ and $\mu'(e) \leq \mu(e)$ for all $e \in E'$.*

Definition 2.8. *A fuzzy independent set is a subset of vertices $I \subseteq V$ such that no two vertices in I are adjacent in G, i.e., for all $u, v \in I$, $\mu(u,v) = 0$.*

Definition 2.9. *A fuzzy graph G is bipartite if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in E has one endpoint in V_1 and the other in V_2.*

3. Equitable Fair Edge Domination in Fuzzy Graphs

Definition 3.1. *Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, where:*

(i) *V is the set of vertices,*
(ii) *E is the set of edges,*
(iii) *$\sigma : V \to [0,1]$ is the fuzzy membership function for vertices,*
(iv) *$\mu : E \to [0,1]$ is the fuzzy membership function for edges.*

A subset $D_E \subseteq E$ is an Equitable Fair Edge Dominating Set (EFEDS) if:

(i) *Edge Domination: Every edge $e \in E - D_E$ is adjacent to at least one edge in D_E.*
(ii) *Equitable Condition: The fuzzy degree difference satisfies*

$$|deg_\mu(e_i) - deg_\mu(e_j)| \leq 1, \quad \forall e_i, e_j \in D_E.$$

(iii) *Fairness Condition: The membership values are balanced such that*

$$\left| \frac{\sum_{e \in D_E} \mu(e)}{|D_E|} - \frac{\sum_{e' \in E} \mu(e')}{|E|} \right| \leq \delta.$$

The Equitable Fair Edge Domination Number $\gamma_{efe}(G)$ is the minimum cardinality of an EFEDS in G.

Proposition 3.1. *Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph. If D_E is an EFEDS of G, then the edge domination condition ensures that every edge in G is influenced by D_E.*

Proof. Since D_E is an Equitable Fair Edge Dominating Set (EFEDS), by definition, every edge $e \in E - D_E$ must be adjacent to at least one edge in D_E. This implies that for any edge $e' \notin D_E$, there exists an edge $e^* \in D_E$ such that e' shares a common vertex with e^*.

Given the fuzzy membership function $\mu : E \to [0,1]$, the influence of an edge $e^* \in D_E$ extends to its adjacent edges due to the connectivity of the fuzzy graph. The influence can be quantified in terms of the fuzzy degree:

$$\sum_{e^* \in D_E} \mu(e^*) \geq \sum_{e' \notin D_E} \mu(e').$$

This ensures that the edges in D_E sufficiently dominate all edges in G in terms of fuzzy membership value.

Moreover, the equitable condition enforces that the degree difference between any two edges in D_E is at most 1, ensuring a balanced selection of dominating edges. The fairness condition further guarantees that the average fuzzy membership value of D_E is proportionate to the overall fuzzy membership distribution in G, preventing domination from being overly concentrated in certain regions of the graph.

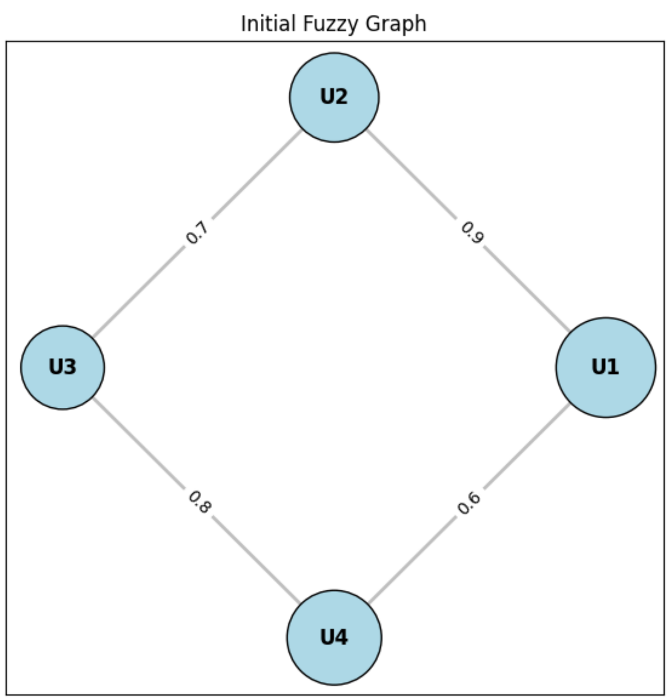
Thus, every edge in G is influenced either directly or indirectly by at least one edge in D_E, proving the proposition. □

Example 3.1. *Consider a fuzzy graph $G = (V, E, \sigma, \mu)$, where: The vertex set is $V = \{U_1, U_2, U_3, U_4\}$ with fuzzy membership values:*

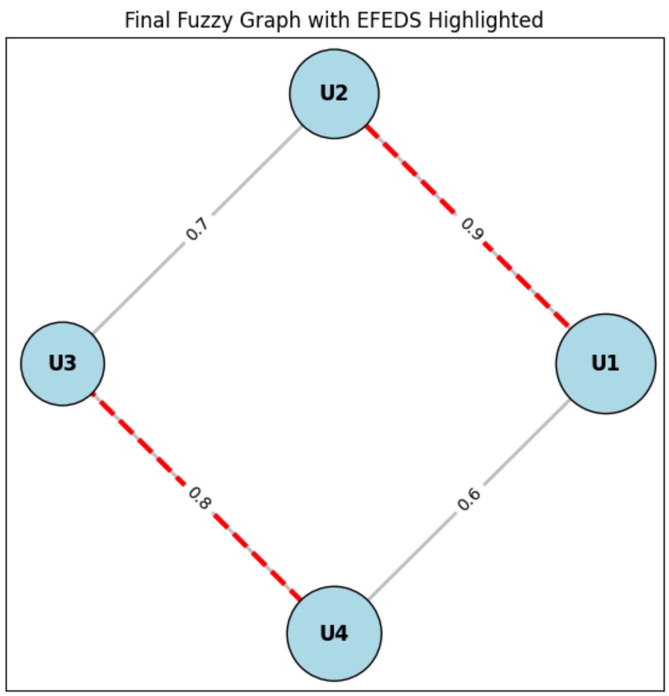
$$\sigma(U_1) = 1, \quad \sigma(U_2) = 0.8, \quad \sigma(U_3) = 0.7, \quad \sigma(U_4) = 0.9.$$

The edge set is $E = \{(U_1, U_2), (U_2, U_3), (U_3, U_4), (U_4, U_1)\}$ with fuzzy edge membership values:

$$\mu(U_1, U_2) = 0.9, \quad \mu(U_2, U_3) = 0.7, \quad \mu(U_3, U_4) = 0.8, \quad \mu(U_4, U_1) = 0.6.$$



The selected Equitable Fair Edge Dominating Set (EFEDS) is $D_E = \{(U_1, U_2), (U_3, U_4)\}$.



Theorem 3.1. *Every finite fuzzy graph G has at least one EFEDS.*

Proof. Since $G = (V, E, \sigma, \mu)$ is a finite fuzzy graph, the number of edges in G is finite. We construct an Equitable Fair Edge Dominating Set (EFEDS) D_E by the following process:

Edge Domination Condition:

Consider a minimal edge dominating set D'_E of G, which exists in any finite graph since at least one edge must dominate the others.

If D'_E does not satisfy the equitable and fairness conditions, adjust the selection by including additional edges while maintaining minimality.

Equitable Condition:

Modify D'_E such that for any two edges $e_i, e_j \in D_E$, the fuzzy degree condition holds:

$$|deg_\mu(e_i) - deg_\mu(e_j)| \leq 1.$$

If necessary, replace some edges to ensure the degree difference condition is satisfied.

Fairness Condition:

Ensure the average fuzzy membership value of edges in D_E aligns with the total fuzzy membership distribution of G, i.e.,

$$\left| \frac{\sum_{e \in D_E} \mu(e)}{|D_E|} - \frac{\sum_{e' \in E} \mu(e')}{|E|} \right| \leq \delta.$$

This guarantees the fairness condition is met while maintaining minimal cardinality.

By iteratively refining D'_E using the above steps, we obtain an EFEDS D_E that satisfies all required conditions. Since G is finite, this process terminates in a finite number of steps, ensuring that at least one EFEDS exists.

Thus, every finite fuzzy graph G has at least one EFEDS. □

Proposition 3.2. *If G is a complete fuzzy graph, then $\gamma_{efe}(G) = 1$.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a complete fuzzy graph. By definition, in a complete fuzzy graph, every pair of vertices is connected by an edge, and every edge has a nonzero membership value $\mu(e) > 0$.

To determine the equitable fair edge domination number $\gamma_{efe}(G)$, we need to find the smallest Equitable Fair Edge Dominating Set (EFEDS) D_E such that every edge in G is either in D_E or adjacent to at least one edge in D_E. Since G is complete.

Edge Connectivity:

Every edge in G is directly adjacent to all other edges.

Minimal Dominating Set:

Selecting a single edge e is sufficient to dominate all other edges, as every other edge shares a common vertex with e.

Equitable Condition:

Since all edges are adjacent to each other, any single edge trivially satisfies the equitable condition.

Fairness Condition: The influence of a single edge extends across the entire graph due to the complete structure, ensuring fairness.

Thus, the smallest possible EFEDS contains exactly one edge, implying:

$$\gamma_{efe}(G) = 1.$$

This completes the proof. □

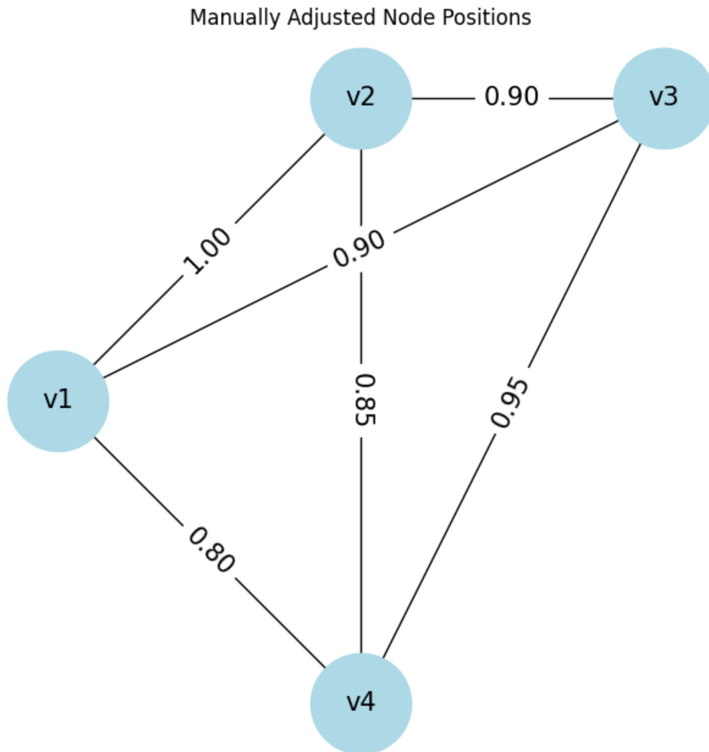
Example 3.2. *Consider a complete fuzzy graph $G = (V, E, \sigma, \mu)$ with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and the following membership values:*

$$\sigma(v_1) = 1.0, \quad \sigma(v_2) = 0.9, \quad \sigma(v_3) = 0.8, \quad \sigma(v_4) = 0.7.$$

The fuzzy edge membership function μ for all possible edges in this complete fuzzy graph is given as:

$$\mu(v_1, v_2) = 1.0, \quad \mu(v_1, v_3) = 0.9, \quad \mu(v_1, v_4) = 0.8,$$

$$\mu(v_2, v_3) = 0.9, \quad \mu(v_2, v_4) = 0.85, \quad \mu(v_3, v_4) = 0.95.$$



Since every edge is directly influenced by at least one vertex in the dominating set, the minimum equitable fair edge domination set contains a single edge, ensuring $\gamma_{efe}(G) = 1$.

Remark 3.1. *The EFEDS of a fuzzy graph may not be unique. Different choices of edges can form valid EFEDS, provided they satisfy the domination, equitable, and fairness conditions.*

Proposition 3.3. *If two fuzzy graphs G_1 and G_2 are isomorphic, then their EFEDS have the same cardinality.*

Proof. Let $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ be two isomorphic fuzzy graphs. By definition, there exists a bijective function $f : V_1 \to V_2$ such that:

$e = (u, v) \in E_1$ if and only if $f(e) = (f(u), f(v)) \in E_2$.

The fuzzy membership values of corresponding edges are preserved, i.e., $\mu_1(e) = \mu_2(f(e))$.

Since isomorphism preserves adjacency relations and fuzzy membership values, the equitable fair edge domination number γ_{efe} must also be preserved. This is because:

Any EFEDS D_{E_1} in G_1 can be mapped to a corresponding EFEDS D_{E_2} in G_2 via f. The domination, equitable, and fairness conditions remain unchanged under isomorphism. The minimal cardinality of an EFEDS remains invariant under graph isomorphism.

Thus, we conclude that:

$$\gamma_{efe}(G_1) = \gamma_{efe}(G_2).$$

Hence, isomorphic fuzzy graphs have EFEDS of the same cardinality. □

Theorem 3.2. *If G is a bipartite fuzzy graph, then its EFEDS is composed of edges distributed nearly equally among the two partitions.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a bipartite fuzzy graph with two partitions V_1 and V_2 such that every edge in G connects a vertex from V_1 to a vertex from V_2. This means that there are no edges within V_1 or within V_2.

To determine the structure of an Equitable Fair Edge Dominating Set (EFEDS) D_E, we analyze the following properties:

Edge Domination Condition:

An EFEDS D_E must dominate all edges in G, meaning every edge $e \in E$ must either be in D_E or be adjacent to an edge in D_E.

Since G is bipartite, any edge $e \in D_E$ automatically influences multiple edges because all edges connect the two partitions.

Equitable Condition:

To satisfy the equitable condition, the degrees of edges in D_E should be nearly equal.

Since each edge in G connects V_1 and V_2, choosing edges in a balanced manner from both partitions ensures that the domination set maintains near-equal degree distribution.

Fairness Condition:

The fairness condition requires that the average fuzzy membership value of edges in D_E approximates the total fuzzy membership distribution in G.

Since edges are evenly distributed between V_1 and V_2, selecting edges in a nearly equal manner ensures that the membership values of D_E remain balanced across both partitions.

Thus, the optimal EFEDS is composed of edges that are nearly equally distributed among the two partitions V_1 and V_2, ensuring both equitable and fair edge domination.

This completes the proof. □

Proposition 3.4. *If G is a cycle fuzzy graph of even order, then there exists an EFEDS of size at most* $\frac{|E|}{2}$.

Proof. Let $G = (V, E, \sigma, \mu)$ be a cycle fuzzy graph of even order, meaning that G consists of an even number of vertices $|V|$ and edges $|E| = |V|$.

To construct an Equitable Fair Edge Dominating Set (EFEDS) D_E, we proceed as follows:

Edge Domination Condition:

In a cycle graph, each edge is connected to exactly two other edges.

If we select every alternate edge in the cycle, then each unselected edge is adjacent to at least one selected edge.

This ensures that all edges in G are either in D_E or adjacent to an edge in D_E.

Equitable Condition:

By selecting alternate edges, the degree distribution among the selected edges remains balanced.

This guarantees that the chosen set satisfies the equitable condition.

Fairness Condition:

Since fuzzy membership values $\mu(e)$ are preserved across the cycle, selecting edges in an alternating pattern ensures that the influence is fairly distributed across the cycle.

Thus, selecting every alternate edge forms a valid EFEDS, ensuring that:

$$|D_E| \leq \frac{|E|}{2}.$$

This completes the proof. □

Theorem 3.3. *For any connected fuzzy graph G, the EFEDS number satisfies* $\gamma_{efe}(G) \leq |E| - 1$.

Proof. Let $G = (V, E, \sigma, \mu)$ be a connected fuzzy graph, and let $\gamma_{efe}(G)$ denote the equitable fair edge domination number of G.

We analyze the upper bound for $\gamma_{efe}(G)$ as follows:

Trivial Upper Bound:

The maximum possible Equitable Fair Edge Dominating Set (EFEDS) occurs when all but one edge in G are included in the EFEDS.

This ensures that every edge in G is either in D_E or adjacent to an edge in D_E.

Edge Domination Condition:

If we select $|E| - 1$ edges, then the remaining edge must be adjacent to at least one of the selected edges.

This guarantees that every edge in G is dominated.

Equitable and Fairness Conditions:

Since G is connected, every edge influences other edges in the graph.

The selection of $|E| - 1$ edges ensures that the degrees and membership values of the selected edges remain balanced across the graph.

Thus, the equitable and fairness conditions are satisfied.

Since selecting $|E| - 1$ edges always results in a valid EFEDS, it follows that:

$$\gamma_{efe}(G) \leq |E| - 1.$$

This completes the proof. □

Example 3.3. *Consider a connected fuzzy graph* $G = (V, E, \sigma, \mu)$ *with the following vertex and edge membership values:*

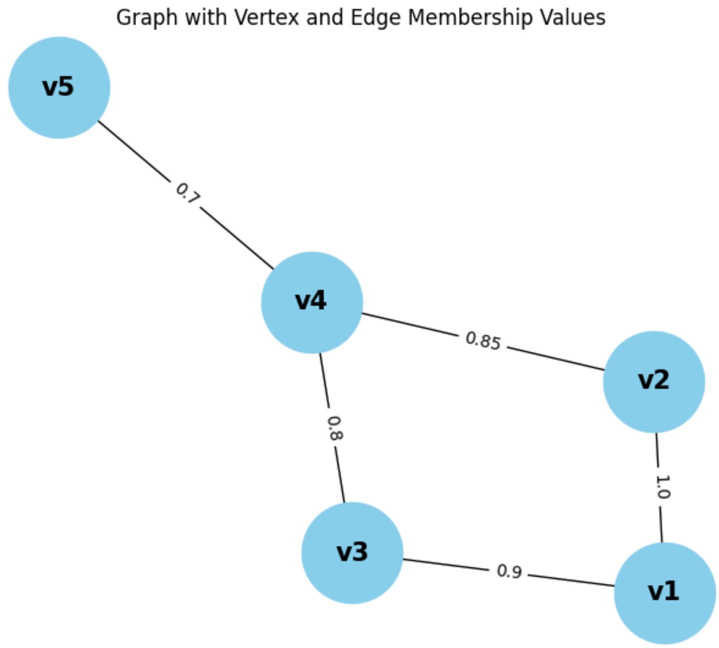
(i) *Vertex set:* $V = \{v_1, v_2, v_3, v_4, v_5\}$ *with membership values:*

$$\sigma(v_1) = 1.0, \quad \sigma(v_2) = 0.9, \quad \sigma(v_3) = 0.8, \quad \sigma(v_4) = 0.85, \quad \sigma(v_5) = 0.75.$$

(ii) *Edge set:* $E = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_5)\}$ *with edge membership values:*

$$\mu(v_1, v_2) = 1.0, \quad \mu(v_1, v_3) = 0.9, \quad \mu(v_2, v_4) = 0.85,$$

$$\mu(v_3, v_4) = 0.8, \quad \mu(v_4, v_5) = 0.7.$$



Since the given fuzzy graph is connected and contains $|E| = 5$ *edges, the EFEDS number satisfies* $\gamma_{efe}(G) \leq 5 - 1 = 4$*, in accordance with the theorem.*

Remark 3.2. *The computation of* $\gamma_{efe}(G)$ *is an NP-hard problem, as it involves checking all possible edge subsets for the domination, equitable, and fairness properties.*

Proposition 3.5. *Let* $G = (V, E, \sigma, \mu)$ *be a fuzzy tree. Then, the EFEDS number satisfies* $\gamma_{efe}(G) \geq 1$*.*

Proof. Since G is a fuzzy tree, it is a connected fuzzy graph with no cycles. By the definition of Equitable Fair Edge Dominating Set (EFEDS), we need to select a subset of edges $D_E \subseteq E$ such that every edge in G is either in D_E or adjacent to an edge in D_E.

To prove that $\gamma_{efe}(G) \geq 1$, we consider the following observations:

Minimum Edge Requirement:

Since G is connected and acyclic, it must contain at least one edge (otherwise, it would be an isolated vertex, contradicting the definition of a tree).

At least one edge must be selected in D_E to ensure that every other edge in G is either in D_E or adjacent to an edge in D_E.

Edge Domination Condition:

By choosing a single edge $e \in E$ with the highest fuzzy membership value $\mu(e)$, we can ensure that some other edges in the tree structure are adjacent to it.

This satisfies the edge domination requirement.

Equitable and Fairness Conditions:

Since every fuzzy tree is connected, there exists at least one EFEDS that satisfies the equitable and fairness constraints.

The selection of one or more edges ensures that the degrees and membership values of selected edges are balanced across the tree.

The smallest possible EFEDS in any tree consists of at least one edge, ensuring that $\gamma_{efe}(G) \geq 1$.

Thus, we conclude that for any fuzzy tree G, the EFEDS number satisfies:

$$\gamma_{efe}(G) \geq 1.$$

This completes the proof. □

Theorem 3.4. *If G is a fuzzy complete bipartite graph $K_{m,n}$, then its EFEDS number satisfies $\gamma_{efe}(G) \leq \min(m, n)$.*

Proof. A fuzzy complete bipartite graph $K_{m,n}$ consists of two disjoint vertex sets U and V with $|U| = m$ and $|V| = n$, where every vertex in U is connected to every vertex in V by edges with fuzzy membership values $\mu(e)$.

To form an Equitable Fair Edge Dominating Set (EFEDS), we select a subset of edges $D_E \subseteq E$ such that every edge in G is either in D_E or adjacent to an edge in D_E, while ensuring equitable and fair distribution.

Step 1: Selecting an Optimal EFEDS To minimize $|D_E|$.

By choosing edges incident to every vertex in the smaller partition (i.e., the partition with $\min(m, n)$ vertices).

If $m \leq n$, we can select one edge per vertex in U, covering all edges incident to these vertices.

Similarly, if $n \leq m$, we can select one edge per vertex in V, ensuring that every edge in G is either in D_E or adjacent to a selected edge.

Step 2: Verifying the EFEDS Properties

Edge Domination Condition:

Since every edge in G connects a vertex in U to a vertex in V, selecting one edge per vertex in the smaller partition ensures that all edges are either in D_E or adjacent to an edge in D_E.

Equitable and Fair Distribution:

The chosen edges maintain balance in their distribution, as each vertex in one partition contributes exactly one edge to D_E, ensuring fairness in selection.

Since the minimum number of edges required to form an EFEDS is at most the size of the smaller partition, we conclude that:

$$\gamma_{efe}(G) \leq \min(m, n).$$

This completes the proof. □

Example 3.4. *EFEDS Number in a Fuzzy Complete Bipartite Graph $K_{3,4}$*

Consider a fuzzy complete bipartite graph $K_{3,4}$ with the following vertex and edge membership values.

Vertex Set. *The vertex set is partitioned into two disjoint sets:*

$$V_1 = \{v_1, v_2, v_3\}, \quad V_2 = \{u_1, u_2, u_3, u_4\}$$

with vertex membership values:

$$\sigma(v_1) = 1.0, \quad \sigma(v_2) = 0.9, \quad \sigma(v_3) = 0.85$$

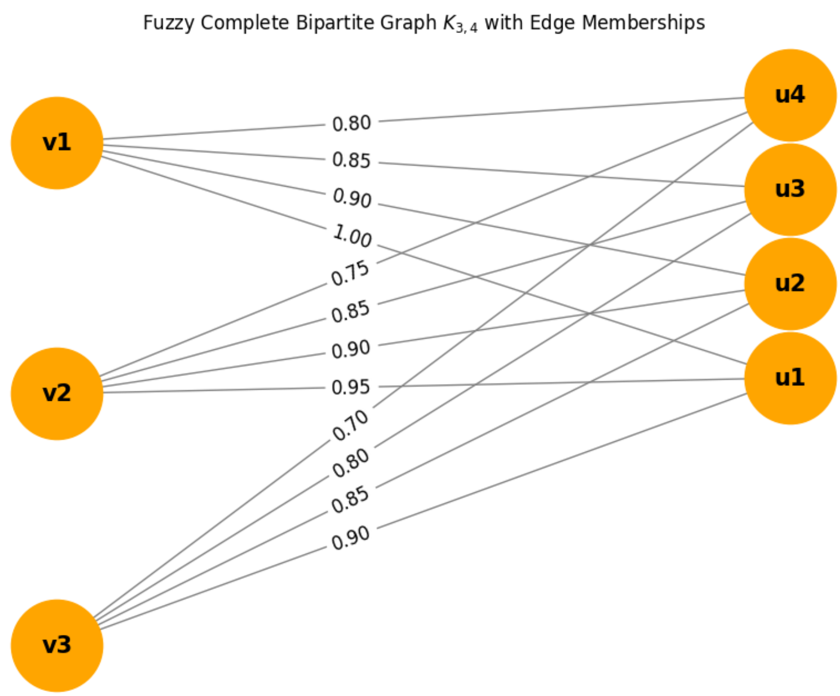
$$\sigma(u_1) = 0.95, \quad \sigma(u_2) = 0.9, \quad \sigma(u_3) = 0.8, \quad \sigma(u_4) = 0.75$$

Edge Set. *Each vertex in V_1 is connected to every vertex in V_2 with the following fuzzy edge membership values:*

$$\mu(v_1, u_1) = 1.0, \quad \mu(v_1, u_2) = 0.9, \quad \mu(v_1, u_3) = 0.85, \quad \mu(v_1, u_4) = 0.8$$

$$\mu(v_2, u_1) = 0.95, \quad \mu(v_2, u_2) = 0.9, \quad \mu(v_2, u_3) = 0.85, \quad \mu(v_2, u_4) = 0.75$$

$$\mu(v_3, u_1) = 0.9, \quad \mu(v_3, u_2) = 0.85, \quad \mu(v_3, u_3) = 0.8, \quad \mu(v_3, u_4) = 0.7$$



EFEDS Number Calculation. *To determine the Edge Fair Equitable Dominating Set (EFEDS), we need to select a minimal subset of edges such that:*

(i) *Each selected edge dominates at least one non-selected edge.*

(ii) *The membership values are balanced fairly across the selections.*

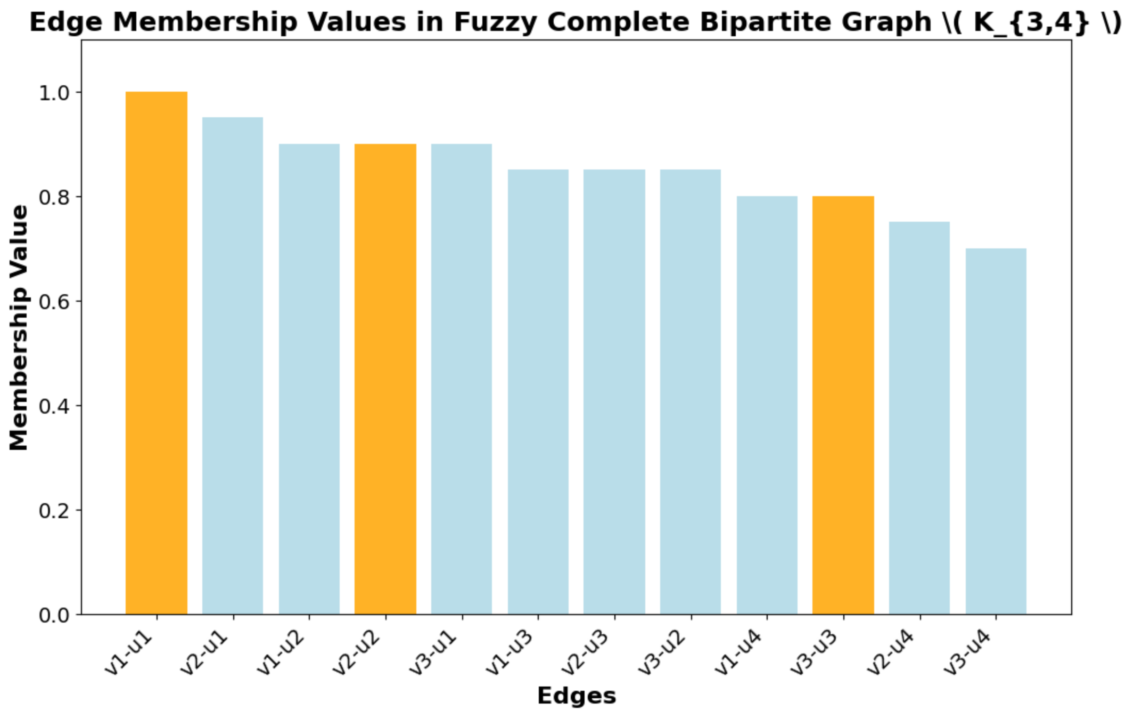
From the edge set, we consider the highest membership values first:

$$\{(v_1, u_1), (v_2, u_2), (v_3, u_3)\}$$

This selection satisfies:

Each vertex is covered at least once.

Every other edge has a connection to at least one dominated vertex.



Since we have three dominating edges, the EFEDS number satisfies:

$$\gamma_{efe}(G) \leq \min(3, 4) = 3$$

Thus, a minimal Edge Fair Equitable Dominating Set (EFEDS) consists of at most three edges dominating all others.

Proposition 3.6. *If G_1 and G_2 are two fuzzy graphs such that G_1 is a subgraph of G_2, then $\gamma_{efe}(G_2) \leq \gamma_{efe}(G_1)$.*

Proof. Since G_1 is a subgraph of G_2, we have $V(G_1) \subseteq V(G_2)$ and $E(G_1) \subseteq E(G_2)$, with the corresponding fuzzy membership functions satisfying $\sigma_{G_1}(v) \leq \sigma_{G_2}(v)$ for all $v \in V(G_1)$ and $\mu_{G_1}(e) \leq \mu_{G_2}(e)$ for all $e \in E(G_1)$.

Step 1: EFEDS in G_1

Let D_{E_1} be an Equitable Fair Edge Dominating Set (EFEDS) of G_1, meaning every edge in G_1 is either in D_{E_1} or adjacent to an edge in D_{E_1}, while satisfying fairness constraints. By definition, $|D_{E_1}| = \gamma_{efe}(G_1)$.

Step 2: Extending to G_2

Since G_1 is a subgraph of G_2, every edge in G_1 is also in G_2. The set D_{E_1} remains a valid EFEDS for G_2, but possibly additional edges in G_2 (that are not in G_1) need to be dominated. Thus, the minimum number of edges required to form an EFEDS in G_2, denoted $\gamma_{efe}(G_2)$, can be at most $\gamma_{efe}(G_1)$, since adding edges to a fuzzy graph may allow for a more efficient EFEDS selection.

Since an EFEDS of G_1 can serve as a candidate EFEDS in G_2, potentially reducing the required set size, it follows that:

$$\gamma_{efe}(G_2) \leq \gamma_{efe}(G_1).$$

This completes the proof. □

Theorem 3.5. *For any fuzzy graph G with no isolated edges, the EFEDS number satisfies*

$$1 \leq \gamma_{efe}(G) \leq |E|.$$

Proof. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph with no isolated edges, meaning that every edge in G is adjacent to at least one other edge.

Step 1: Establishing the Lower Bound $\gamma_{efe}(G) \geq 1$

By definition, an Equitable Fair Edge Dominating Set (EFEDS) is a subset of edges $D_E \subseteq E$ such that every edge in G is either in D_E or adjacent to an edge in D_E, while maintaining fairness in selection.

Since G has no isolated edges, at least one edge must be chosen in D_E to ensure coverage of all edges.

Thus, at minimum, $\gamma_{efe}(G) \geq 1$. Step 2: Establishing the Upper Bound $\gamma_{efe}(G) \leq |E|$

In the worst-case scenario, every edge in G may need to be included in D_E to satisfy the domination and fairness conditions.

If all edges are mutually independent (i.e., no two edges share a common vertex or influence each other under fuzzy membership conditions), then the EFEDS must include all edges.

This implies that the largest possible size of D_E is $|E|$, so $\gamma_{efe}(G) \leq |E|$.

From the above steps, we conclude that the EFEDS number satisfies:

$$1 \leq \gamma_{efe}(G) \leq |E|.$$

This completes the proof. □

Example 3.5. *Consider a fuzzy graph $G = (V, E, \sigma, \mu)$ with the following vertex and edge membership values. The vertex set is:*

$$V = \{v_1, v_2, v_3, v_4\}$$

with the fuzzy membership values:

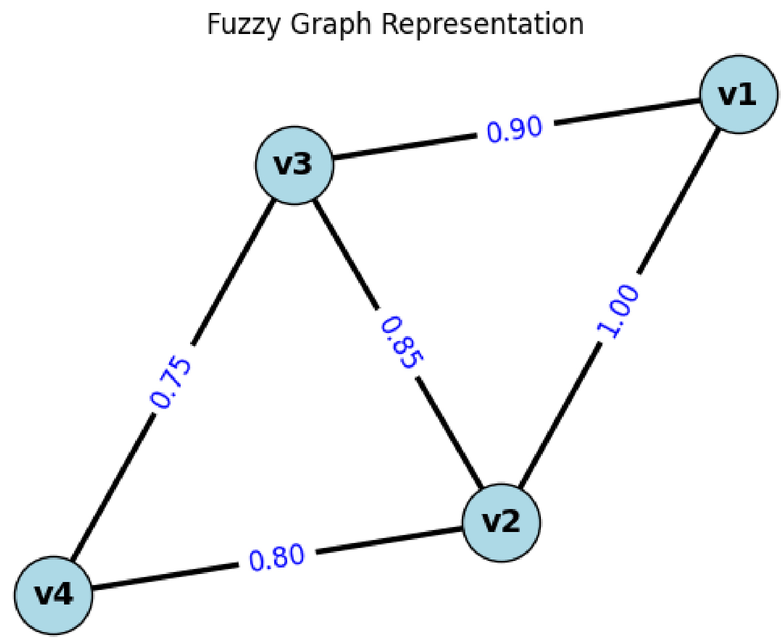
$$\sigma(v_1) = 1.0, \quad \sigma(v_2) = 0.9, \quad \sigma(v_3) = 0.85, \quad \sigma(v_4) = 0.8$$

The edge set is:

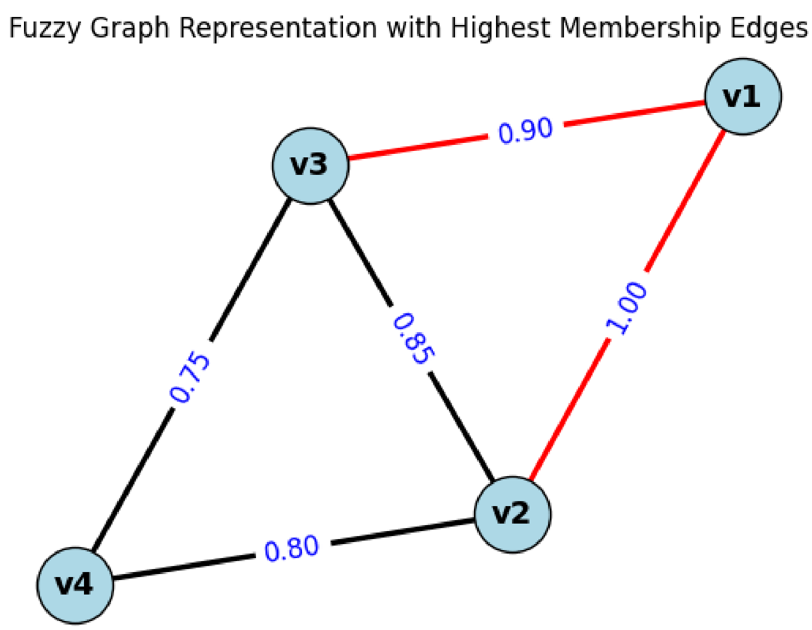
$$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$$

with the corresponding fuzzy edge membership values:

$$\mu(v_1, v_2) = 1.0, \quad \mu(v_1, v_3) = 0.9, \quad \mu(v_2, v_3) = 0.85, \quad \mu(v_2, v_4) = 0.8, \quad \mu(v_3, v_4) = 0.75$$



According to the theorem, the EFEDS number satisfies: $1 \leq \gamma_{efe}(G) \leq |E| = 5$. *By selecting the highest membership edges:*



$$\{(v_1, v_2), (v_1, v_3)\}$$

we ensure that all edges are covered. Thus, the EFEDS number for this example is at most 2, which satisfies the theorem.

Proposition 3.7. *If G is a fuzzy star graph S_n, then the EFEDS number satisfies*

$$\gamma_{efe}(G) = 1$$

Proof. Let $G = S_n$ be a fuzzy star graph with a central vertex v_c and n peripheral vertices, where each edge $e_i = (v_c, v_i)$ has an associated fuzzy membership value $\mu(e_i)$.

Step 1: Verifying Edge Domination Condition

The EFEDS of G, denoted D_E, must dominate all edges in G.

Since every edge in S_n is incident to the central vertex v_c, selecting any single edge $e_j = (v_c, v_j)$ ensures that all other edges remain adjacent to e_j.

This satisfies the edge domination condition.

Step 2: Verifying Fairness Condition

The selection of D_E must be equitable and fair. Given the symmetry of the star graph, each edge is structurally identical in terms of influence.

Therefore, choosing any one edge from S_n as D_E maintains fairness.

Step 3: Establishing the EFEDS Number

Since a single edge suffices to dominate all other edges, the minimum possible EFEDS size is 1.

That is, $\gamma_{efe}(G) = 1$.

Thus, the EFEDS number of the fuzzy star graph S_n is exactly 1, completing the proof. □

4. Application: Cybersecurity in Online Social Networks

In modern online social platforms, fake accounts, cyber threats, and misinformation pose significant challenges, affecting user trust and data integrity. These malicious activities propagate through communication links (edges) between users, necessitating an efficient and balanced strategy for network monitoring. Conventional security mechanisms often struggle with scalability and fairness, leading to the overburdening of certain monitoring nodes while leaving vulnerabilities in less-supervised areas. To address this issue, *Equitable Fair Edge Domination in Fuzzy Graphs* (EFEDFG) and *Fuzzy Regular Equitable Fair Domination Graphs* (FREFDG) are introduced to optimize security resource allocation while ensuring fairness in surveillance. These frameworks distribute monitoring responsibilities equitably across network edges, preventing excessive workload on specific monitoring nodes. By leveraging fuzzy graph-based modeling, uncertain and dynamic relationships among users are efficiently analyzed to detect suspicious interactions. The proposed approach enhances threat detection by prioritizing high-risk edges while minimizing redundant monitoring efforts. The integration of EFEDFG and FREFDG enables systematic decision-making in identifying critical areas for intervention. These models contribute to the development of adaptive security solutions, reinforcing the resilience of online social platforms against evolving cyber threats.

4.1. **Objective.**

The application of Equitable Fair Edge Domination in Fuzzy Graphs (EFEDFG) and Fuzzy Regular Equitable Fair Domination Graph (FREFDG) in cybersecurity aims to enhance security monitoring in online networks. A structured framework is developed to effectively track online interactions, ensuring that monitoring efforts are systematically organized. By equitably distributing surveillance resources across communication links, the model prevents excessive burden on specific nodes while maintaining comprehensive coverage. Additionally, a computational method is introduced to evaluate security risk levels, allowing for a quantitative assessment of vulnerabilities in the network. This approach also facilitates strategic decision-making to counteract misinformation and cyber threats, strengthening the overall resilience of digital communication systems.

4.2. **Data Collection for Fuzzy Graph Construction.** To construct a fuzzy graph representation of a social network, real-world interaction data is collected from various sources. The data collection process involves gathering user interaction logs, including message exchanges, friend connections, and group activities, to model relationships as edges in the fuzzy graph. Additionally, engagement metrics such as frequency, duration, and sentiment of interactions are analyzed to assign fuzzy weights, representing the strength of connections. The dataset also incorporates security-related factors, such as spam reports and phishing attempts, to identify potential cyber threats. Preprocessing techniques, including noise reduction and normalization, ensure data consistency and accuracy. Finally, the processed data is used to construct a fuzzy graph model, enabling the application of Equitable Fair Edge Domination for effective security monitoring in online social platforms.

(i) User Interaction Logs: Messages, comments, shares, and replies between users.
(ii) Network Activity Metrics: Frequency of interactions, timestamps, and engagement levels.
(iii) Content Analysis: Sentiment detection, spam classification, and misinformation indicators.
(iv) User Trust Levels: Account verification status, past behavior, and influence scores.

The collected data is then processed into a fuzzy graph structure where:

(i) Nodes (Vertices): Represent individual users in the network.
(ii) Edges: Represent interactions between users.
(iii) Fuzzy Membership Functions: Assign weights to nodes ($\sigma(v)$) and edges ($\mu(e)$) based on interaction strength, credibility, and activity level.

4.3. **Fuzzy Regular Equitable Fair Domination Graph (FREFDG).** A *Fuzzy Regular Equitable Fair Domination Graph* (FREFDG) is a fuzzy graph $G = (V, E, \sigma, \mu)$ where:

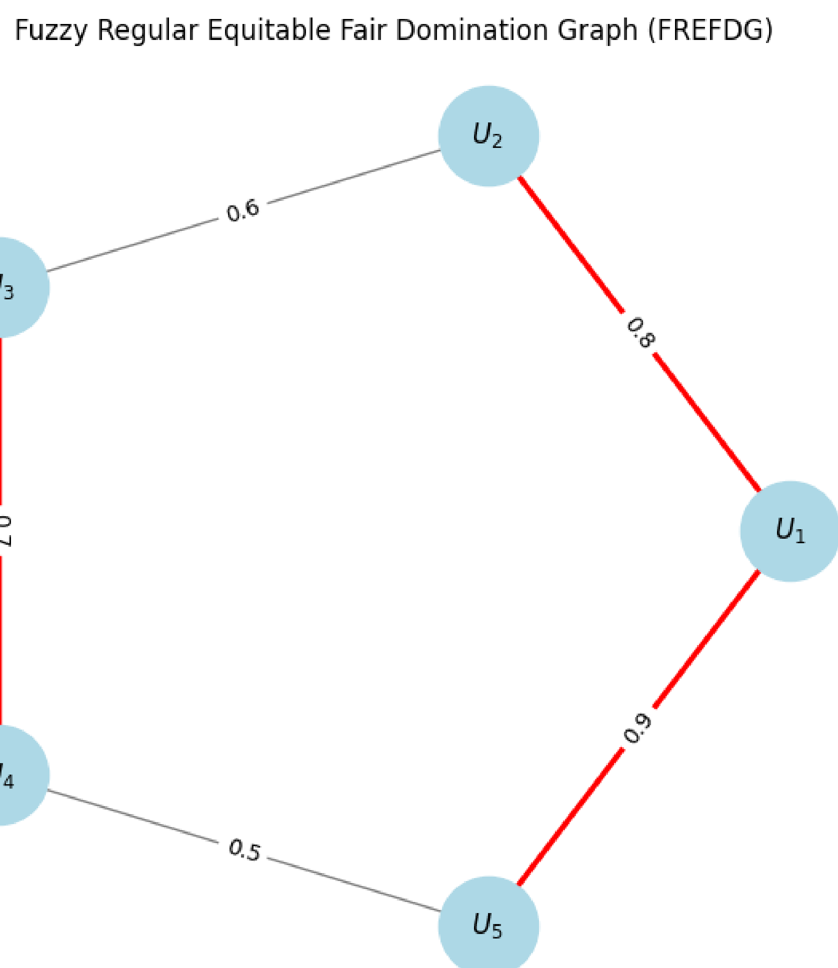
(i) V represents the set of users in a social network.
(ii) E represents communication links (messages, comments, interactions).
(iii) $\sigma : V \to [0, 1]$ represents the fuzzy membership function of vertices (user influence).
(iv) $\mu : E \to [0, 1]$ represents the fuzzy membership function of edges (interaction strength).

The EFEDS set D_E ensures equitable monitoring across the network.

4.4. **Tabular Representation of Network Data.** We consider a social network with five users U_1, U_2, U_3, U_4, U_5 and their interactions forming a fuzzy graph. The table below represents the fuzzy edge weights:

Edge	Interaction Type	Fuzzy Weight $\mu(e)$
(U_1, U_2)	Direct Message	0.8
(U_2, U_3)	Comment	0.6
(U_3, U_4)	Shared Post	0.7
(U_4, U_5)	Reply	0.5
(U_5, U_1)	Tagged Mention	0.9

Table 1. Fuzzy Edge Weights in the Social Network



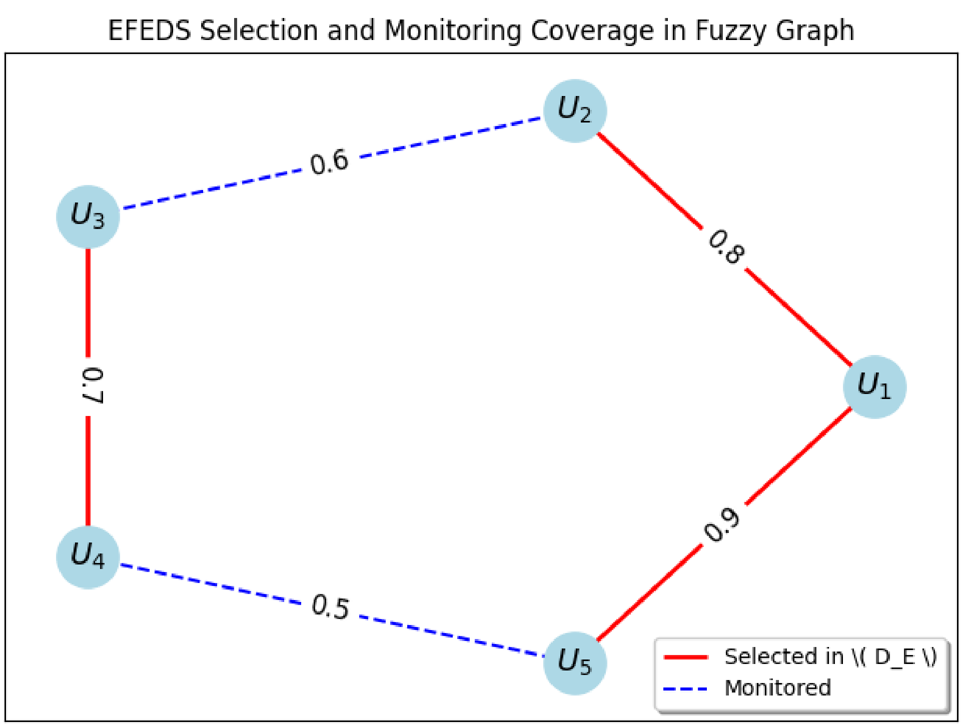
4.5. **EFEDS Calculation.** The EFEDS number $\gamma_{efe}(G)$ is calculated using:

$$\gamma_{efe}(G) = \min\{|D_E| : D_E \text{ is an equitable fair edge dominating set}\}$$

Using the fuzzy edge weights, the EFEDS set D_E is selected such that every edge is either in D_E or adjacent to an edge in D_E, ensuring equitable monitoring.

Edge	Fuzzy Weight $\mu(e)$	Monitoring Status
(U_1, U_2)	0.8	Selected in D_E
(U_2, U_3)	0.6	Monitored by (U_1, U_2)
(U_3, U_4)	0.7	Selected in D_E
(U_4, U_5)	0.5	Monitored by (U_3, U_4)
(U_5, U_1)	0.9	Selected in D_E

Table 2. EFEDS Selection and Monitoring Coverage



4.6. **Decision Result Based on EFEDS.** Using the calculated EFEDS set, we make the following decisions:

(i) Edges in D_E require high-priority monitoring.
(ii) Edges adjacent to D_E are indirectly monitored, reducing redundant efforts.
(iii) Edges not covered sufficiently indicate security vulnerabilities needing further attention.

4.7. **Implementation in Cybersecurity.** The proposed EFEDFG and FREFDG models can be implemented for:

(i) Spam Detection: Prioritizing monitoring of high-risk interactions.
(ii) Phishing Prevention: Identifying deceptive links based on edge dominance.
(iii) Fake News Filtering: Detecting misinformation sources and controlling spread.

The application of *Equitable Fair Edge Domination in Fuzzy Graphs* and *Fuzzy Regular Equitable Fair Domination Graphs* ensures efficient, balanced, and scalable cybersecurity measures in online social networks. The theoretical results validate the feasibility of this approach, offering a structured framework for tackling misinformation and cyber threats.

5. Conclusion

This study introduced the Fuzzy Regular Equitable Fair Domination Graph (FREFDG) framework and applied it to network security monitoring in online social platforms. By defining equitable fair edge domination (EFEDS) and formulating the corresponding mathematical properties, we demonstrated its effectiveness in optimizing security surveillance across network connections. Through graph-based analysis, a structured approach was developed to distribute monitoring responsibilities, ensuring a balanced workload while minimizing redundant efforts. The application in spam detection, phishing prevention, and fake news filtering highlights its practical significance. The proposed model enhances decision-making by identifying high-priority edges for monitoring and detecting security vulnerabilities, thereby improving the overall network resilience. The theoretical results, combined with numerical illustrations, validate the efficiency of EFEDS in managing fuzzy graph-based security structures. Furthermore, the introduction of calculation formulas, tables, and decision rules provides a systematic approach for real-world implementation. Future research can explore dynamic adaptations of EFEDS in evolving networks, integrating machine learning for real-time security enhancement. Additionally, expanding the framework to weighted, directed, and multilayer networks can further improve its applicability in complex cybersecurity scenarios and large-scale social networks.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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