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OPIAL TYPE INTEGRAL INEQUALITIES FOR WIDDER DERIVATIVES AND LINEAR DIFFERENTIAL OPERATORS

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Abstract. In this paper we establish Opial type integral inequalities for Widder derivatives and linear differential operator. Also, for applications we construct some related inequalities as special cases.

1. Introduction

The following inequality established in 1960. by Opial [15] :

Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt, \tag{1.1}$$

where constant $\frac{h}{4}$ is the best possible.

Over the last 50 years, Opial inequality (1.1) is studied by many mathematicians and extended, generalized in different ways. It is recognized as fundamental result in the theory of differential equations (see the monograp[1]). Opial inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations, (see, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 18, 19]).

Following theorems by Mitrinović and Pečarić, include such generalizations of Opial's inequality given in [16, page, 237–238] and for them we need next characterization:

We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U_1(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t)v(t)\,dt$$

where v is a continuous function and K is an arbitrary non-negative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and are finite.

Theorem 1.1. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v,K)$ where*

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$\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leq M$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

(1.2)
$$\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx \leq \frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx.$$

If the function $\phi(x^{\frac{1}{q}})$ *is concave, then the reverse inequality holds.*

A similar result follows by using another class $U_2(v,K)$ of functions $u:[a,b]\longrightarrow\mathbb{R}$ which admits representation

$$u(x)=\int_x^b K(x,t)v(t)dt.$$

Theorem 1.2. *Let* $\phi:[0,\infty)\longrightarrow\mathbb{R}$ *be a differentiable function such that for* $q>1$ *the function* $\phi(x^{\frac{1}{q}})$ *is convex and* $\phi(0)=0$. *Let* $u\in U_2(v,K)$ *where* $\left(\int_x^b (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leq N$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

(1.3)
$$\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q dx \leq \frac{q}{N^q(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}N|v(x)|\right)dx\,.$$

If the function $\phi(x^{\frac{1}{q}})$ *is concave, then reverse inequality holds.*

In [5] we gave extensions of above Mitrinović–Pečarić inequalities which are stated in the following theorems.

Theorem 1.3. *Let* $\phi:[0,\infty)\longrightarrow\mathbb{R}$ *be a differentiable function such that for* $q>1$ *the function* $\phi(x^{\frac{1}{q}})$ *is convex and* $\phi(0)=0$. *Let* $u\in U_1(v,K)$ *where* $\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leq M$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\begin{aligned}\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx &\leq \frac{q}{M^q}\,\phi\Big(M\Big(\int_a^b |v(x)|^q\,dx\Big)^{\frac{1}{q}}\Big)\\ &\leq \frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx\,. \qquad (1.4)\end{aligned}$$

If the function $\phi(x^{\frac{1}{q}})$ *is concave, then reverse inequalities hold.*

Theorem 1.4. *Let* $\phi:[0,\infty)\longrightarrow\mathbb{R}$ *be a differentiable function such that for* $q>1$ *the function* $\phi(x^{\frac{1}{q}})$ *is convex and* $\phi(0)=0$. *Let* $u\in U_2(v,K)$ *where* $\left(\int_x^b (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leq M$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\begin{aligned}\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q dx &\leq \frac{q}{M^q}\,\phi\Big(M\Big(\int_a^b |v(x)|^q\,dx\Big)^{\frac{1}{q}}\Big)\\ &\leq \frac{q}{M^q(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx\,. \qquad (1.5)\end{aligned}$$

If the function $\phi(x^{\frac{1}{q}})$ *is concave, then reverse inequalities hold.*

In view of the great importance of Taylor's formula in analysis, it may be regarded as extremely surprising that so few attempts at generalization have been made.

The problem of the representation of an arbitrary function by means of linear combinations of prescribed functions has received no small amount of attention (see the introduction of [17]). Therefore, in 1927 Widder gave generalization of Taylor's formula [17].
In this paper we are interested to give Opial type integral inequalities by Mitrinović and Pečarić for Widder derivatives and linear differential operators. Also, we give their extensions and as applications discuss their special cases and examples.

2. Mitrinović-Pečarić Inequalities for Widder Derivatives

In this section we give Opial type integral inequalities for Widder derivatives. We extend these inequalities, also give their special cases and provide some examples. The following are taken from [17].
Let $f, u_0, u_1, ..., u_n \in C^{n+1}([a, b]), n \geq 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), ..., u_n(x)] := \begin{vmatrix} u_0(x) & u_1(x) & . & . & . & u_i(x) \\ u_0^{'}(x) & u_1^{'}(x) & . & . & . & u_i^{'}(x) \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ u_0^{i}(x) & u_1^{i}(x) & . & . & . & u_i^{i}(x) \end{vmatrix}, \tag{2.1}$$

$i = 1, ..., n$. Here $W_0(x) = u_0(x)$. Assume $W_0(x) > 0$ over $[a, b], i = 1, ..., n$.
For $i \geq 0$, the differential operator of order i (Widder derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), ..., u_{i-1}(x), f(x)]}{W_{i-1}(x)} \tag{2.2}$$

$i = 1, ..., n + 1; L_0 f(x) := f(x), \forall x \in [a, b]$. Consider also

$$g_i(x, t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & u_1(t) & . & . & . & u_i(t) \\ u_0^{'}(t) & u_1^{'}(t) & . & . & . & u_i^{'}(t) \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ u_0^{i-1}(t) & u_1^{i-1}(t) & . & . & . & u_{i-1}^{i}(x) \\ u_0(x) & u_1(x) & . & . & . & u_i(x) \end{vmatrix}, \tag{2.3}$$

$i = 1, ..., n; g_0(x, t) := \frac{u_0(x)}{u_0(t)}, \forall x, t \in [a, b]$.

Example 2.1. *Sets of the form $\{u_0, u_1, ..., u_n\}$ are $\{u_0, u_1, ..., u_n\}$, $\{1, sinx, -cosx, -sin2x, cos2x, ..., (-1)^n sinnx, (-1)^n cosnx\}$, etc.*

We also mention the generalized Widder-Talylor's formula, see, [17] also [3].

Theorem 2.2. *Let the functions $f, u_0, u_1, ..., u_n \in C^{n+1}([a, b])$, and the Wronskians $W_0(x), W_1(x), ..., W_n(x) > 0$ on $[a, b]$, $x \in [a, b]$. Then for $t \in [a, b]$ we have*

$$f(x) = y(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t)g_1(x, t) + ... + L_n f(t)g_n(x, t) + R_n(x), \tag{2.4}$$

where

$$R_n(x) := \int_t^x g_n(x, s)L_{n+1}f(s)ds$$

For example [17] one could take $u_0(x) = c > 0$. If $u_i(x) = x^i, i = 0, 1, ..., n$, defined on $[a, b]$, then
$L_i y(t) = f^i(t)$ and $g_i(x,t) = \frac{(x-t)^i}{i!}$, $t \in [a,b]$.

We need the following result.

Corollary 2.3. *By additionally assuming for fixed $x_0 \in [a, b]$ that $L_i f(x_0) = 0$, $i = 0, 1, ..., n$, we get that*

$$f(x) = \int_{x_0}^{x} g_n(x,s) L_{n+1} f(s) ds. \tag{2.5}$$

Note that all results of this section are under the assumptions of Theorem 2.2 and Corollary 2.3.

Theorem 2.4. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \geq x_0 \in [a, b]$ and $p \in (1, \infty)$ such that $\left(\int_{x_0}^{x} |g_n(x,t)|^p dt\right)^{\frac{1}{p}} \leq M, \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{x_0}^{b} |f(x)|^{1-q} \phi'(|f(x)| |L_{n+1} f(x)|^q \, dx \leq \frac{q}{M^q} \phi \left(M \left(\int_{x_0}^{b} |L_{n+1} f(t)|^q dt \right)^{\frac{1}{q}} \right). \tag{2.6}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

Proof. As f has representation

$$f(x) = \int_{x_0}^{x} g_n(x,t) L_{n+1} f(t) dt.$$

By applying Holder's inequality we have

$$\begin{aligned} |f(x)| &= \left| \int_{x_0}^{x} g_n(x,t) L_{n+1} f(t) dt \right| \\ &\leq \left(\int_{x_0}^{x} |g_n(x,t)|^p dt \right)^{\frac{1}{p}} \left(\int_{x_0}^{x} |L_{n+1} f(t)|^q dt \right)^{\frac{1}{q}} \\ &\leq M \left(\int_{x_0}^{x} |L_{n+1} f(t)|^q dt \right)^{\frac{1}{q}} \end{aligned} \tag{2.7}$$

Let

$$p(x) = \int_{x_0}^{x} |L_{n+1} f(t)|^q dt.$$

Then $p'(x) = |L_{n+1} f(x)|^q$. Now from (2.7) we can have $|f(x)| \leq M \left(p(x)\right)^{\frac{1}{q}}$. From the convexity of $\phi(x^{\frac{1}{q}})$ it follows that the function $x^{1-q}\phi'(x)$ is increasing. Thus we have

$$\begin{aligned}\int_{x_0}^{b} |f(x)|^{1-q}\phi'(|f(x)|)|L_{n+1}f(x)|^q dx &\leq \int_{x_0}^{b} M^{1-q}\,(p(x))^{\frac{1}{q}-1}\,\phi'\left(M\,(p(x))^{\frac{1}{q}}\right)p'(x)dx\\ &= \frac{q}{M^q}\int_{x_0}^{b}\phi'\left(M(p(x))^{\frac{1}{q}}\right)d(M(p(x)^{\frac{1}{q}})\\ &= \frac{q}{M^q}\phi\left(M\,(p(b))^{\frac{1}{q}}\right)\\ &= \frac{q}{M^q}\phi\left(M\left(\int_{x_0}^{b}|L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}}\right).\end{aligned}$$

□

Next we give extension of above theorem.

Theorem 2.5. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let $x\geq x_0\in[a,b]$ and $p\in(1,\infty)$ such that $\left(\int_{x_0}^{x}|g_n(x,t)|^p dt\right)^{\frac{1}{p}}\leq M$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{x_0}^{b}|f(x)|^{1-q}\phi'(|f(x)||L_{n+1}f(x)|^q\,dx \leq \frac{q}{M^q}\phi\left(M\left(\int_{x_0}^{b}|L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{q}{M^q\,(b-x_0)}\int_{x_0}^{b}\phi\left((b-x_0)^{\frac{1}{q}}M|L_{n+1}f(t)|\right)dt\,. \tag{2.8}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. Inequality (2.6) holds by Theorem 2.4. Since $\phi(x^{\frac{1}{q}})$ is convex, the following Jensen's inequality holds

$$\phi\Big(\Big(\frac{1}{b-a}\int_{x_0}^{b}g(t)\,dt\Big)^{\frac{1}{q}}\Big)\leq\frac{1}{b-a}\int_{x_0}^{b}\phi\left(g^{\frac{1}{q}}(t)\right)dt\,. \tag{2.9}$$

Applying (2.9) on (2.6) we get (2.8). □

The counter part of above theorem is given in the following.

Theorem 2.6. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let $x\leq x_0\in[a,b]$ and $p\in(1,\infty)$ such that $\left(\int_{x}^{x_0}|g_n(x,t)|^p dt\right)^{\frac{1}{p}}\leq M$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{a}^{x_0}|f(x)|^{1-q}\phi'(|f(x)||L_{n+1}f(x)|^q\,dx \leq \frac{q}{M^q}\phi\left(M\left(\int_{a}^{x_0}|L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}}\right). \tag{2.10}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

Proof. As f has representation

$$f(x)=\int_{x_0}^{x}g_n(x,t)L_{n+1}f(t)dt.$$

By applying Holder's inequality we have

$$
\begin{aligned}
|f(x)| &= \left|\int_x^{x_0} g_n(x,t)L_{n+1}f(t)dt\right| \\
&\le \left(\int_x^{x_0} |g_n(x,t)|^p dt\right)^{\frac{1}{p}} \left(\int_x^{x_0} |L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}} \\
&\le M\left(\int_x^{x_0} |L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}} \qquad (2.11)
\end{aligned}
$$

Let

$$p(x) = \int_x^{x_0} |L_{n+1}f(t)|^q dt.$$

Then $-p'(x) = |L_{n+1}f(x)|^q \ge 0$. Now from (2.11) we can have $|f(x)| \le M\,(p(x))^{\frac{1}{q}}$. From the convexity of $\phi(x^{\frac{1}{q}})$ it follows that the function $x^{1-q}\phi'(x)$ is increasing. Thus we have

$$
\begin{aligned}
\int_a^{x_0} |f(x)|^{1-q}\phi'(|f(x)|)|L_{n+1}f(x)|^q dx &\le -\int_a^{x_0} M^{1-q}\,(p(x))^{\frac{1}{q}-1}\,\phi'\left(M\,(p(x))^{\frac{1}{q}}\right)p'(x)dx \\
&= -\frac{q}{M^q}\int_a^{x_0} \phi'\left(M(p(x))^{\frac{1}{q}}\right) d(M(p(x)^{\frac{1}{q}}) \\
&= \frac{q}{M^q}\phi\left(M\,(p(a))^{\frac{1}{q}}\right) \\
&= \frac{q}{M^q}\phi\left(M\left(\int_a^{x_0} |L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

□

Next we give extension of above theorem.

Theorem 2.7. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \le x_0 \in [a,b]$ and $p \in (1,\infty)$ such that $\left(\int_x^{x_0} |g_n(x,t)|^p dt\right)^{\frac{1}{p}} \le M$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$
\begin{aligned}
\int_a^{x_0} |f(x)|^{1-q}\phi'(|f(x)||L_{n+1}f(x)|^q\,dx &\le \frac{q}{M^q}\phi\left(M\left(\int_a^{x_0} |L_{n+1}f(t)|^q dt\right)^{\frac{1}{q}}\right) \\
&\le \frac{q}{M^q\,(b-x_0)}\int_a^{x_0} \phi\left((b-x_0)^{\frac{1}{q}}M|L_{n+1}f(t)|\right)dt\,. \qquad (2.12)
\end{aligned}
$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. As in proof of the Theorem 2.7, inequalities follow from Theorem 2.6 and Jensen's inequality (2.9). □

Remark 2.8. *If directly we replace u by f, v by $L_{n+1}f$ and general kernal $K(x,t)$ by $g_n(x,t)$ in (1.2), (1.3), (1.4) and (1.5) we can see above results.*

Next we discuss extreme cases.

Theorem 2.9. *Let $p = 1, q = \infty$ and $x \ge x_0 \in [a,b]$. Then we have*

$$\frac{1}{b-x_0}\int_{x_0}^{b} |f(x)||L_{n+1}f(x)|\,dx \le M||L_{n+1}f||_\infty^2. \qquad (2.13)$$

Proof. As f has representation

$$f(x)=\int_{x_0}^{x} g_n(x,t)L_{n+1}f(t)dt.$$

From this we get

$$\begin{aligned}|f(x)| &\leq \left(\int_{x_0}^{x}|g_n(x,t)|dt\right)||L_{n+1}f||_\infty\\ &\leq M||L_{n+1}f||_\infty\end{aligned}$$

and using $|L_{n+1}f(t)|\leq ||L_{n+1}f||_\infty$ we have

$$|f(x)||L_{n+1}f(t)|\leq M||L_{n+1}f||_\infty^2.$$

Now integrating the last inequality on $[x_0,b]$, we get (2.13). □

Theorem 2.10. *Let $p=1,q=\infty$ and $x\leq x_0\in[a,b]$. Then we have*

$$\frac{1}{x_0-a}\int_{a}^{x_0}|f(x)||L_{n+1}f(x)|\,dx\leq M||L_{n+1}f||_\infty^2. \tag{2.14}$$

Proof. As f has representation

$$f(x)=\int_{x}^{x_0} g_n(x,t)L_{n+1}f(t)dt.$$

From this we get

$$\begin{aligned}|f(x)| &\leq \left(\int_{x_0}^{x}|g_n(x,t)|dt\right)||L_{n+1}f||_\infty\\ &\leq M||L_{n+1}f||_\infty\end{aligned}$$

and using $|L_{n+1}f(t)|\leq ||L_{n+1}f||_\infty$ we have

$$|f(x)||L_{n+1}f(t)|\leq M||L_{n+1}f(t)||_\infty^2.$$

Now integrating the last inequality on $[a,x_0]$, we get (2.14). □

Corollary 2.11. *Let $p,q\in(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$\int_{x_0}^{x}|f(x)|^q|L_{n+1}f(x)|^q\,dx\leq\frac{M^q}{2}\left(\int_{x_0}^{x}|L_{n+1}f(t)|^q dt\right)^2. \tag{2.15}$$

Proof. By setting $\phi(t)=t^{2q}$ and $b=x\geq x_0$ in Theorem 2.4 one can get (2.15). □

Corollary 2.12. *Let $p,q\in(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$\int_{x}^{x_0}|f(x)|^q|L_{n+1}f(x)|^q\,dx\leq\frac{M^q}{2}\left(\int_{x}^{x_0}|L_{n+1}f(t)|^q dt\right)^2. \tag{2.16}$$

Proof. By setting $\phi(t)=t^{2q}$ and $a=x\leq x_0$ in Theorem 2.5 one can get (2.16). □

Corollary 2.13. *Let $p,q\in(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$\left|\int_{x_0}^{x}|f(x)|^q|L_{n+1}f(x)|^q\,dx\right|\leq\frac{M^q}{2}\left(\int_{x_0}^{x}|L_{n+1}f(t)|^q dt\right)^2. \tag{2.17}$$

Proof. From (2.15) and (2.16) one can easily get (2.17). □

Example 2.14. *If we take* $u_0(x) = c > 0$ *and* $u_n(x) = x^n, n = 0, 1, 2, ..., n$ *defined on* $[a, b]$, *then* $L_n f(x) = f^n(x)$ *and* $g_n(x, t) = \frac{(x-t)^n}{n!}, t \in [a, b],$. *Here we can have* $M = \frac{(b-x_0)^{\frac{np+1}{p}}}{n!(np+1)^{\frac{1}{p}}}$ *and the inequality (2.6) becomes*

$$\int_{x_0}^{b} |f(x)|^{1-q}\phi'\left(|f(x)||f^{n+1}(x)|^q\right) dx \leq \frac{n!q(np+1)^{\frac{q}{p}}}{(b-x_0)^{\frac{q(np+1)}{p}}}\phi\left(\frac{(b-x_0)^{\frac{np+1}{p}}}{n!(np+1)^{\frac{1}{p}}}\left(\int_{x_0}^{b}|f^{n+1}(t)|^q dt\right)^{\frac{1}{q}}\right).$$

Also extension of (2.6) becomes

$$\begin{aligned}\int_{x_0}^{b} |f(x)|^{1-q}\phi'\left(|f(x)||f^{n+1}(x)|^q\right) dx &\leq \frac{n!q(np+1)^{\frac{q}{p}}}{(b-x_0)^{\frac{q(np+1)}{p}}}\phi\left(\frac{(b-x_0)^{\frac{np+1}{p}}}{n!(np+1)^{\frac{1}{p}}}\left(\int_{x_0}^{b}|f^{n+1}(t)|^q dt\right)^{\frac{1}{q}}\right)\\ &\leq \frac{n!q(np+1)^{\frac{q}{p}}}{(b-x_0)^{q(n+1)}}\int_{x_0}^{b}\phi\left(\frac{(b-x_0)^{n+1}}{n!(np+1)^{\frac{1}{p}}}|L_{n+1}f(t)|\right) dt\,.\end{aligned}$$

Remark 2.15. *Examples similar to Example 2.14 can be obtained by using other inequalites given in above results. We omit here such examples.*

3. Mitrinović-Pečarić Inequalities for Linear differential operators

In this section we give Opial type integral inequalities for Linear differential operators. We extend these inequalities, also give some special cases.

Here we follow [12, page, 145–154].

Let I be a closed interval of $\mathbb{R}$. Let $a_i(x), i = 0, 1, ..., n-1 (n \in \mathbb{N}), h(x)$ be continuous functions on I and let $L = D^n + a_{n-1}(x)D^{n-1} + ... + a_0(x)$ be a fixed linear differential operator on $C^n(I)$. Let $y_1(x), ..., y_n(x)$ be a set of linear independent solutions to $Ly = 0$. Here the associated Green's function for L is

$$H(x,t) := \left|\begin{matrix} y_1(t) & \cdots & y_n(t)\\ y_1'(t) & \cdots & y_n'(t)\\ \vdots & & \\ y_1^{n-2}(t) & \cdots & y_n^{n-2}(t)\\ y_1(x) & \cdots & y_n(x)\end{matrix}\right| \Bigg/ \left|\begin{matrix} y_1(t) & \cdots & y_n(t)\\ y_1'(t) & \cdots & y_n'(t)\\ \vdots & & \\ y_1^{n-2}(t) & \cdots & y_n^{n-2}(t)\\ y_1(t) & \cdots & y_n(t)\end{matrix}\right|, \tag{3.1}$$

which is a continuous function on I^2. Consider fixed $x_0 \in I$, then

$$y(x) = \int_{x_0}^{x} H(x,t)h(t)dt, x \in I \tag{3.2}$$

is the unique solution to the initial value problem

$$Ly = h;\ y^{(i)}(x_0) = 0,\ i = 0, 1, ..., n-1. \tag{3.3}$$

Theorem 3.1. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \geq x_0 \in [a, b]$ and $p \in (1, \infty)$ such that $\left(\int_{x_0}^{x} (H(x,t))^p dt\right)^{\frac{1}{p}} \leq M$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{x_0}^{b} |y(x)|^{1-q} \phi'(|y(x)||(Ly)(x)|^q \, dx \leq \frac{q}{M^q} \phi \left(M \left(\int_{x_0}^{b} |(Ly)(t)|^q dt \right)^{\frac{1}{q}} \right). \tag{3.4}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

Proof. Here y has representation

$$y(x) = \int_{x_0}^{x} H(x,t) h(t) dt.$$

rest of proof follows from the proof of Theorem 2.4. □

Next we give extension of above theorem.

Theorem 3.2. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \geq x_0 \in [a, b]$ and $p \in (1, \infty)$ such that $\left(\int_{x_0}^{x} (H(x,t))^p dt\right)^{\frac{1}{p}} \leq M$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}\int_{x_0}^{b} |y(x)|^{1-q} \phi'(|y(x)||(Ly)(x)|^q \, dx &\leq \frac{q}{M^q} \phi \left(M \left(\int_{x_0}^{b} |(Ly)(t)|^q dt \right)^{\frac{1}{q}} \right) \\ &\leq \frac{q}{M^q \, (b - x_0)} \int_{x_0}^{b} \phi \left((b - x_0)^{\frac{1}{q}} M |(Ly)(t)| \right) dt \, .\end{aligned}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. Proof is similar to the proof of Theorem 2.5. □

The counter part of above theorem is given in the following.

Theorem 3.3. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \leq x_0 \in [a, b]$ and $p \in (1, \infty)$ such that $\left(\int_{x}^{x_0} (H(x,t))^p dt\right)^{\frac{1}{p}} \leq M$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{a}^{x_0} |y(x)|^{1-q} \phi'(|y(x)||(Ly)(x)|^q \, dx \leq \frac{q}{M^q} \phi \left(M \left(\int_{a}^{x_0} |(Ly)(t)|^q dt \right)^{\frac{1}{q}} \right). \tag{3.5}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

Proof. Proof is similar to the proof of Theorem 2.6. □

Next we give extension of above theorem.

Theorem 3.4. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $x \le x_0 \in [a,b]$ and $p \in (1,\infty)$ such that $\left(\int_x^{x_0} (H(x,t))^p dt\right)^{\frac{1}{p}} \le M$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^{x_0} |y(x)|^{1-q}\phi'(|y(x)||(Ly)(x)|^q\, dx \le \frac{q}{M^q}\phi\left(M\left(\int_a^{x_0} |(Ly)(t)|^q dt\right)^{\frac{1}{q}}\right)$$

$$\le \frac{q}{M^q\,(b-x_0)}\int_a^{x_0} \phi\left((b-x_0)^{\frac{1}{q}} M|(Ly)(t)|\right) dt\,. \tag{3.6}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. Proof is similar to the proof of Theorem 2.7. □

Remark 3.5. *If directly we replace u by y, v by Ly, and general kernal $K(x,t)$ by $H(x,t)$ in (1.2), (1.3), (1.4) and (1.5) we can see above results.*

Next we discuss extreme cases.

Theorem 3.6. *Let $p = 1, q = \infty$ and $x \ge x_0 \in [a,b]$. Then we have*

$$\frac{1}{b-x_0}\int_{x_0}^{b} |y(x)||(Ly)(x)|\, dx \le M||Ly||_\infty^2. \tag{3.7}$$

Proof. As f has representation

$$y(x) = \int_{x_0}^{x} H(x,t)(Ly)(t)dt.$$

From this we get

$$\begin{aligned} |y(x)| &\le \left(\int_{x_0}^{x} |H(x,t)|dt\right)||Ly||_\infty \\ &\le M||Ly||_\infty \end{aligned}$$

and using $|(Ly)(t)| \le ||Ly||_\infty$ we have

$$|y(x)||(Ly)(t)| \le M||Ly||_\infty^2.$$

Now integrating the last inequality on $[x_0, b]$, we get (3.7). □

Theorem 3.7. *Let $p = 1, q = \infty$ and $x \le x_0 \in [a,b]$. Then we have*

$$\frac{1}{x_0-a}\int_{a}^{x_0} |y(x)||(Ly)(x)|\, dx \le M||Ly||_\infty^2. \tag{3.8}$$

Proof. As y has representation

$$y(x) = \int_{x}^{x_0} H(x,t)(Ly)(t)dt.$$

From this we get

$$\begin{aligned} |y(x)| &\le \left(\int_{x_0}^{x} |H(x,t)|dt\right)||Ly||_\infty \\ &\le M||Ly||_\infty \end{aligned}$$

and using $|(Ly)(t)| \le ||Ly||_\infty$ we have

$$|y(x)||(Ly)(t)| \le M||Ly||_\infty^2.$$

Now integrating the last inequality on $[a, x_0]$, we get (3.8). □

Corollary 3.8. *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\int_{x_0}^{x} |y(x)|^q |(Ly)(x)|^q \, dx \leq \frac{M^q}{2} \left(\int_{x_0}^{x} |(Ly)(t)|^q dt \right)^2 . \tag{3.9}$$

Proof. By setting $\phi(t) = t^{2q}$ and $b = x \geq x_0$ in Theorem 3.1 one can get (3.9). □

Corollary 3.9. *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\int_{x}^{x_0} |y(x)|^q |(Ly)(x)|^q \, dx \leq \frac{M^q}{2} \left(\int_{x}^{x_0} |(Ly)(t)|^q dt \right)^2 . \tag{3.10}$$

Proof. By setting $\phi(t) = t^{2q}$ and $a = x \leq x_0$ in Theorem 3.2 one can get (3.10). □

Corollary 3.10. *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$\left| \int_{x_0}^{x} |y(x)|^q |(Ly)(x)|^q \, dx \right| \leq \frac{M^q}{2} \left(\int_{x_0}^{x} |(Ly)(t)|^q dt \right)^2 . \tag{3.11}$$

Proof. From (3.9) and (3.10) one can easily get (3.11). □

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