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Nullity Distributions Associated with Hashiguchi Connection

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Abstract. In this paper, we employ the Klein-Grifone formalism to study the nullity distributions associated with the curvature tensors of the Hashiguchi connection. We establish several important results regarding the nullity distribution $\mathcal{N}_{R^\star}$ of the h-curvature tensor $\overset{\star}{R}$, demonstrating that $\mathcal{N}_{R^\star}$ is completely integrable and that its corresponding foliation consists of auto-parallel leaves. An illustrative example shows that the nullity distribution $\mathcal{N}_{P^\star}$, associated with the hv-curvature tensor $\overset{\star}{P}$, is not generally completely integrable. Furthermore, we determine necessary and sufficient conditions under which $\mathcal{N}_{P^\star}$ becomes completely integrable.

1. Introduction

In 1952, Chern and Kuiper [5] introduced a distribution on a Riemannian manifold M by assigning to each point $x \in M$ the subspace

$$\mathcal{N}_R(x) = \{U \in T_xM \mid R(U,V) = 0, \text{ for all } V \in T_xM\},$$

where R denotes the curvature tensor of the Riemannian connection. This subspace is referred as the *nullity space* at x, and the associated distribution across M is called the *nullity distribution* $\mathcal{N}$. The dimension $\mu_x = \dim \mathcal{N}_x$ is called the *index of nullity* at x. Chern and Kuiper showed that if μ_x is constant in a neighborhood of x, then $\mathcal{N}$ forms a completely integrable distribution, and the leaves of the corresponding foliation are flat submanifolds. Later, Maltz [11] extended this result by proving that these leaves are also totally geodesic.

In 1972, Akbar-Zadeh [1,2] generalized this concept to Finsler geometry by studying the nullity distribution of the classical curvature tensor associated with the Cartan connection using the

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pullback formalism. More recently, Bidabad and Refie-Rad [3] investigated a broader concept known as the *k-nullity distribution* within the same framework.

In a parallel development, in 1982, N. L. Youssef [13] studied the nullity distribution of various curvature tensors within the Klein-Grifone approach to Finsler geometry. Specifically, he established the nullity distribution corresponding to the curvature of the Barthel connection, and in [14], he addressed the nullity distribution of the Berwald connection.

Within the framework of the Klein-Grifone (KG) approach to Finsler geometry, Youssef and his collaborators, in [17, 19], conducted a detailed investigation of the nullity distributions corresponding to the curvature tensors of both the Cartan and Chern connections. Their work focused on analyzing the geometric and integrability properties of these distributions, shedding light on the underlying structure of the Finsler manifold as governed by these classical connections.

In the present work, we investigate the nullity distributions associated with the curvature tensors, specifically the h- and hv-curvatures of the Hashiguchi connection within the framework of the Klein-Grifone approach to Finsler geometry. Unlike previous studies, our focus lies on the intrinsic geometric nature of these nullity distributions and their implications for the broader geometry of the Finsler manifold. Moreover, since the v-curvature of Hashiguchi connection coincide with the v-curvature of Cartan connection which was studied in [19], so we do not consider its distribution here.

This paper is organized as follows. Section 2 introduces the necessary preliminaries, including a concise overview of the Klein-Grifone formalism in Finsler geometry, along with essential tools from the FrÃűlicher-Nijenhuis theory of vector-valued forms and derivations, which are instrumental in our analysis. Section 3 is devoted to a detailed discussion of the fundamental properties and key identities of curvature tensors, with a particular emphasis on the structure of fundamental linear connections in the KG-setting. Section 4 focuses on the nullity distribution $\mathcal{N}_{R^\star}$ associated with the h-curvature tensor $\overset{\star}{R}$. We demonstrate that $\mathcal{N}_{R^\star}$ is contained within the nullity distribution of the curvature tensor of the Barthel connection. Moreover, we prove that $\mathcal{N}_{R^\star}$ is completely integrable, and that the leaves of the corresponding foliation are auto-parallel and, consequently, totally geodesic. Section 5 establishes the nullity distributions corresponding to the hv-curvature tensor $\overset{\star}{P}$. Using explicit examples, we show that these distributions are not generally completely integrable. However, we also establish necessary and sufficient conditions under which complete integrability can be achieved.

2. Preliminaries: The Klein-Grifone Approach

In this section, we briefly outline the fundamental notions of the Klein-Grifone approach to global Finsler geometry. For more comprehensive treatments, we refer the reader to [7–9]. All geometric objects considered are assumed to be of class C^∞.

Throughout this paper, we adopt the following notations: M denotes a smooth real manifold of dimension n; $\mathfrak{F}(M)$ is the $\mathbb{R}$-algebra of smooth real-valued functions on M; $\mathfrak{X}(M)$ represents the $\mathfrak{F}(M)$-module of smooth vector fields on M; $\pi_M : TM \to M$ is the tangent bundle of M, while $\pi : \mathcal{T}M \to M$ refers to the subbundle of nonzero tangent vectors on M; $V(TM)$ designates the vertical subbundle of the double tangent bundle TTM; i_U denotes the interior product with respect to a vector field $U \in \mathfrak{X}(M)$; df stands for the exterior derivative of a function f; $d_L := [i_L, d]$ is the exterior derivative associated with a vector form L; and $\mathcal{L}_U$ denotes the Lie derivative with respect to $U \in \mathfrak{X}(M)$.

We have the short exact sequence of vector bundles:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms are defined as follows:

$$\rho := (\pi_{\mathcal{T}M}, d\pi), \quad \gamma(u,v) := j_u(v),$$

with j_u denoting the natural isomorphism $j_u : T_{\pi_M(v)}M \to T_u(T_{\pi_M(v)}M)$. The vector 1-form $J := \gamma \circ \rho$ on TM is called the *natural almost tangent structure*. The vertical vector field C on TM given by $C := \gamma \circ \overline{\eta}$, where $\overline{\eta}(u) = (u,u)$, is known as the *canonical* or *Liouville vector field*.

We will make use of the FrÃűlicherâĂŞNijenhuis bracket in specific cases [6]:

If L is a vector ℓ-form and $U \in \mathfrak{X}(M)$, then for all $V_1, \ldots, V_\ell \in \mathfrak{X}(M)$,

$$[U,L](V_1,\ldots,V_\ell) = [U, L(V_1,\ldots,V_\ell)] - \sum_{i=1}^{\ell} L(V_1,\ldots,[U,V_i],\ldots,V_\ell).$$

In particular, if L is a vector 1-form,

$$[U,L]V = [U,LV] - L[U,V].$$

If K, L are vector 1-forms, then for all $U, V \in \mathfrak{X}(M)$,

$$\begin{aligned}[K,L](U,V) = {} & [KU,LV] + [LU,KV] + KL[U,V] + LK[U,V] \\ & - K[LU,V] - K[U,LV] - L[KU,V] - L[U,KV].\end{aligned}$$

The Nijenhuis torsion N_K of a vector 1-form K is defined by

$$N_K := \frac{1}{2}[K,K](U,V) = [KU,KV] + K^2[U,V] - K[KU,V] - K[U,KV]. \tag{2.1}$$

The natural almost tangent structure J satisfies:

$$J^2 = 0, \quad [J,J] = 0, \quad [C,J] = -J, \quad \mathrm{Im}(J) = \ker(J) = V(TM). \tag{2.2}$$

A scalar p-form ω is said to be *semi-basic* if $i_{JU}\omega = 0$ for all $U \in \mathfrak{X}(TM)$. A vector ℓ-form L is semi-basic if $JL = 0$ and $i_{JU}L = 0$ for all $U \in \mathfrak{X}(TM)$.

A scalar p-form ω is *homogeneous of degree r* if $\mathcal{L}_C\omega = r\omega$. The structure J is homogeneous of degree 0, i.e., J is h(0). A vector ℓ-form L is *homogeneous of degree r*, written h(r), if $[C,L] = (r-1)L$.

A *semispray* on M is a vector field S on TM, of class C^∞ on $\mathcal{T}M$ and C^1 on TM, such that $JS = C$. A semispray S satisfying $[C,S] = S$ is called a *spray*.

A *nonlinear connection* on M is a vector 1-form Γ on TM, of class C^∞ on $\mathcal{T}M$ and C^0 on TM, such that

$$J\Gamma = J, \qquad \Gamma J = -J.$$

The associated vertical and horizontal projectors v and h are defined by

$$v := \frac{1}{2}(I - \Gamma), \qquad h := \frac{1}{2}(I + \Gamma).$$

The decomposition $TTM = V(TM) \oplus H(TM)$ naturally arises, where the vertical subbundle is given by $V(TM) := \mathrm{Im}(v) = \ker(h)$, and the horizontal subbundle by $H(TM) := \mathrm{Im}(h) = \ker(v)$, both determined by the nonlinear connection Γ. For any vector field $U \in \mathfrak{X}(TM)$, its horizontal and vertical projections are denoted hU and vU, respectively. These projections satisfy the relations:

$$Jv = 0, \quad vJ = J, \quad Jh = J, \quad hJ = 0.$$

A nonlinear connection Γ is called *homogeneous* if it commutes with the canonical vector field C, i.e., $[C, \Gamma] = 0$. Such a connection determines a semispray S by the condition $S = hS'$, for any semispray S'. When Γ is homogeneous, the resulting semispray becomes a spray.

The *torsion* associated with a nonlinear connection Γ is defined by the vector-valued 2-form

$$t := \frac{1}{2}[J, \Gamma],$$

and the corresponding *curvature* is expressed as

$$\mathfrak{R} := -\frac{1}{2}[h, h].$$

An almost complex structure F on TM is associated with Γ via:

$$F^2 = -I, \quad FJ = h, \quad Fh = -J,$$

and defines an isomorphism on each fiber $T_z(TM)$ for $z \in TM$.

Definition 2.1 ([9]). *A* Finsler space *of dimension n is a pair (M, E), where M is a smooth n-dimensional manifold, and $E : TM \to \mathbb{R}$ is a function, referred as the energy, which satisfies the following conditions:*

- **(a):** *$E(u) > 0$ for every $u \in \mathcal{T}M$ and vanishes at the zero section, i.e., $E(0) = 0$,*
- **(b):** *E is smooth on $\mathcal{T}M$ and continuously differentiable on TM,*
- **(c):** *E is positively homogeneous of degree 2, meaning that $\mathcal{L}_C E = 2E$,*
- **(d):** *The 2-form $\Omega := dd_J E$, known as the* fundamental form, *is of maximal rank.*

It is worth mentioning that if the energy function E satisfies the aforementioned conditions on a conic subset of $\mathcal{T}M$, then the pair (M, E) is referred as a *conic Finsler manifold*.

Theorem 2.1 ([9]). *Given a Finsler space (M, E), there exists a vector field $S \in \mathfrak{U}(TM)$ satisfying $i_S\Omega = -dE$. This vector field S defines a spray and is called the* canonical spray *associated with the Finsler structure.*

Theorem 2.2 ([9]). *For a Finsler space (M, E), there is a unique nonlinear connection that is conservative (i.e., $d_h E = 0$), homogeneous, and torsion-free. This connection is determined by:*

$$\Gamma = [J, S],$$

where S denotes the canonical spray. It is commonly referred as the canonical connection, Barthel connection, *or the* Cartan nonlinear connection *corresponding to (M, E).*

It is worth mentioning that the semispray related to the Barthel connection coincides with the canonical spray.

3. Fundamental Linear Connections

This section provides a concise overview of the key concepts and properties of the Berwald $\overset{\circ}{D}$, Cartan D, and Hashiguchi connections $\overset{\star}{D}$, which serve as foundational tools for the developments in this work. For more comprehensive discussions, the reader may consult [8, 14, 15, 18].

Theorem 3.1 ([8]). *Let (M, E) be a Finsler space. Then, there exists a unique linear connection $\overset{\circ}{D}$ on TM satisfying the following conditions:*

(a) $\overset{\circ}{D} J = 0.$

(b) $\overset{\circ}{D} C = v.$

(c) $\overset{\circ}{D} \Gamma = 0 \quad (\Longleftrightarrow \overset{\circ}{D} h = \overset{\circ}{D} v = 0).$

(d) $\overset{\circ}{D}_{JU} JV = J[JU, V].$

(e) $\overset{\circ}{T}(JU, V) = 0,$

where h and v are the horizontal and vertical projectors associated with the Barthel connection $\Gamma = [J, S]$, and $\overset{\circ}{T}$ is the classical torsion of $\overset{\circ}{D}$. This connection is known as the Berwald connection.

The Berwald connection $\overset{\circ}{D}$ is explicitly given by:

$$\begin{cases} \overset{\circ}{D}_{JU} JV &= J[JU, V], \\ \overset{\circ}{D}_{hU} JV &= v[hU, JV], \\ \overset{\circ}{D} F &= 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *The torsion of the Berwald connection satisfies*

$$\overset{\circ}{T}(hU, hV) = \mathfrak{R}(U, V),$$

where $\mathfrak{R}$ is the curvature tensor of the Barthel connection.

Let (M, E) be a Finsler space and set $\Omega := dd_J E$. Define a map $\overline{g}$ on $V(TM)$ by

$$\overline{g}(JU, JV) := \Omega(JU, V), \quad \text{for all } U, V \in T(TM).$$

This induces a metric g on $T(TM)$ given by:

$$g(U, V) = \overline{g}(JU, JV) + \overline{g}(vU, vV) = \Omega(U, FV). \tag{3.2}$$

Theorem 3.2 ([8]). *Let (M, E) be a Finsler space. Then, there exists a unique linear connection D on TM satisfying:*

(a) $DJ = 0$.
(b) $DC = v$.
(c) $D\Gamma = 0$ ($\Longleftrightarrow Dh = Dv = 0$).
(d) $Dg = 0$.
(e) $T(JU, JV) = 0$.
(f) $JT(hU, hV) = 0$.

This connection is referred as the Cartan connection.

The Cartan connection D is given explicitly by:

$$\begin{cases} D_{JU}JV &= \overset{\circ}{D}_{JU}JV + C(U,V), \\ D_{hU}JV &= \overset{\circ}{D}_{hU}JV + C'(U,V), \\ DF &= 0, \end{cases} \tag{3.3}$$

where C and C' are tensor fields on TM defined by:

$$\Omega(C(U,V),Z) = \tfrac{1}{2}(\mathcal{L}_{JU}(J^*g))(V,Z), \qquad \Omega(C'(U,V),Z) = \tfrac{1}{2}(\mathcal{L}_{hU}g)(JV,JZ),$$

with $(J^*g)(V,Z) := g(JV,JZ)$. The tensors C and C', known as the first and second Cartan tensors, are symmetric, semi-basic, and satisfy:

$$C(U,S) = C'(U,S) = 0. \tag{3.4}$$

We present the following lemmas.

Lemma 3.2. *The (h)h-torsion $T(hU,hV)$ and (h)v-torsion $T(hU,JV)$ of Cartan connection are given respectively by*

$$T(hU,hV) = \mathfrak{R}(U,V), \quad T(hU,JV) = (C' - FC)(U,V),$$

where $\mathfrak{R}$ is the curvature of the Barthel connection.

Lemma 3.3. *The h-curvature R, hv-curvature P and v-curvature Q of Cartan connection are given respectively by*

(a): $R(U,V)Z = \overset{\circ}{R}(U,V)Z + (D_{hU}C')(V,Z) - (D_{hV}C')(U,Z) + C'(FC'(U,Z),V) - C'(FC'(V,Z),U) + C(F\mathfrak{R}(U,V),Z)$.

(b): $P(U,V)Z = \overset{\circ}{P}(U,V)Z + (D_{hU}C)(V,Z) - (D_{JV}C')(U,Z) + C(FC'(U,Z),V) + C(FC'(U,V),Z) - C'(FC(V,Z),U) - C'(FC(U,V),Z)$.

(c): $Q(U,V)Z = C(FC(U,Z),V) - C(FC(V,Z),U)$,

where $\overset{\circ}{R}$ and $\overset{\circ}{P}$ are respectively the h-curvature and hv-curvature of Berwald connection.

Lemma 3.4. *For Cartan connection, the following properties hold:*

(a): $R(U,V)S = \mathfrak{R}(U,V)$.

(b): $P(U,V)S = C'(U,V)$.

(c): $P(S,U)V = P(U,S)V = 0$.

(d): $Q(S,U)V = Q(U,S)V = Q(U,V)S = 0$.

Lemma 3.5. *A semi spray S satisfies the following relation*

$$J[JU,S] = JU, \text{ for all } U \in \mathfrak{X}(TM).$$

Lemma 3.6. *For a homogenous connection* $\mathbf{\Gamma}$*, its horizontal projector* $\boldsymbol{h}$ *satisfies*

$$[C, \boldsymbol{h}U] = \boldsymbol{h}[C, U], \ \textit{for all} \ \ U \in \mathfrak{X}(TM).$$

Proof. Since $\mathbf{\Gamma}$ is homogenous, then $\mathbf{h}$ is h(1). Thus, $[C, \mathbf{h}] = 0$ and hence

$$\begin{aligned} 0 &= [C, \mathbf{h}]U \\ &= [C, \mathbf{h}U] - \mathbf{h}[C, U]. \end{aligned}$$

□

Theorem 3.3. *The Hashiguchi connection* $\overset{\star}{D}$ *is uniquely determined by the following relations:*

- **(a):** $\overset{\star}{D}_{JU} JV = J[JU, V] + C(U, V)$.
- **(b):** $\overset{\star}{D}_{hU} JV = v[hU, JV]$.
- **(c):** $\overset{\star}{D} F = 0$.

Lemma 3.7. *The (h)h-torsion* $\overset{\star}{T}(hU, hV)$ *and (h)v-torsion* $\overset{\star}{T}(hU, JV)$ *are given by:*

- **(a):** $\overset{\star}{T}(hU, hV) = \mathfrak{R}(U, V)$.
- **(b):** $\overset{\star}{T}(hU, JV) = -FC(U, V)$.

Lemma 3.8. *The h-curvature* $\overset{\star}{R}$*, mixed curvature* $\overset{\star}{P}$ *and the v-curvature* $\overset{\star}{Q}$ *of the Hashiguchi connection, are given by*

- **(a):** $\overset{\star}{R}(U, V)Z = \overset{\circ}{R}(U, V)Z + C(F\mathfrak{R}(U, V), Z)$.
- **(b):** $\overset{\star}{P}(U, V)Z = \overset{\circ}{P}(U, V)Z + (\overset{\star}{D}_{hU} C)(V, Z)$.
- **(c):** $\overset{\star}{Q}(U, V)Z = Q(U, V)Z = C(FC(U, Z), V) - C(FC(V, Z), U)$.

Lemma 3.9. *The h-curvature* $\overset{\star}{R}$ *and mixed curvature* $\overset{\star}{P}$ *of the Hashiguchi connection have the following properties:*

- **(a):** $\overset{\star}{R}(U, V)S = \mathfrak{R}(U, V)$.
- **(b):** $\overset{\star}{P}(U, V)S = \overset{\star}{P}(U, S)V = \overset{\star}{P}(S, U)V = 0$.
- **(c):** $\overset{\star}{Q}(U, V)S = \overset{\star}{Q}(U, S)V = \overset{\star}{Q}(S, U)V = 0$.

Lemma 3.10. *The h-curvature* $\overset{\star}{R}$*, the mixed curvature* $\overset{\star}{P}$ *and the v-curvature* $\overset{\star}{Q}$ *of the Hashiguchi connection satisfy the following properties:*

$$\overset{\star}{D}_C \overset{\star}{R} = 0, \qquad \overset{\star}{D}_C \overset{\star}{P} = -\overset{\star}{P}, \qquad \overset{\star}{D}_C \overset{\star}{Q} = -2\overset{\star}{Q}.$$

Lemma 3.11. *The Bainchi identities for Hashiguchi connection are given by:*

- **(a):** $\mathfrak{S}_{U,V,Z}\{\overset{\star}{R}(U, V)Z\} = \mathfrak{S}_{U,V,Z}\{C(F\mathfrak{R}(U, V), Z)\}$.
- **(b):** $\mathfrak{S}_{U,V,Z}\{\overset{\star}{Q}(U, V)Z\} = 0$.
- **(c):** $C(F\mathfrak{R}(U, V), Z) = \mathfrak{R}(FC(U, Z), V) - \mathfrak{R}(FC(V, Z), U)$.
- **(d):** $\mathfrak{S}_{U,V,Z}\{(\overset{\star}{D}_{hU}\mathfrak{R})(V, Z)\} = 0$.
- **(e):** $\mathfrak{S}_{U,V,Z}\{(\overset{\star}{D}_{hU} \overset{\star}{R})(V, Z)\} = \mathfrak{S}_{U,V,Z}\{\overset{\star}{P}(U, F\mathfrak{R}(V, Z))\}$.

(f): $(\overset{\star}{D}_{hU}\overset{\star}{P})(V,Z)-(\overset{\star}{D}_{hV}\overset{\star}{P})(U,Z)+(\overset{\star}{D}_{JZ}\overset{\star}{R})(U,V)=\overset{\star}{R}(FC(V,Z),U)$
$-\overset{\star}{R}(FC(U,Z),V)-\overset{\star}{Q}(F\mathfrak{R}(U,V),Z).$

(g): $(\overset{\star}{D}_{hU}\overset{\star}{Q})(V,Z)-(\overset{\star}{D}_{JV}\overset{\star}{P})(U,Z)+(\overset{\star}{D}_{JZ}\overset{\star}{P})(U,V)=\overset{\star}{P}(FC(U,V),Z)$
$-\overset{\star}{P}(FC(Z,U),V).$

(h): $\mathfrak{S}_{U,V,Z}\{(\overset{\star}{D}_{JU}\overset{\star}{Q})(V,Z)\}=0.$

Lemma 3.12. *The mixed curvature $\overset{\star}{P}$ of the Hashiguchi connection satisfies that*

$$\overset{\star}{P}(U,V)Z=\overset{\star}{P}(U,Z)V.$$

Lemma 3.13. *The hv-curvature $\overset{\star}{P}$ and the v-curvature $\overset{\star}{Q}$ of the Hashiguchi connection satisfy the following properties:*

$$(\overset{\star}{D}_{hU}\overset{\star}{P})(V,S)=0,\qquad(\overset{\star}{D}_{JU}\overset{\star}{P})(V,S)=\overset{\star}{P}(V,U),$$
$$(\overset{\star}{D}_{hU}\overset{\star}{Q})(V,S)=0,\qquad(\overset{\star}{D}_{JU}\overset{\star}{Q})(V,S)=\overset{\star}{Q}(V,U).$$

Lemma 3.14. *For all $U,V\in\mathfrak{X}(TM)$, we have*

(a): $[JU,JV]=J(\overset{\star}{D}_{JU}V-\overset{\star}{D}_{JV}U).$

(b): $[hU,JV]=J(\overset{\star}{D}_{hU})V-h(\overset{\star}{D}_{JV}U)+FC(U,V).$

(c): $[hU,hV]=h(\overset{\star}{D}_{hU}V-\overset{\star}{D}_{hV}U)-\mathfrak{R}(U,V).$

4. Nullity Distribution of Hashiguchi h-Curvature

We are now prepared to investigate the nullity distribution of the Hashiguchi connection. Our first focus is the nullity distribution of its h-curvature tensor. It is worth mentioning that Youssef has previously studied the nullity distributions of the Barthel and Berwald connections [13, 14]. Moreover, Youssef et al. have studied the nullity distributions of the Cartan and Chern connections [17, 19].

Definition 4.1. *Let $\overset{\star}{R}$ denote the h-curvature tensor of the Hashiguchi connection. The* nullity space *of $\overset{\star}{R}$ at a point $z\in TM$ is the subspace of $H_z(TM)$ defined by*

$$\mathcal{N}_{R^\star}(z):=\{U\in H_z(TM):\ \overset{\star}{R}(U,V)=0,\quad \textit{for all}\quad V\in T_z(TM)\}.$$

The dimension of $\mathcal{N}_{R^\star}(z)$, denoted by $\mu_{R^\star}(z)$, is called the index of nullity *of $\overset{\star}{R}$ at z.*

If this index is constant on an open set, then the assignment

$$z\mapsto\mathcal{N}_{R^\star}(z)$$

defines a distribution $\mathcal{N}_{R^\star}$ of dimension $\mu_{R^\star}$, called the nullity distribution *of $\overset{\star}{R}$. Any vector field tangent to this distribution is called a* nullity vector field*. We denote by $\Gamma(\mathcal{N}_{R^\star})$ the $C^\infty(TM)$-module of the nullity vector fields.*

Proposition 4.1. *The nullity distribution $\mathcal{N}_{R^\star}$ satisfies the following properties:*

(a) $\mathcal{N}_{R^\star}\neq\emptyset$.

(b) $\mathcal{N}_{R^\star} \subseteq \mathcal{N}_{\Re}$, *where* $\mathcal{N}_{\Re}$ *denotes the nullity distribution of the curvature* $\Re$.

(c) *If* $U \in \Gamma(\mathcal{N}_{R^\star})$, *then the Lie bracket* $[C, U]$ *also belongs to* $\Gamma(\mathcal{N}_{R^\star})$ *and consequently to* $\mathcal{N}_{\Re}$.

Proof. **(b)** Let $U \in \Gamma(\mathcal{N}_{R^\star})$. Then, for all $V, Z \in \mathfrak{X}(TM)$,

$$\overset{\star}{R}(U, V)Z = 0.$$

In particular, taking $Z = S$ (the canonical section), we have

$$\overset{\star}{R}(U, V)S = 0, \quad \text{for all} \quad V \in \mathfrak{X}(TM)$$

which implies

$$\Re(U, V) = 0, \quad \text{for all} \quad V \in \mathfrak{X}(TM).$$

Therefore, $U \in \Gamma(\mathcal{N}_{R^\star})$.

(c) Let $U \in \Gamma(\mathcal{N}_{R^\star})$. By Lemma 3.10,

$$(\overset{\star}{D}_C \overset{\star}{R})(U, V) = 0,$$

which yields

$$\overset{\star}{R}(\overset{\star}{D}_C U, V) = 0.$$

Using Theorem 3.3, it follows that

$$\overset{\star}{R}([C, U], V) = 0.$$

Since h is homogeneous of degree 1, we have $[C, U] = [C, hU] = h[C, U]$, so $[C, U]$ is horizontal, and thus

$$[C, U] \in \Gamma(\mathcal{N}_{R^\star}) \subseteq \Gamma(\mathcal{N}_{\Re}).$$

□

Definition 4.2. *The* conullity space *of the h-curvature tensor at z, denoted by* $\mathcal{N}^{\perp}_{R^\star}(z)$, *is the orthogonal complemmaent of* $\mathcal{N}_{R^\star}(z)$ *in* $H_z(TM)$ *with respect to the metric g.*

Proposition 4.2. *For each point* $z \in TM$, *either* $\mu_{R^\star}(z) = n$ *or* $\mu_{R^\star}(z) \leq n - 2$.

Proof. Suppose $\mu_{R^\star}(z) \neq n$. Then there exists a nonzero vector $U \notin \mathcal{N}_{R^\star}(z)$. Hence, there exists $V \in H_z(TM)$ such that

$$\overset{\star}{R}(U, V) \neq 0,$$

and so

$$\overset{\star}{R}(V, U) \neq 0.$$

Thus, both $U, V \notin \Gamma(\mathcal{N}_{R^\star})$, which implies $U, V \in \Gamma(\mathcal{N}^{\perp}_{R^\star})$. Consequently,

$$\dim \mathcal{N}^{\perp}_{R^\star} \geq 2,$$

and therefore

$$\mu_{R^\star}(z) \leq n - 2.$$

□

Theorem 4.1. *The nullity distributions of the h-curvature tensors of the Hashiguchi connection $\mathcal{N}_{R^\star}$ and the Berwald connection $\mathcal{N}_{R^\circ}$ coincide:*

$$\mathcal{N}_{R^\star} = \mathcal{N}_{R^\circ}.$$

Proof. Let $U \in \Gamma(\mathcal{N}_{R^\star})$. By Proposition 3.8 (a) and Proposition 4.1 (b), we have $U \in \Gamma(\mathcal{N}_{R^\circ})$. Hence,

$$\mathcal{N}_{R^\star} \subseteq \mathcal{N}_{R^\circ}.$$

Conversely, if $U \in \Gamma(\mathcal{N}_{R^\circ})$, then by Proposition 3.8 (a) and the inclusion $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_{\mathfrak{R}}$ [14], we get

$$U \in \Gamma(\mathcal{N}_{R^\star}).$$

Thus,

$$\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_{R^\star},$$

and the two nullity distributions coincide. □

As an immediate consequence of this theorem and the known complete integrability of $\mathcal{N}_{R^\circ}$ [14], we have the following corollary.

Corollary 4.1. *If the index of nullity $\mu_{R^\star}$ is constant on an open subset $U \subseteq TM$, then the nullity distribution*

$$z \mapsto \mathcal{N}_{R^\star}(z)$$

is completely integrable.

Remark 4.1. *It is worth mentioning that the nullity distribution $\mathcal{N}$_$\mathfrak{R}$ associated with the curvature of the Barthel connection is also completely integrable, as established by Youssef [13].*

When the index of nullity $\mu_{R^\star}(z)$ is constant, Frobenius' theorem guarantees the existence of a foliation of TM by maximal connected submanifolds (called *leaves*) whose tangent spaces at each point coincide with $\mathcal{N}_{R^\star}(z)$. We refer to this foliation as the *nullity foliation*.

Theorem 4.2. *The leaves of the nullity foliations $\mathcal{N}_{R^\star}$ and $\mathcal{N}_{\mathfrak{R}}$ are auto-parallel submanifolds with respect to the Hashiguchi connection.*

Proof. To prove that $\mathcal{N}_{R^\star}$ is auto-parallel with respect to $\overset{\star}{D}$, we must show that if $U, V \in \Gamma(\mathcal{N}_{R^\star})$, then

$$\overset{\star}{D}_U V \in \Gamma(\mathcal{N}_{R^\star}).$$

Let $U, V \in \Gamma(\mathcal{N}_{R^\star})$. Since $U, V \in \Gamma(\mathcal{N}_{\mathfrak{R}})$ and are horizontal, and since $\overset{\star}{D} h = 0$, we have

$$\overset{\star}{D}_U (hV) = h \overset{\star}{D}_U V,$$

which implies $\overset{\star}{D}_U V$ is horizontal.

Using Lemma 3.11 (e), the first Bianchi identity yields

$$\mathfrak{S}_{U,V,Z}\{(\overset{\star}{D}_U \overset{\star}{R})(V,Z)\} = 0,$$

and consequently,

$$\mathfrak{S}_{U,V,Z}\{ \overset{\star}{R}(\overset{\star}{D}_U V, Z)\} = 0.$$

Since $U, V, Z \in \Gamma(\mathcal{N}_{R^\star})$, this implies

$$\overset{\star}{R}(\overset{\star}{D}_U V, Z) = 0 \quad \text{for all } \ Z \in \mathfrak{X}(TM),$$

and thus $\overset{\star}{D}_U V \in \Gamma(\mathcal{N}_{R^\star})$.

Similarly, applying Lemma 3.11 (d) for $\mathfrak{R}$, we conclude that if $U, V \in \Gamma(\mathcal{N}_{\mathfrak{R}})$, then $\overset{\star}{D}_U V \in \Gamma(\mathcal{N}_{\mathfrak{R}})$. □

In Riemannian geometry, it is well established that the concepts of auto-parallel and totally geodesic submanifolds are equivalent [10]. While this equivalence does not generally extend to broader settings, it is known that every auto-parallel submanifold is necessarily totally geodesic [4]. Based on this, we derive the following corollary.

Corollary 4.2. *The nullity foliations $\mathcal{N}_{\mathfrak{R}}$ and $\mathcal{N}_{R^\star}$ form totally geodesic submanifolds with respect to the Hashiguchi connection.*

5. Nullity Distribution of Hashiguchi *hv*-Curvature

In this section, we study the nullity distribution associated with the mixed curvature tensor $P^\star$ of the Hashiguchi connection $\overset{\star}{D}$. It is shown that the distribution $\mathcal{N}_{P^\star}$ is, in general, not completely integrable. To achieve integrability, a particular condition is introduced. Additionally, we identify a class of Finsler spaces in which this condition is automatically satisfied.

Definition 5.1. *Let $\overset{\star}{P}$ represent the mixed curvature tensor of the Hashiguchi connection $\overset{\star}{D}$. The* nullity space *of $\overset{\star}{P}$ at a point $z \in TM$ is defined as the subspace of $H_z(TM)$ given by*

$$\mathcal{N}_{P^\star}(z) := \left\{ U \in H_z(TM) : \ \overset{\star}{P}(U, V) = 0, \quad \textit{for all } V \in T_z(TM) \right\}.$$

The quantity $\mu_{P^\star}(z) := \dim \mathcal{N}_{P^\star}(z)$ is referred as the index of nullity *of $\overset{\star}{P}$ at z.*

Proposition 5.1. *The nullity distribution $\mathcal{N}_{P^\star}$ satisfies the following properties:*

(a) $\mathcal{N}_{P^\star} \neq \emptyset$.
(b) *The spray S belongs to $\Gamma(\mathcal{N}_{P^\star})$.*
(c) *If $U \in \Gamma(\mathcal{N}_{P^\star})$, then the Lie bracket $[C, U]$ also lies in $\Gamma(\mathcal{N}_{P^\star})$.*

Proof. **(a)** This follows since the zero vector field trivially belongs to $\Gamma(\mathcal{N}_{P^\star})$.

(b) By Proposition 3.13, we have $\overset{\star}{P}(S, U)V = 0$, implying $S \in \Gamma(\mathcal{N}_{P^\star})$.

(c) Suppose $U \in \Gamma(\mathcal{N}_{P^\star})$. Using Proposition 3.10, we obtain

$$(\overset{\star}{D}_C \overset{\star}{P})(U, V) = 0,$$

which implies

$$\overset{\star}{P}(\overset{\star}{D}_C U, V) = 0.$$

Applying Theorem 3.3, it follows that

$$\overset{\star}{P}([C,U], V) = 0.$$

Since h is homogeneous of degree 1, we have $[C,U] = [C,hU] = h[C,U]$, so $[C,U]$ is horizontal. Hence, $[C,U] \in \Gamma(\mathcal{N}_{P^\star})$. □

A Finsler manifold (M,E) is called *Landesberg* if its second Cartan tensor vanishes, i.e. $C' = 0$, equivalently $P = 0$ [12].

Proposition 5.2. *If (M,E) is a Landesberg space, then the index of nullity $\mu_{P^\star}$ attains its maximal value.*

Proof. If (M,E) is Landesberg, then $C' = 0$. By (3.3) and Theorem 3.3, the horizontal covariant derivatives of the Cartan and Hashiguchi connections coincide. Thus, by Propositions 3.3 and 3.8, the mixed curvature vanishes. Consequently, $\mu_{P^\star}$ reaches its maximal value. □

In general, the nullity distribution $\mathcal{N}_{P^\star}$ is not completely integrable, as the following example illustrates. Detailed computations corresponding to this example are presented in the supplementary materials, including a PDF document and Maple worksheets generated using the Finsler package [16], available at: https://github.com/salahelgendi/Example1_ND_Hashiguchi_connection.git.

Example 5.1. *Let $\mathcal{U} = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3\} \subset TM$, where $M := \mathbb{R}^3$. Define E on $\mathcal{U}$ by*

$$E(x,y) := \sqrt{(y^1)^4 + e^{-x^2 x^3} y^3 (y^2)^3}.$$

Locally, the equation $\overset{\star}{P}(U,V)Z = 0$ for all $V, Z \in H(TM)$ can be expressed as

$$X^j \, \overset{\star}{P}{}^h_{ijk} = 0.$$

For this conic Finsler space, $\overset{\star}{P}{}^h_{ijk} = 0$. Also, the non-zero components R^h_{ij} are

$$R^2_{23} = -\frac{1}{3} y^2, \quad R^3_{23} = y^3.$$

That is, $\mathcal{N}_{P^\star}$ coincide with the horizontal distribution. Moreover, the solutions of the system

$$X^j \, \overset{\star}{P}{}^h_{ijk} = 0.$$

can be written as

$$X^1 = t_1, \quad X^2 = t_2, \quad X^3 = t_3, \quad t_1, t_3 \in \mathbb{R}.$$

Hence, the nullity vector X is given by

$$X = t_1 h_1 + t_2 h_2 + t_3 h_3,$$

and so $\mu_{P^\star} = 3$. Consider, for example,

$$X = h_2, \quad V = h_3.$$

A direct computation gives

$$[X,V]=[h_2,h_3]=R^i_{23}\frac{\partial}{\partial y_i}=-\frac{1}{3}y^2\frac{\partial}{\partial y_2}+y^3\frac{\partial}{\partial y_3},$$

which is vertical. Therefore, $\mathcal{N}_{P^\star}$ is not completely integrable since $[X,V]$ is not horizontal. □

Nevertheless, we have the following result.

Theorem 5.1. *Suppose the index of nullity $\mu_{P^\star}$ is constant on an open subset $U\subset TM$. Then, the nullity distribution $\mathcal{N}_{P^\star}$ is completely integrable if and only if*

$$\Re(U,V)=0,\quad (\overset{\star}{D}_{JZ}\overset{*}{R})(U,V)=0,\quad \textit{for all }\ U,V\in\mathcal{N}_{P^\star},\quad \textit{for all }\ Z\in\mathfrak{X}(TM).$$

Proof. Let $U,V\in\Gamma(\mathcal{N}_{P^\star})$ with $\Re(U,V)=0$ and $(\overset{\star}{D}_{JZ}\overset{*}{R})(U,V)=0$ for all $Z\in\mathfrak{X}(TM)$. Since $\Re(U,V)=0$, the bracket $[hU,hV]$ is horizontal. Applying Lemma 3.11 **(d)**, we get

$$(\overset{\star}{D}_{hU}\overset{\star}{P})(V,Z)-(\overset{\star}{D}_{hV}\overset{\star}{P})(U,Z)=0\implies \overset{\star}{P}(\overset{\star}{D}_UV-\overset{\star}{D}_VU,Z)=0.$$

Hence,

$$\overset{\star}{P}([U,V]+\Re(U,V),Z)=0\implies \overset{\star}{P}([U,V],Z)=0,$$

which implies $[U,V]\in\Gamma(\mathcal{N}_{P^\star})$. Thus, $\mathcal{N}_{P^\star}$ is completely integrable.

Conversely, assume $\mathcal{N}_{P^\star}$ is completely integrable. For any $U,V\in\Gamma(\mathcal{N}_{P^\star})$, the bracket $[hU,hV]$ is horizontal, hence $\Re(U,V)=0$. Using Lemma 3.11 **(d)** and

$$\overset{\star}{P}([hU,hV],Z)=(\overset{\star}{D}_{hU}\overset{\star}{P})(V,Z)-(\overset{\star}{D}_{hV}\overset{\star}{P})(U,Z)=0,$$

we conclude $(\overset{\star}{D}_{JZ}\overset{\star}{R})(U,V)=0$ for all $U,V\in\Gamma(\mathcal{N}_{P^\star})$ and $Z\in\mathfrak{X}(TM)$. □

Remark 5.1. *The conditions in Theorem 5.1 are not difficult to satisfy. For example, Finsler spaces where the h-curvature of the Chern connection vanishes satisfy these conditions. In such spaces, the nullity distribution $\mathcal{N}_R$ coincides with the entire horizontal distribution $H(TM)$, and hence by Theorem 5.1, $\mathcal{N}_{P^\star}$ is completely integrable.*

Theorem 5.2. *If $\mathcal{N}_{P^\star}$ is completely integrable and satisfies $hF[hU,JV]\in\Gamma(\mathcal{N}_{P^\star})$ for all $U,V\in\Gamma(\mathcal{N}_{P^\star})$, then the nullity foliation $\mathcal{N}_{P^\star}$ is auto-parallel.*

Proof. To show that $\mathcal{N}_{P^\star}$ is auto-parallel with respect to the Chern connection, we need to prove that if $U,V\in\mathcal{N}_{P^\star}$, then $\overset{\star}{D}_UV\in\mathcal{N}_{P^\star}$. Given $U,V\in\mathcal{N}_{P^\star}$ and $Z\in HTM$, and assuming $hF[hU,JV]\in\mathcal{N}_{P^\star}$, Theorem 3.3 and Proposition 5.1 yield

$$\overset{\star}{P}(\overset{\star}{D}_{hU}hV,Z)=\overset{\star}{P}(Fv[hU,JV]+FC'(U,FV),Z)=\overset{\star}{P}(hF[hU,JV],Z)=0.$$

□

Corollary 5.1. *If $\mathcal{N}_{P^\star}$ is completely integrable and for all $U,V\in\Gamma(\mathcal{N}_{P^\star})$, we have $hF[hU,JV]\in\Gamma(\mathcal{N}_{P^\star})$, then the nullity foliation $\mathcal{N}_{P^\star}$ is totally geodesic.*

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