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Laplace Decomposition Shooting Method for Solving Two-Point Boundary Value Problems

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ABSTRACT. This research aims to present an efficient computational method for finding approximate solutions to the boundary-value problems (BVPs) of ordinary differential equations (ODEs), specifically, the class of two-point BVPs. This method is based on the combination of the shooting method with the mixture of the Laplace transform and the Adomian decomposition method. In addition, the study examined several BVPs in both the linear and nonlinear cases of the second- and third-order ODEs. For validating the suggested method, the acquired results are contrasted with the actual solutions and those of the other approximation methods. Moreover, the convergence of the proposed method to the actual analytical solutions has been noted to be very high. Additionally, this method provides the most accurate numerical results for BVPs per this study's findings.

1. Introduction

Two-point boundary value problems (BVPs) arise in various science and technological fields, including electrical engineering, optimization theory, theoretical physics, applied mathematics and elasticity to state but a few. In this regard, as not all of these BVPs can be analyzed analytically, many scientists have in the past and present times proposed various approximation and numerical methods, including the shooting method [1], which is one of the most widely known and used methods for this class of equations. Moreover, shooting methods are characterized by different pluses, including a reduced computational size of the system, rapid

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convergence, and also needing less computational time to attain optimal approximate solutions for dissimilar nonlinear BVPs to mention a few. In fact, these computational advantages of the shooting method are what are geared toward the successful combination of the shooting method with the Laplace Adomian decomposition method [2]. Besides, the Laplace Adomian decomposition method is used to solve a range of applications, including the regularized long-wave (RLW) evolution equation [3], and certain forms of the Korteweg-de Vries (KdV) equations [4]. In addition, the approach has been successfully applied to solve diverse nonlinear integral equations, including the mixture of the Volterra integral equations with differential equations [5].

However, the current manuscript attempts to develop a symbolic algorithm to find the approximate computational solution of the generalized nth-order ODEs of the following form

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right), \qquad 0 \le x \le b, \tag{1}$$

amidst the presence of the following two-point boundary data

$$y^{(i)}(0) = \alpha_i, \quad y(b) = \beta, \qquad i = 0,1 \dots, n-2, \tag{2}$$

by coupling the shooting method [1] with the Laplace Adomian decomposition method [2] to obtain an efficient method called the Laplace Decomposition Shooting Method (LDSM). Indeed, the two beseeched approaches have been exhaustively used in the literature to analyze various classes of BVP; see [6-7] for various considerations using the Laplace Adomian decomposition method, and [8-15] for some documentations on the applications of shooting method. The LDSM basically intends to approximate the solution of linear and nonlinear BVPs of the second- and third-order, where α_i, β are given real constants and $n = 2$ or 3. In the same vein, the effectiveness of the present method will be evaluated using actual and other approximate solutions acquired using different techniques [16-34], and a conclusion regarding the efficacy of the devised method shall be reported at the end.

2. General description of the Laplace Adomian decomposition method

The Laplace Adomian decomposition method (LADM) is a blend of the Laplace integral transform with the Adomian decomposition method. More precisely, this method is an approximate or rather a semi-analytical method for solving linear/nonlinear initial value problems [6-7]. Now, to demonstrate how this method works, let us consider an nth-order nonlinear ODE below

$$Ly + Ry + Ny = r(x), \tag{3}$$

together with the following initial conditions

$$y^{(i)}(0) = \alpha_i, \quad y^{(n-1)}(0) = t, \; i = 0,1 \dots, n-2, \tag{4}$$

where L represents the highest-order derivatives of the second- or third-order with the initial conditions (4), Ry is a linear differential operator and Ny denotes the nonlinear differential operator both have order less than 2 or 3, respectively and $r(x)$ is a given source function, while

α_i and t are given constants. Therefore, the technique then goes by applying the Laplace integral transform on both sides of (3) to get

$$\mathcal{L}[Ly] + \mathcal{L}[Ry] + \mathcal{L}[Ny] = \mathcal{L}[r(x)],$$

or further when explicitly expressing $\mathcal{L}[Ly]$ via the Laplace transform formula the following

$$s^n\mathcal{L}[y] - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0) + \mathcal{L}[Ry] + \mathcal{L}[Ny] = \mathcal{L}[r(x)]. \tag{5}$$

Next, on using the initial conditions (4) in (5), we have

$$s^n\mathcal{L}[y] - s^{n-1}\alpha_0 - \cdots - s\alpha_{n-2} - t + \mathcal{L}[Ry] + \mathcal{L}[Ny] = \mathcal{L}[r(x)],$$

or

$$\mathcal{L}[y] = \frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{t}{s^n} + \frac{1}{s^n}\mathcal{L}[r(x)] - \frac{1}{s^n}\mathcal{L}[Ry] - \frac{1}{s^n}\mathcal{L}[Ny]. \tag{6}$$

In addition, LADM proceeds to decompose the solution $y(x)$ via an infinite series representation of the following form

$$y(x) = \sum_{m=0}^{\infty} y_m(x), \tag{7}$$

while the nonlinear term Ny is obtained using the following recurrent formula

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{m}\lambda^i y_i\right)\right]_{\lambda=0}, \quad m = 0, 1, 2, \dots . \tag{8}$$

Moreover, when (7) - (8) are substituted into (6), one gets

$$\mathcal{L}\left[\sum_{m=0}^{\infty} y_m\right] = \frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{t}{s^n} + \frac{1}{s^n}\mathcal{L}[r(x)] - \frac{1}{s^n}\mathcal{L}\left[R\left(\left[\sum_{m=0}^{\infty} y_m\right]\right)\right] - \frac{1}{s^n}\mathcal{L}\left[\sum_{m=0}^{\infty} A_m\right],$$

or equally upon deploying the linearity of the Laplace transform the following

$$\sum_{n=0}^{\infty}\mathcal{L}[y_m] = \frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{t}{s^n} + \frac{1}{s^n}\mathcal{L}[r(x)] - \frac{1}{s^n}\sum_{m=0}^{\infty}\mathcal{L}[Ry_m] - \frac{1}{s^n}\sum_{m=0}^{\infty}\mathcal{L}[A_m].$$

Therefore, from the above equation, one acquires the resulting recursive scheme in the Laplace transform domain as follows

$$\begin{aligned} \mathcal{L}[y_0] &= \frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{t}{s^n} + \frac{1}{s^n}\mathcal{L}[r(x)] = K(s), \\ \mathcal{L}[y_{m+1}] &= -\frac{1}{s^n}\mathcal{L}[Ry_m] - \frac{1}{s^n}\mathcal{L}[A_m], \quad m \geq 0. \end{aligned} \tag{9}$$

Lastly, when the inverse Laplace transform is applied on (9), the overall recursive relationship is obtained in the following form

$$\begin{aligned} y_0 &= K(x), \\ y_{m+1} &= -\mathcal{L}^{-1}\left[\frac{1}{s^n}\mathcal{L}[Ry_m] + \frac{1}{s^n}\mathcal{L}[A_m]\right], \quad m \geq 0, \end{aligned} \tag{10}$$

where $K(x)$ denotes the terms originating from the prescribed initial data and the source term. Further, one obtains the explicit solution components $y_1, y_2, \dots, y_m$from (10) upon which the approximate solution of the governing model in (3) - (4) is obtained as follows

$$y_{M+1} = \sum_{m=0}^{M} y_m(x) .$$

In the same way, the linear case follows the same steps of the solution as in the nonlinear case, only that we should exclude the nonlinear term. Thus, one rewrites (3) as follows

$$Ly + Ry = r(x), \tag{11}$$

with the same initial conditions in (4). Then, on using LADM, the recursive relation for (11) with the initial condition (4) is obtained as follows

$$\begin{aligned} y_0 &= K(x), \\ y_{m+1} &= -\mathcal{L}^{-1}\left[\frac{1}{s^n}\mathcal{L}\left[Ry_m\right]\right], \quad m \geq 0. \end{aligned} \tag{12}$$

Therefore, one obtains an approximate closed-form solution for the linear model (11) together with the initial data (4) as follows

$$y_{M+1} = \sum_{m=0}^{M} y_m(x)\,.$$

3. Laplace decomposition shooting method

The shooting method [1, 8-15] is an iterative method that has been used for solving different BVPs by initially transforming the given BVPs into a system of coupled initial value problems (IVPs) with specified initial conditions. Besides, none of these resulting IVPs is treated analytically; so, the solution will be approximated using the one-step methods or multistep methods or directly using the ADM. In this paper, we will use the LDSM directly to solve IVPs in both the linear and nonlinear cases of the second- and third-order differential equations.

3.1 Linear case

Consider the generalized nth-order linear two-point BVPs as follows

$$y^{(n)}(x) = \sum_{j=1}^{n-1} P_{n-j+1}(x)y^{(n-j)}(x) + P_1(x)y(x) + r(x), \;\; 0 \leq x \leq b, \tag{13}$$

subject to the following two-point boundary data

$$y^{(i)}(0) = \alpha_i, \quad y(b) = \beta, \quad i = 0,1 \;\ldots, n-2, \tag{14}$$

where $P_{n-j+1}(x),\; j = 1,2,\ldots,n$ are prescribed known functions. Therefore, shooting method starts off by converting the governing nth-order BVP into two IVPs, upon which the endowed boundary data in (14) are changed as given in the following resulting IVPs

$$\begin{aligned} u^{(n)}(x) &= \sum_{j=1}^{n-1} P_{n-j+1}(x)u^{(n-j)}(x) + P_1(x)u(x) + r(x), \\ u^{(i)}(0) &= \alpha_i,\; u^{(n-1)}(0) = 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} v^{(n)}(x) &= \sum_{j=1}^{n-1} P_{n-j+1}(x)v^{(n-j)}(x) + P_1(x)v(x), \\ v^{(i)}(0) &= 0,\; v^{(n-1)}(0) = 1. \end{aligned} \tag{16}$$

Therefore, on using LADM, we get

$$\mathcal{L}\left[\sum_{m=0}^{\infty} u_m(x)\right] = \frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{1}{s^n}\mathcal{L}[r(x)] - \frac{1}{s^n}\mathcal{L}\Big(\sum_{j=1}^{n-1} P_{n-j+1}(x)\sum_{m=0}^{\infty} u_m^{(n-j)}(x) + P_1(x)\sum_{m=0}^{\infty} u_m(x)\Big),$$

and

$$\mathcal{L}\left[\sum_{m=0}^{\infty} v_m(x)\right] = \frac{1}{s^n} - \frac{1}{s^n}\mathcal{L}\Big(\sum_{j=1}^{n-1} P_{n-j+1}(x)\sum_{m=0}^{\infty} v_m^{(n-j)}(x) + P_1(x)\sum_{m=0}^{\infty} v_m(x)\Big).$$

Therefore, the recursive relationships will be as follows

$$\begin{cases} u_0 = \mathcal{L}^{-1}\left[\frac{\alpha_0}{s} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{1}{s^n}\mathcal{L}[r(x)]\right], \\ u_{m+1} = -\mathcal{L}^{-1}\left[\frac{1}{s^n}\mathcal{L}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)\, u_m^{(n-j)}(x) + P_1(x)\, u_m(x)\right)\right], \end{cases} \tag{17}$$

and

$$\begin{cases} v_0 = \mathcal{L}^{-1}\left[\frac{1}{s^n}\right], \\ v_{m+1} = -\mathcal{L}^{-1}\left[\frac{1}{s^n}\mathcal{L}\left(\sum_{j=1}^{n-1} P_{n-j+1}(x)\, v_m^{(n-j)}(x) + P_1(x)\, v_m(x)\right)\right], \end{cases} \tag{18}$$

where $m \geq 0$.

In addition, if $u(x)$ and $v(x)$ represent the solutions for the IVPs in (15) - (16) correspondingly, one further defines

$$z(x) = u(x) + \frac{\beta - u(b)}{v(b)} v(x), \ v(b) \neq 0, \tag{19}$$

as the approximate solution to governing linear BVP in (13) - (14).

3.2 Nonlinear case

Without further delay, the procedure of the shooting method on nonlinear BVPs is similar to that of the corresponding linear BVPs only that the solution to the nonlinear BVPs cannot be presumed as a linear sum of the solutions to two IVPs as in the case of linear BVPs. Thus, one approximates the solution to the BVP through the use of the solutions to several IVPs in t. So, we will convert the nth-order BVP in (1) - (2) to IVPs where we replace the boundary conditions with specific initial conditions. Precisely, this problem has the form

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right), \qquad 0 \leq x \leq b, \tag{20}$$

with the initial data given as follows

$$y^{(i)}(0) = \alpha_i, \ y^{(n-1)}(0) = t, \ i = 0,1 \ \dots, n-2. \tag{21}$$

We will then use the LDSM directly to treat (20) - (21) by selecting parameters $t = t_s$ in such a way that

$$\lim_{s\to\infty} y(b, t_s) = y(b) = \beta, \tag{22}$$

where $y(x, t_s)$ presents the solution to the IVP (20) - (21) with $t = t_s$, while $y(x)$ represents the solution to the BVP (1) - (2). Further, we seek an initial guess of the form $t_0 = \frac{\beta - \alpha}{b - a}$ in order to construct the solution of the first IVP alongside utilizing the Newton's method for finding the value for t_1 as follows

$$t_1 = t_0 - \frac{y(b, t_0) - \beta}{\frac{dy}{dt}(b, t_0)}. \tag{23}$$

Next, to determine the value of $\frac{dy}{dt}(b, t_0)$, we scale the IVP expressed in (20) - (21) to depend on x and t variables as follows

$$y^{(n)} = f\left(x, y(x,t), y'(x,t), \dots, y^{(n-1)}(x,t)\right), \tag{24}$$

with the initial data as follows

$$y^{(i)}(0,t) = \alpha_i, \quad y^{(n-1)}(0,t) = t, \ i = 0,1 \ \dots, n-2. \tag{25}$$

We then proceed to differentiate (24) – (25) partially in t. Then, if we let $z(x,t) = \frac{\partial y}{\partial t}(x,t)$, the above model now becomes

$$\begin{aligned} z^{(n)} = {} & \frac{\partial f}{\partial y}\left(x, y(x,t), y'(x,t), \dots, y^{(n-1)}(x,t)\right) z(x,t) + \\ & \frac{\partial f}{\partial y'}\left(x, y(x,t), y'(x,t), \dots, y^{(n-1)}(x,t)\right) z'(x,t) + \cdots + \\ & \frac{\partial f}{\partial y^{(n-1)}}\left(x, y(x,t), y'(x,t), \dots, y^{(n-1)}(x,t)\right) z^{(n-1)}(x,t), \end{aligned} \tag{26}$$

for $0 \le x \le b$, with the following transformed initial conditions as follows

$$z^{(i)}(0) = 0, \quad z^{(n-1)}(0) = 1. \tag{27}$$

Lastly, the IVP given above will be directly solved at t_s using the LADM, which gives $\frac{dy}{dt}(b, t_0) = z(x, t_0)$. Also, to determine the complete sequence, the guessed points t_s for $s = 2, 3, \dots$ together with the nonlinear function $y(b,t) - \beta = 0$ are thus obtained through the utilization of the Secant iterative method as follows

$$t_s = t_{s-1} - \frac{(y(b,t_{s-1}) - \beta)(t_{s-1} - t_{s-2})}{y(b,t_{s-1}) - y(b,t_{s-2})}, \qquad s = 2,3, \dots . \tag{28}$$

Moreover, the current study takes the following as the stopping criteria

$$|y(b, t_s) - \beta| \le tolerance. \tag{29}$$

4. Numerical examples

The current section demonstrates the efficacy of the devised LDSM scheme on several BVPs of both the second- and third-order ODEs that comprise various linear and nonlinear equations. In addition, the section equally beseeched various computational methods to assess the efficiency of the devised LDSM scheme, including the known Runge-Kutta method of order-fourth (SRKM4), and several other methods in the open literature [16-34]. In addition, some interesting helpful Figures 1-5 and Tables 1-10 have been reported; reporting various treads of the resulting absolute error differences that exist between the proposed LDSM scheme and various computational solutions posed by dissimilar methods.

Example 1 Consider the following linear second-order BVP [16-18]

$$y''(x) + 2\,y'(x) + 5y(x) = 6\cos(2x) - 7\sin(2x), \qquad y(0) = 4, \quad y\left(\frac{\pi}{4}\right) = 1.$$

The actual analytical solution for the present BVP is given as follows

$$y(x) = \sin(2x) + 2(1 + e^{-x})\cos(2x).$$

Accordingly, we consider the two IVPs as follows

$$u'' + 2\,u' + 5u = 6\cos(2x) - 7\sin(2x), \qquad u(0) = 4, \quad u'(0) = 0, \tag{30}$$

and

$$v'' + 2\,v' + 5v = 0, \qquad v(0) = 0, \quad v'(0) = 1. \tag{31}$$

Then, on applying the Laplace transform to (30) - (31) alongside using the new prescribed initial conditions, one gets the following

$$\mathcal{L}[u(x)] = \frac{4s+8}{s^2+2s+5} + \frac{6s-14}{(s^2+2s+5)(s^2+4)}, \tag{32}$$

and

$$\mathcal{L}[v(x)] = \frac{1}{s^2+2s+5}. \tag{33}$$

So, upon applying the LDSM procedure, one obtains from the IVPs (32) - (33) the following

$$\begin{cases} \mathcal{L}[u_0(x)] = \frac{4s+8}{s^2+2s+5} + \frac{6s-14}{(s^2+2s+5)(s^2+4)}, \\ \mathcal{L}[u_{m+1}(x)] = \frac{1}{s^2+2s+5}\mathcal{L}[0], \qquad m \geq 0, \end{cases} \tag{34}$$

and

$$\begin{cases} \mathcal{L}[v_0(x)] = \frac{1}{s^2+2s+5}, \\ \mathcal{L}[v_{m+1}(x)] = \frac{1}{s^2+2s+5}\mathcal{L}[0], \qquad m \geq 0. \end{cases} \tag{35}$$

Next, on taking the inverse Laplace transform of (34) - (35), one obtains the recursive relations as follows

$$\begin{cases} u_0(x) = 2(1+e^{-x})\cos(2x) + \sin(2x), \\ u_{m+1}(x) = 0, \qquad m \geq 0, \end{cases}$$

and

$$\begin{cases} v_0(x) = \frac{1}{2}e^{-x}\sin(2x), \\ v_{m+1}(x) = 0, \qquad m \geq 0. \end{cases}$$

Then, the solutions of the IVPs in (30) - (31) are thus obtained from the above schemes upon taking the respective series summations. Lastly, the approximate computational solution of the governing BVP is obtained as $z(x)$ when $M = 20$ and $h = \frac{\pi}{80}$. Therefore, for $x_k = kh$ for $k = 0,1,\ldots,M$, the approximate solution then takes the following approximation $z(x_k) = u(x_k) + \frac{1-u\left(\frac{\pi}{4}\right)}{v\left(\frac{\pi}{4}\right)} v(x_k)$.

The absolute errors of LDSM and SRKM4 solutions when $h = \frac{\pi}{80}$ are presented in Table 1. It is clear that that the proposed method agreed with the exact solutions.

$\mathbf{x}$	E_{SRKM4}	E_{LDSM}
$\mathbf{0}$	0	0
$\boldsymbol{\frac{\pi}{40}}$	1.3×10^{-7}	1.0×10^{-39}
$\boldsymbol{\frac{\pi}{20}}$	2.4×10^{-7}	1.0×10^{-39}
$\boldsymbol{\frac{3\pi}{40}}$	3.3×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{\pi}{10}}$	3.9×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{\pi}{8}}$	4.1×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{3\pi}{20}}$	3.9×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{7\pi}{40}}$	3.4×10^{-7}	3.0×10^{-39}
$\boldsymbol{\frac{\pi}{5}}$	2.6×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{9\pi}{40}}$	1.4×10^{-7}	2.0×10^{-39}
$\boldsymbol{\frac{\pi}{4}}$	0	2.0×10^{-39}

Table 1: Absolute error of LDSM and SRKM4 solutions when $h = \frac{\pi}{80}$.

In Table 2, the comparison of maximum errors between the proposed method and the mentioned methods is given. Obviously, our method generated more accurate results compared with the others.

Numerical Methods	**CBS [16]**	**NCBS[17]**	**new CBS [18]**	**SRKM4**	**LDSM**
Maximum Error	6.3×10^{-5}	5.1×10^{-6}	9.3×10^{-8}	4.1×10^{-7}	3.0×10^{-39}

Table 2: Comparison of maximum errors when $h = \frac{\pi}{80}$.

Figure 1 graphically depicts all three solutions for graphical visualization, including the actual solution, the proposed LDSM solution, and the beseeched SRKM4 solution for computational validation.

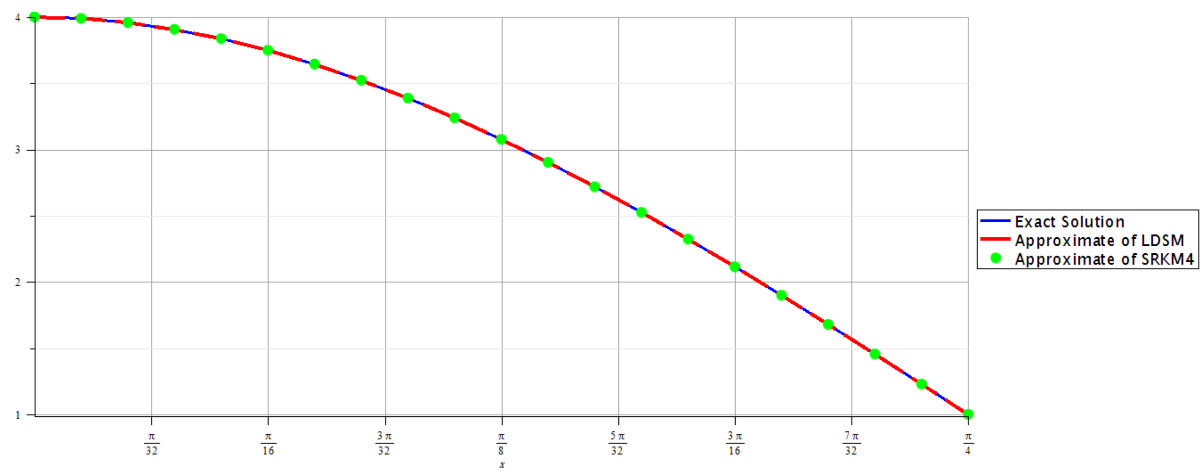

Figure 1: Graphical comparison, depicting the actual and competing approximate solutions with $h = \frac{\pi}{80}$.

Example 2 Consider the following linear third-order BVP [19-20]

$$y'''(x) + y(x) = (x-4)\sin(x) + (1-x)\cos(x), \quad y(0) = 0, \quad y'(0) = -1, \quad y(1) = 0.$$

The actual solution of the present two-point BVP is further reported as follows

$$y(x) = (x-1)\sin(x).$$

Accordingly, we consider the two IVPs to have following forms

$$u''' + u = (x-4)\sin(x) + (1-x)\cos(x), \quad u(0) = 0, \ u'(0) = -1, \ u''(0) = 0, \tag{36}$$

and

$$v''' + v = 0, \qquad v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 1. \tag{37}$$

Therefore, on deploying the proposed method, the IVPs in (36) - (37) are recurrently expressed as follows

$$\begin{cases} \mathcal{L}[u_0(x)] = -\frac{s}{s^3+1} + \frac{s^3-5s^2+3s-3}{(s^3+1)(s^2+1)^2}, \\ \mathcal{L}[u_{m+1}(x)] = \frac{1}{s^3+1}\mathcal{L}[0], \qquad m \geq 0, \end{cases} \tag{38}$$

and

$$\begin{cases} \mathcal{L}[v_0(x)] = \frac{1}{s^3+1}, \\ \mathcal{L}[v_{m+1}(x)] = \frac{1}{s^3+1}\mathcal{L}[0], \qquad m \geq 0. \end{cases} \tag{39}$$

upon which when applying the inverse Laplace transform to (38) and (39) yields the following recursive scheme

$$\begin{cases} u_0(x) = -\frac{2}{3}e^{-x} + \frac{2}{3}e^{\frac{1}{2}x}\left(-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}x\right) + \cos\left(\frac{\sqrt{3}}{2}x\right)\right) + (x-1)\sin(x), \\ u_{m+1}(x) = 0, \qquad m \geq 0, \end{cases}$$

and

$$\begin{cases} v_0(x) = \frac{1}{3}e^{-x} + \frac{1}{3}e^{\frac{1}{2}x}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}x\right) - \cos\left(\frac{\sqrt{3}}{2}x\right)\right), \\ v_{m+1}(x) = 0, \qquad m \geq 0. \end{cases}$$

Then, the solutions to the IVPs in (36) - (37) are thus obtained from the above schemes upon taking the respective series summations. Consequently, one obtains the approximate computational solution via the help of the deployed LDSM $z(x)$ with $M = 8$ and $h = \frac{1}{8}$, using $z(x_k) = u(x_k) + \frac{-u(1)}{v(1)} v(x_k)$, $x_k = kh$, $k = 0,1,\ldots,M$.

In Table 3, the absolute errors for $h = \frac{1}{8}$ are provided. We can say that the proposed method fit well with the exact solutions.

x	E_{SRKM4}	E_{LDSM}
0	0	0
$\frac{1}{8}$	5.3×10^{-7}	0
$\frac{1}{4}$	1.0×10^{-6}	3.0×10^{-40}
$\frac{3}{8}$	1.4×10^{-6}	1.0×10^{-40}
$\frac{1}{2}$	1.6×10^{-6}	1.0×10^{-40}
$\frac{5}{8}$	1.7×10^{-6}	2.0×10^{-40}
$\frac{3}{4}$	1.4×10^{-6}	1.0×10^{-40}
$\frac{7}{8}$	8.8×10^{-7}	2.2×10^{-40}
1	1.0×10^{-40}	2.0×10^{-40}

Table 3: Absolute error of LDSM and SRKM4 solutions when $h = \frac{1}{8}$.

The maximum errors between the proposed method and the other methods are demonstrated in Table 4. Evidently, our results were better than the others.

Numerical Methods	**2PQS [19]**	**4PQS [19]**	**6PQS [19]**	**SC3 − PGM [20]**	**SC4 − PGM[20]**	**SRKM4**	**LDSM**
Maximum Error	1.3×10^{-2}	3.6×10^{-8}	2.7×10^{-9}	8.6×10^{-10}	6.7×10^{-10}	1.7×10^{-6}	3.0×10^{-40}

Table 4: Comparison of maximum errors when $h = \frac{1}{8}$.

Further, when the actual, LDSM and SRKM4 solutions are graphically depicted as shown in Figure 2, one may observe an excellent agreement among the three solutions.

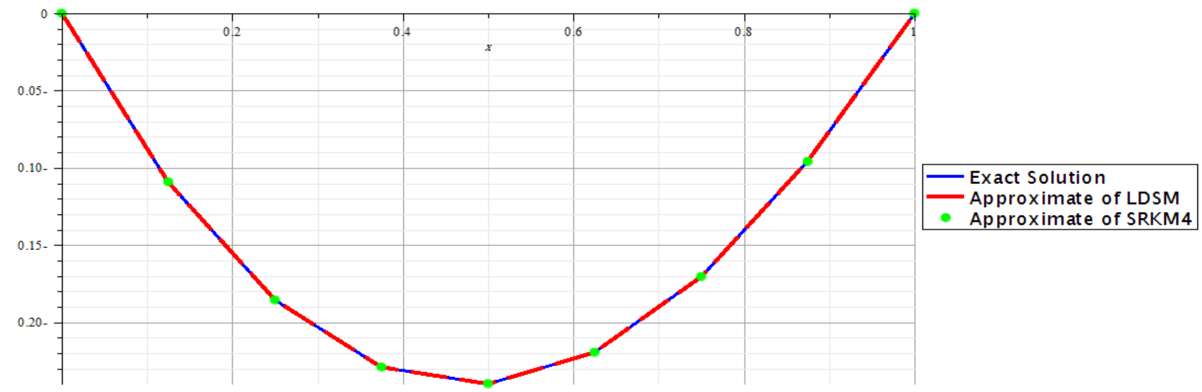

Figure 2: Graphical comparison, depicting the actual and competing approximate solutions with $h = \frac{1}{8}$.

Example 3 Consider of the following nonlinear second-order BVP [21]

$$y''(x) + x\,y^2(x)y'(x) + x^2\,y^3(x) = r(x),\ y(0) = 0,\quad y(1) = 0, \tag{40}$$

where $r(x) = (1-x)\left(6(-1+x) + 6x + (-1+x)^8x^5 + (-1+x)^7x^3(-1+4x)\right)$.

This nonlinear BVP admits the following actual analytical solution

$$y(x) = x(1-x)^3.$$

Therefore, for the given BVP, one considers the two IVPs as follows

$$y'' = -x\,y^2(x)y'(x) - x^2\,y^3(x) + r(x), \qquad y(0) = 0,\quad y'(0) = t_s, \tag{41}$$

and

$$z'' = \left(-2xy(x)y'(x) - 3x^2\,y^2(x)\right)z(x) - x\,y^2(x)z'(x), \qquad z(0) = 0,\quad z'(0) = 1. \tag{42}$$

Accordingly, when deploying the proposed method on the IVPs in (41) - (42) and upon using the initial conditions, then the recursive equations are obtained as follows

$$\begin{cases} \mathcal{L}[y_0(x)] = \frac{t_s}{s^2} - \frac{6}{s^{17}}(s^{14} - 3s^{13} + 4s^{12} - s^{11} + 48s^{10} - 1220s^9 + 21240s^8 \\ \quad -277200s^7 + 2822400s^6 - 22861440s^5 + 148780800s^4 \\ \quad -778377600s^3 + 3193344000s^2 - 9340531200s + 14529715200), \\ \mathcal{L}[y_{m+1}(x)] = -\frac{1}{s^2}\mathcal{L}[xA_m(x)] - \frac{1}{s^2}\mathcal{L}[x^2B_m(x)], \qquad m \geq 0, \end{cases} \tag{43}$$

and

$$\begin{cases} \mathcal{L}[z_0(x)] = \frac{1}{s^2}, \\ \mathcal{L}[z_{m+1}(x)] = -\frac{1}{s^2}\mathcal{L}\left[\left(2xy_m(x)y'_m(x) + 3x^2\,{y_m}^2(x)\right)z_m(x)\right] - \frac{1}{s^2}\mathcal{L}[x{y_m}^2(x)z'_m(x)],\ \ m \geq 0, \end{cases} \tag{44}$$

where A_m and B_m in the above schemes denote the Adomian polynomials for y^2y' and y^3, respectively, which are the nonlinear terms present in the acquired IVPs. Next, on applying the inverse Laplace transform to (43) - (44), we obtain

$$\begin{cases} y_0(x) = t_s x - \dfrac{1}{240240}x^2(1001x^{14} - 10296x^{13} + 52800x^{12} - 180180x^{11} + 447720x^{10} \\ \quad -825552x^9 + 1121120x^8 - 1101100x^7 + 759330x^6 - 348920x^5 \\ \quad +96096x^4 - 12012x^3 + 240240x^2 - 720720x + 720720), \\ y_{m+1}(x) = \mathcal{L}^{-1}\left[-\dfrac{1}{s^2}\mathcal{L}[xA_m(x)] - \dfrac{1}{s^2}\mathcal{L}[x^2B_m(x)]\right], \qquad m \geq 0, \end{cases}$$

and

$$\begin{cases} z_0(x) = x, \\ z_{m+1}(x) = \mathcal{L}^{-1}\left[-\dfrac{1}{s^2}\mathcal{L}\left[\left(2xy_m(x)y'_m(x) + 3x^2\, {y_m}^2(x)\right)z_m(x)\right] - \dfrac{1}{s^2}\mathcal{L}[x{y_m}^2(x)z'_m(x)]\right], \quad m \geq 0. \end{cases}$$

Consequently, the solution for (41) when $M = 4$ is obtained in a series form as

$$y(x) = \sum_{m=0}^{4} y_m = y_0 + y_1 + \cdots + y_4.$$

Therefore, when using 7 iterations, then $y(x, t_6)$ represents the solution to the second-order BVP (40) with $t = t_6$.

Table 5 lists the absolute error when $h = \frac{1}{4}$. We can say that the proposed method was in good agreement with the exact solutions

x	E_{SRKM4}	E_{LDSM}
0	0	0
$\mathbf{\frac{1}{4}}$	2.7×10^{-6}	1.6×10^{-19}
$\mathbf{\frac{1}{2}}$	3.8×10^{-6}	1.8×10^{-19}
$\mathbf{\frac{3}{4}}$	1.8×10^{-6}	3.4×10^{-19}
1	2.9×10^{-32}	5.6×10^{-29}

Table 5: Absolute error of LDSM and SRKM4 solutions when $h = \frac{1}{4}$.

The maximum errors between the proposed method and SGM method are presented in Table 6. Clearly, our method produced better approximations compared with SGM.

Numerical Methods	SGM [21]	SRKM4	LDSM
Maximum Error	9.9×10^{-3}	3.8×10^{-6}	3.4×10^{-19}

Table 6: Comparison of maximum errors when $h = \frac{1}{4}$.

Figure 3 graphically illustrates the pictorial representation of the devised approach's solution in comparison with those of the contesting SRKM4 and the actual solution. Invariably, the proposed method's solution matches that of the actual solution with good precision.

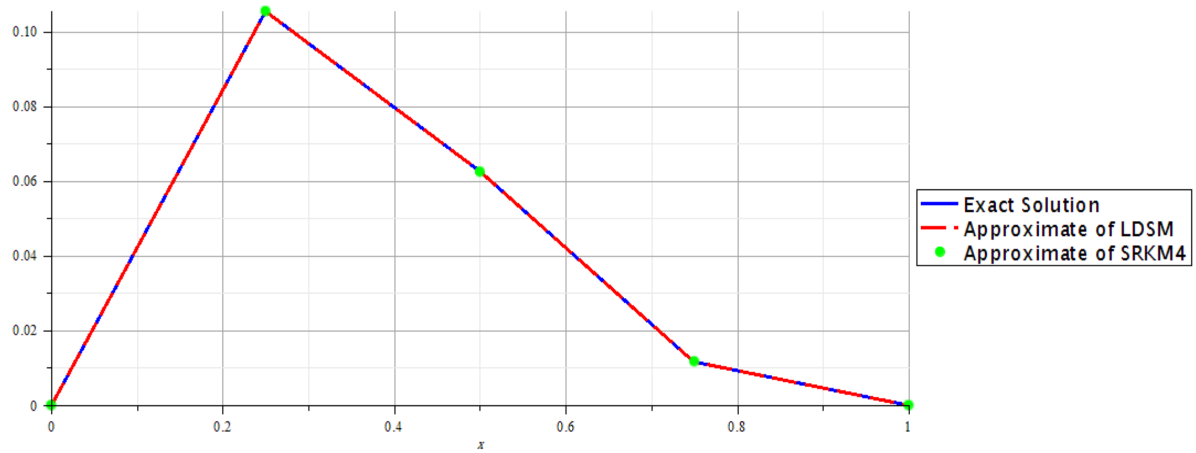

Figure 3: Graphical comparison, depicting the actual and competing approximate solutions with $h = \frac{1}{4}$.

Example 4 Consider the following nonlinear third-order BVP [22]

$$y'''(x) = y^2 - y(e^x + x) + e^x,\ y(0) = 1,\ y'(0) = 2,\ y(1) = e^1 + 1, \tag{45}$$

that is being satisfied by the following actual analytical solution

$$y(x) = e^x + x.$$

Accordingly, we consider the two IVPs as follows

$$y''' = y^2 - y(e^x + x) + e^x,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = t_s, \tag{46}$$

and

$$z''' = 2yz - (e^x + x)\, z, \qquad z(0) = 0,\ z'(0) = 0,\ z''(0) = 1. \tag{47}$$

Moreover, the recursive relations of the above IVPs via LDSM are as follows

$$\begin{cases} y_0 = x + e^x + \frac{x^2}{2}(-1 + t_s), & \\ y_{m+1} = \mathcal{L}^{-1}\left[\frac{1}{s^3}\mathcal{L}[A_m(x)] - \frac{1}{s^3}\mathcal{L}[y_m(e^x + x)]\right], & m \geq 0, \end{cases}$$

and

$$\begin{cases} z_0 = \frac{x^2}{2}, & \\ z_{m+1} = \mathcal{L}^{-1}\left[\frac{1}{s^3}\mathcal{L}[2y_m z_m] - \frac{1}{s^3}\mathcal{L}[(e^x + x)z_m]\right], & m \geq 0, \end{cases}$$

where A_m in the first recursive scheme denotes the Adomian polynomials in favour of the nonlinear term y^2. Further, equation (46) when $M = 2$ admits the following series solution

$$y(x) = \sum_{m=0}^{2} y_m = y_0 + y_1 + y_2.$$

Therefore, when using 6 iterations, $y(x, t_5)$ represents the solution to the third-order BVP (45) with $t = t_5$.

Table 7 displays the absolute error when $h = \frac{1}{2}$. Clearly, the proposed method worked well with the exact solutions.

x	E_{SRKM4}	E_{LDSM}
0	0	0
$\frac{1}{2}$	5.7×10^{-5}	1.6×10^{-30}
1	2.5×10^{-26}	6.5×10^{-30}

Table 7: Absolute error of LDSM and SRKM4 solutions when $h = \frac{1}{2}$.

The maximum errors between the proposed method and the MADM method are demonstrated in Table 8. Again, our method produced better approximations compared with MADM.

Numerical Methods	MADM [22]	SRKM4	LDSM
Maximum Error	9.0×10^{-3}	5.7×10^{-5}	6.5×10^{-30}

Table 8: Comparison of maximum errors

See Figure 4 for the graphical illustration, graphically comparing the proposed method's solution with those of the actual solution, and the competing SRKM4.

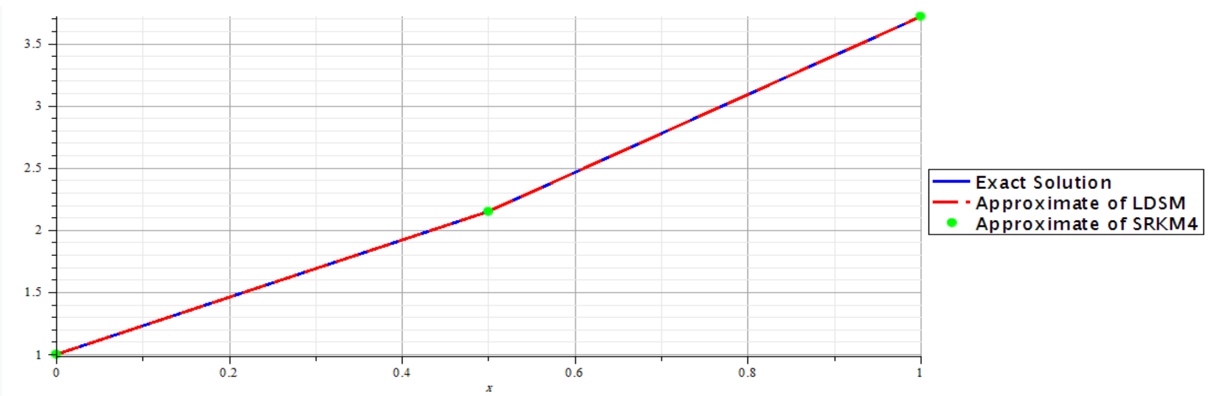

Figure 4: Graphical comparison, depicting the actual and competing approximate solutions with $h = \frac{1}{2}$.

Example 5 Consider the two-point Bratu-type BVP as follows [23-34]

$$y''(x) + \lambda e^{y} = 0, \; y(0) = 0, \quad y(1) = 0. \tag{48}$$

Case 1: when $\lambda =$ 0.1, 0.5, and 1, the actual solution of the BVP (48) is reported as follows

$$y(x) = -2\,ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \text{ where } \theta \text{ satisfies } \theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right).$$

Case 2: when $\lambda = -1$, the actual solution of the BVP (48) is reported as follows

$$y(x) = 2\ln\left(\theta \sec\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)\right) - \ln 2\,,\ \text{with } \theta \text{ satisfying } \theta\sec\left(\frac{\theta}{4}\right) = \sqrt{2}.$$

Consequently, we consider the two IVPs as follows

$$y'' = -\lambda e^{y}, \qquad y(0) = 0,\ y'(0) = t_s, \tag{49}$$

and

$$z'' = -\lambda e^{y}\, z, \qquad z(0) = 0,\ z'(0) = 1. \tag{50}$$

Moreover, the recursive relations of the above IVPs via LDSM are as follows

$$\begin{cases} y_0 = t_s\, x, & \\ y_{m+1} = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[-\lambda A_m(x)]\right], & m \geq 0, \end{cases}$$

and

$$\begin{cases} z_0 = x\,, & \\ z_{m+1} = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[-\lambda e^{y_m} z_m]\right], & m \geq 0, \end{cases}$$

where A_m in the first recursive scheme denotes the Adomian polynomials in favor of the nonlinear term e^{y}. Further, equation (49) when $M = 10$ admits the following series solution

$$y(x) = \sum_{m=0}^{10} y_m = y_0 + y_1 + \cdots + y_{10}.$$

Therefore, when using 7 iterations, then $y(x, t_6)$ represents the solution to the second-order BVP (48) with $t = t_6$. The computed absolute errors at different values of λ are tabulated in Table 9.

λ	0.1		0.5		1		-1	
x	E_{SRKM4}	E_{LDSM}	E_{SRKM4}	E_{LDSM}	E_{SRKM4}	E_{LDSM}	E_{SRKM4}	E_{LDSM}
0	0	0	0	2.0×10^{-29}	0	0	5.0×10^{-40}	1.0×10^{-31}
0.1	6.7×10^{-11}	5.4×10^{-21}	9.5×10^{-9}	6.8×10^{-13}	9.1×10^{-8}	5.3×10^{-9}	5.2×10^{-8}	5.9×10^{-11}
0.2	1.2×10^{-10}	1.1×10^{-20}	1.7×10^{-8}	1.4×10^{-12}	1.7×10^{-7}	1.0×10^{-8}	8.9×10^{-8}	1.2×10^{-10}
0.3	1.6×10^{-10}	1.6×10^{-20}	2.3×10^{-8}	2.0×10^{-12}	2.3×10^{-7}	1.6×10^{-8}	1.1×10^{-7}	1.8×10^{-10}
0.4	1.8×10^{-10}	2.1×10^{-20}	2.7×10^{-8}	2.7×10^{-12}	2.7×10^{-7}	2.1×10^{-8}	1.3×10^{-7}	2.4×10^{-10}
0.5	1.9×10^{-10}	2.7×10^{-20}	2.8×10^{-8}	3.3×10^{-12}	2.8×10^{-7}	2.5×10^{-8}	1.3×10^{-7}	3.0×10^{-10}
0.6	1.8×10^{-10}	3.2×10^{-20}	2.7×10^{-8}	3.9×10^{-12}	2.7×10^{-7}	3.0×10^{-8}	1.3×10^{-7}	3.7×10^{-10}
0.7	1.6×10^{-10}	3.7×10^{-20}	2.3×10^{-8}	4.5×10^{-12}	2.3×10^{-7}	3.4×10^{-8}	1.1×10^{-7}	4.4×10^{-10}
0.8	1.2×10^{-10}	4.1×10^{-20}	1.7×10^{-8}	5.1×10^{-12}	1.7×10^{-7}	3.7×10^{-8}	8.7×10^{-8}	5.1×10^{-10}
0.9	6.8×10^{-11}	4.2×10^{-20}	9.6×10^{-9}	5.1×10^{-12}	9.3×10^{-8}	3.7×10^{-8}	5.1×10^{-8}	5.1×10^{-10}
1	1.2×10^{-41}	4.0×10^{-22}	1.3×10^{-40}	0	2.1×10^{-34}	1.4×10^{-32}	5.5×10^{-36}	3.0×10^{-31}

Table 9: Absolute error of LDSM and SRKM4 solutions when $h = \frac{1}{10}$ and different λ.

A comparison with the other methods reported in the literature are presented in Table 10 shows that our method is computationally effective. Clearly, the LDSM method is more accurate than of other methods reported in the literature.

λ =0.1		**λ =0.5**		**λ =1**		**λ =-1**	
Numerical Methods	**Maximum Error**	**Numerical Methods**	**Maximum Error**	**Numerical Methods**	**Maximum Error**	**Numerical Methods**	**Maximum Error**
LDSM	4.2×10^{-20}	**LDSM**	5.1×10^{-12}	**LDSM**	3.7×10^{-8}	**LDSM**	5.1×10^{-10}
SRKM4	1.9×10^{-10}	**SRKM4**	2.8×10^{-8}	**SRKM4**	2.8×10^{-7}	**SRKM4**	1.3×10^{-7}
BCM [23]	2.4×10^{-15}	**INCFD [26]**	3.1×10^{-8}	**RADM with Taylor [29]**	9.4×10^{-7}	**ADM-RKM [32]**	8.0×10^{-6}
IFD [24]	1.7×10^{-8}	**VIM [27]**	4.2×10^{-5}	**SCM [30]**	6.2×10^{-5}	**NMADM [33]**	1.0×10^{-6}
DESG [25]	2.7×10^{-16}	**BCM [28]**	7.2×10^{-12}	**LGSM [31]**	1.0×10^{-6}	**MAM-ADW [34]**	2.2×10^{-6}

Table 10: Comparison of maximum errors when $h = \frac{1}{10}$ and different λ.

The graph of the actual and the approximate solutions for different values of λ has been plotted in Figure 5.

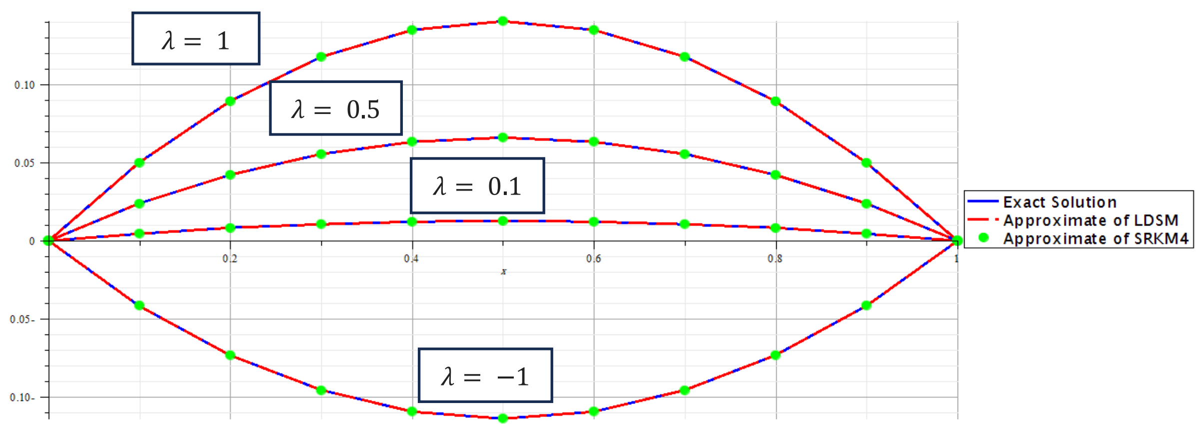

Figure 5: Graphical comparison, depicting the actual and competing approximate solutions with $h = \frac{1}{10}$ and different λ.

5. Conclusions

In conclusion, an efficient modified approximation method has been introduced in this study to efficiently tackle the linear and nonlinear cases of the second- and third-order two-point BVPs of ODEs to find their approximate solutions. The proposed method is noted to be numerically robust and economical, after being applied to various test BVPs and noted to majorly outperform several contesting computational approaches. Lastly, we have supported the findings of the present study with some comparison plots and tables – signifying the usefulness of the devised method. In addition, the initiated technique can be applied to diverse models of real-life applications, including the nonlinear Bratu-typed problems, which emerge in various areas of physical relevance. Lastly, this devised approach is recommended for solving diverse classes of both linear and nonlinear equations, including the complex-valued evolution equations that require robust approximation methods.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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